Far-from-equilibrium hydrodynamics, resummed transport coefficients, and attractor solutions

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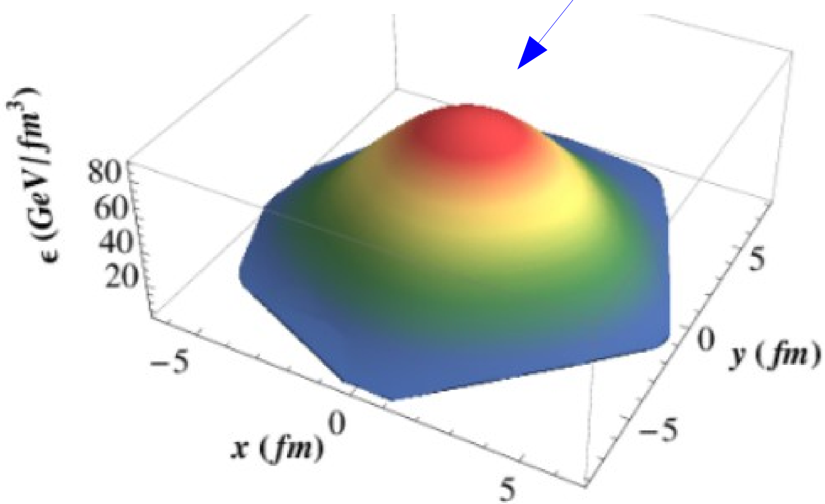
Quark-gluon plasma and relativistic hydrodynamics

At first, it seemed that hydrodynamics was “easily” justifiable

Very smooth fluid over nuclear length scales

quark-gluon plasma



near equilibrium dynamics

macro $\partial\varepsilon/\varepsilon_0 \sim 1/L$

micro $\ell \sim 1/T \sim 1/\Lambda_{QCD}$

Knudsen number

$$K_N \sim \ell\,\partial\varepsilon < 0.1$$

Fluid dynamics at scales of the size of a large nucleus

Standard view (past 100 years): **Gradient expansion**

Hydrodynamics → Effective theory for $\{T, u_\mu\}$ near local equilibrium*

*See G. Denicol's talk for a critical view

Include all possible contributions allowed by symmetries

$$T^{\mu\nu} = T^{\mu\nu}_{ideal} + \mathcal{O}(\partial^\mu u^\nu) + \mathcal{O}(\partial^2 u, \partial^2 T) + \ldots$$

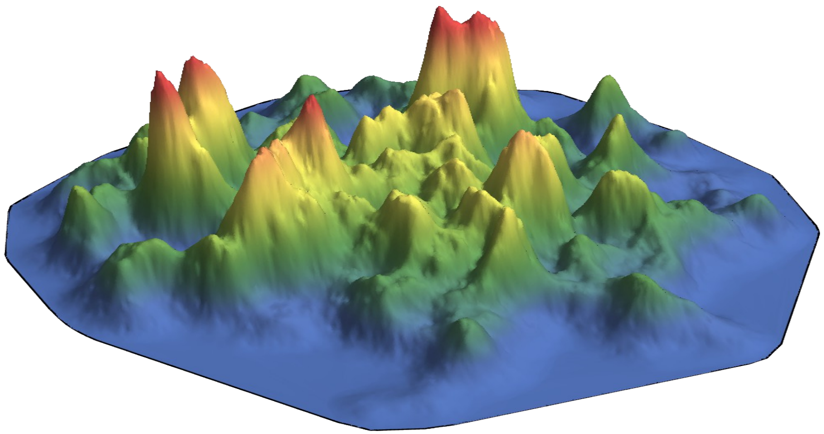
- $T^{\mu\nu}_{ideal}$: Zero viscosity
- $\mathcal{O}(\partial^\mu u^\nu)$: 1st order Kn Navier-Stokes
- $\mathcal{O}(\partial^2 u, \partial^2 T)$: 2nd order Kn^2 (BRSSS)

This is expected to be a divergent series in expanding systems

See M. Heller's talk

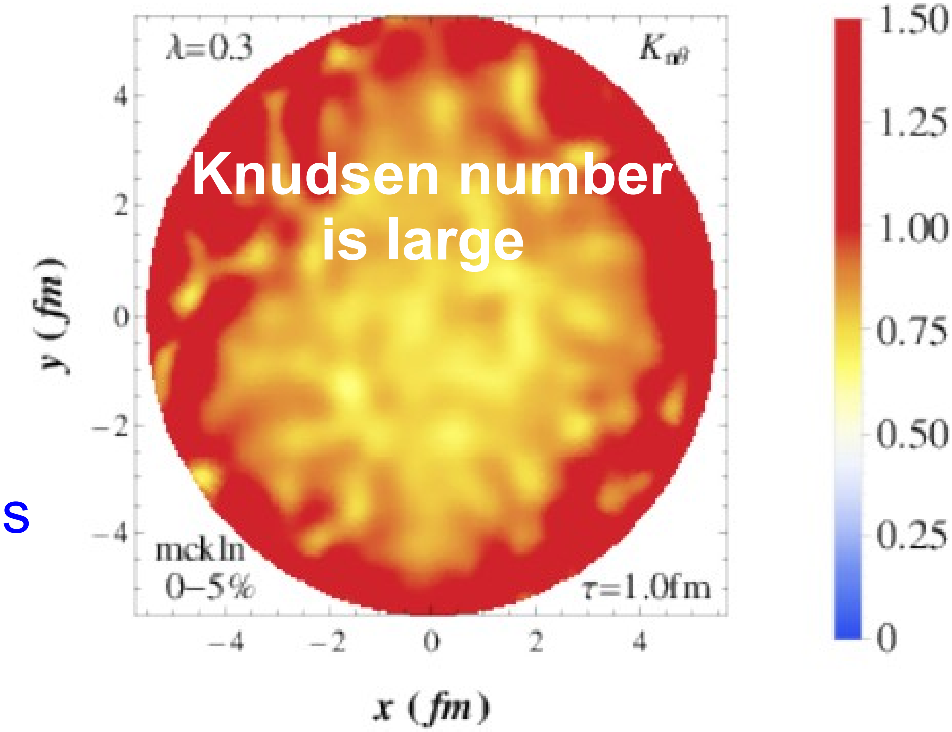
However, even in realistic AA collisions ...

QGP energy density



Large spatial gradients at early times

J. Noronha-Hostler, JN, Gyulassy, PRC 2016



This should be even worse in small systems.

PARADOX? Knudsen number not small but "hydro" works

Incidentally, the theory derived from the gradient expansion was never used in heavy ion simulations.

Israel, Stewart, Ann. Phys. 118, 341 (1979)

“Hydro” in HIC → Israel-Stewart-like theories

*kinetic theory

$T_{\mu\nu}$: variables → $\varepsilon, u_\mu, \pi_{\mu\nu}, \Pi$

Dynamics: $\nabla_\mu T^{\mu\nu} = 0$ (energy-momentum conservation)

$$u^\lambda \nabla_\lambda \pi^{\mu\nu} + F^{\mu\nu}(\varepsilon, \nabla_\alpha u_\beta, \pi^{\alpha\beta}, \Pi) = 0$$ (shear)

$$u^\lambda \nabla_\lambda \Pi + F(\varepsilon, \nabla_\alpha u_\beta, \pi^{\alpha\beta}, \Pi) = 0$$ (bulk)

- Israel-Stewart theories were used because they can be linearly stable and causal around equilibrium (gradient expansion isn't't).

- Causality and existence of solutions can be proven in the nonlinear regime under general conditions (with bulk viscosity).

Bemfica, Disconzi, JN, PRL (2019)

- Israel-Stewart theories resum gradients from the get-go since

$\Pi^{\mu\nu}, \Pi$ **obey differential equations, NOT a polynomial in gradients**

Ex: Sound dispersion relation in Israel-Stewart contains all terms:

Small k

$$\omega(k) = \pm c_s k - ik^2\left(\frac{2}{3}\frac{\eta}{sT} + \frac{1}{2}\frac{\zeta}{sT}\right) + \mathcal{O}(k^3)$$

Large k

$$\omega(k) \sim k$$

Far-from-equilibrium hydrodynamics → Attractors

Talks by M. Heller, S. Schlichting

From paradox to paradigm

The analytical hydrodynamic attractor

G. Denicol, JN, PRD 2018

The equations of motion of Israel-Stewart theory with constant τ_R

$$D\varepsilon + (\varepsilon + P)\theta - \pi^{\mu\nu}\sigma_{\mu\nu} = 0$$

$$(\varepsilon + P)Du^\mu - \Delta^\mu_\lambda \nabla^\lambda P + \Delta^\mu_\lambda \nabla_\mu \pi^{\mu\lambda} = 0$$

$$\tau_R \Delta^{\mu\nu}_{\alpha\beta} D\pi^{\alpha\beta} + \delta_{\pi\pi}\,\theta\pi^{\mu\nu} + \tau_{\pi\pi}\,\Delta^{\mu\nu}_{\alpha\beta}\pi^{\alpha\lambda}\sigma^\beta_\lambda - 2\,\tau_R \Delta^{\mu\nu}_{\alpha\beta}\pi^\alpha_\lambda \omega^{\beta\lambda} + \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu},$$

+ Bjorken flow

$$\varepsilon = 3P$$

can be FULLY solved analytically

Energy density

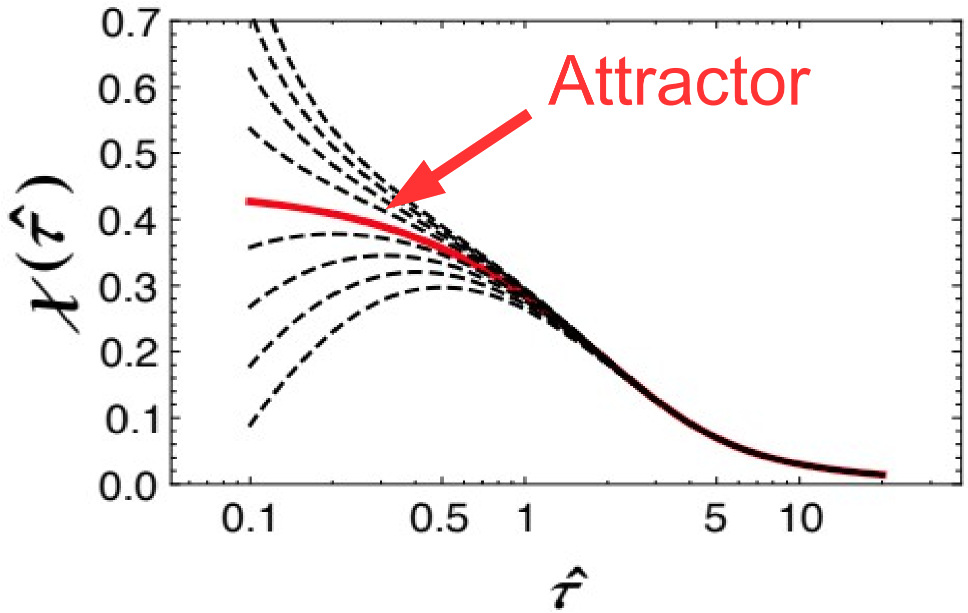
$$\varepsilon(\hat{\tau}) = \varepsilon_0\, e^{-\frac{1}{2}(\hat{\tau}-\hat{\tau}_0)} \left(\frac{\hat{\tau}_0}{\hat{\tau}}\right)^{\frac{5}{6}} \left[\frac{\alpha\left(K_{\sqrt{a}-\frac{1}{2}}\left(\frac{\hat{\tau}}{2}\right) - K_{\frac{1}{2}+\sqrt{a}}\left(\frac{\hat{\tau}}{2}\right)\right) + I_{\sqrt{a}-\frac{1}{2}}\left(\frac{\hat{\tau}}{2}\right) + I_{\frac{1}{2}+\sqrt{a}}\left(\frac{\hat{\tau}}{2}\right)}{\alpha\left(K_{\sqrt{a}-\frac{1}{2}}\left(\frac{\hat{\tau}_0}{2}\right) - K_{\frac{1}{2}+\sqrt{a}}\left(\frac{\hat{\tau}_0}{2}\right)\right) + I_{\sqrt{a}-\frac{1}{2}}\left(\frac{\hat{\tau}_0}{2}\right) + I_{\frac{1}{2}+\sqrt{a}}\left(\frac{\hat{\tau}_0}{2}\right)}\right]$$

Full solution for shear stress tensor

$$\chi(\hat{\tau}) = \frac{\pi}{\varepsilon + P} = \frac{3\sqrt{a}}{4}\left[\frac{\alpha\left(K_{\sqrt{a}-\frac{1}{2}}\left(\frac{\hat{\tau}}{2}\right) + K_{\sqrt{a}+\frac{1}{2}}\left(\frac{\hat{\tau}}{2}\right)\right) + I_{\sqrt{a}-\frac{1}{2}}\left(\frac{\hat{\tau}}{2}\right) - I_{\sqrt{a}+\frac{1}{2}}\left(\frac{\hat{\tau}}{2}\right)}{\alpha\left(K_{\sqrt{a}-\frac{1}{2}}\left(\frac{\hat{\tau}}{2}\right) - K_{\sqrt{a}+\frac{1}{2}}\left(\frac{\hat{\tau}}{2}\right)\right) + I_{\sqrt{a}-\frac{1}{2}}\left(\frac{\hat{\tau}}{2}\right) + I_{\sqrt{a}+\frac{1}{2}}\left(\frac{\hat{\tau}}{2}\right)}\right]$$

First analytical expression for a hydrodynamic attractor

$$\chi(\hat{\tau}) \to \chi_{att}(\hat{\tau}) = \frac{3\sqrt{a}}{4}\left[\frac{I_{\sqrt{a}-\frac{1}{2}}\left(\frac{\hat{\tau}}{2}\right) - I_{\sqrt{a}+\frac{1}{2}}\left(\frac{\hat{\tau}}{2}\right)}{I_{\sqrt{a}-\frac{1}{2}}\left(\frac{\hat{\tau}}{2}\right) + I_{\sqrt{a}+\frac{1}{2}}\left(\frac{\hat{\tau}}{2}\right)}\right]$$



Non-perturbative behavior

$$\exp\{-1/K_N\}$$

Resummation of gradient expansion

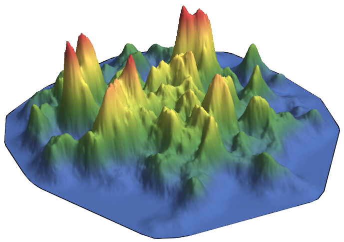
Trans-series

A partial list of the challenges one faces here

- Real fluids have many Knudsen numbers (fields).

Ex: Imagine trying to find leading terms in resurgent series in QFT with many couplings

- Gradient resummation in realistic flows?



- How does a many-body system display attractor behavior?

- Resummed transport coefficients? “Kubo” formulas out-of-equilibrium?

Far-from-equilibrium hydrodynamics for general flows

G. Denicol, JN, to appear (this week?)

Assumption → Dynamics described solely using $\{\varepsilon, u_\mu\}$ $\mu_B = 0$

Attractor → Emergence of constitutive relation for dissipative stress beyond gradient expansion (not a simple truncated polynomial)

$$\pi^{\mu\nu} \sim \sum_{n=0}^{\infty} c_n^{(0),\mu\nu} (K_N)^n + c^{(1),\mu\nu} (K_N)^\beta e^{-S/K_N} + \ldots$$

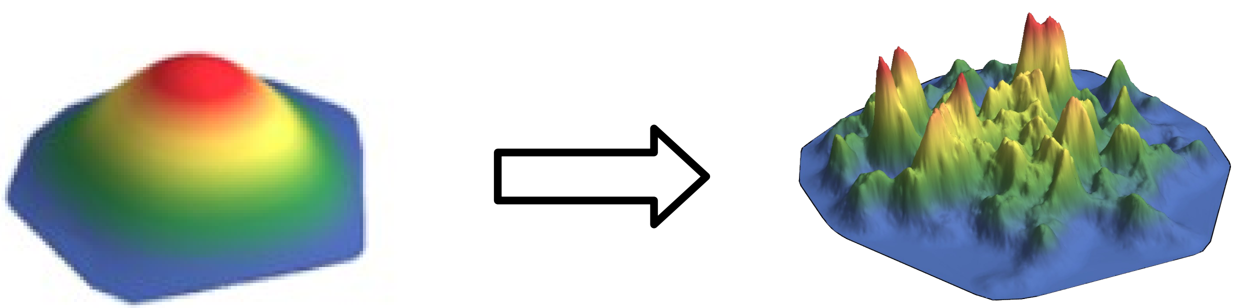
Trans-series (schematic)

How does this come about?

Consider the usual gradient expansion (K_N << 1)

$$\pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} + \eta_1\sigma_\lambda^{\langle\mu}\sigma^{\nu\rangle\lambda} + \eta_2\sigma_\lambda^{\langle\mu}\omega^{\nu\rangle\lambda} + \eta_3\omega_\lambda^{\langle\mu}\omega^{\nu\rangle\lambda} + \eta_4\theta\sigma^{\mu\nu} + \eta_5\nabla_\perp^{\langle\mu}P\nabla_\perp^{\nu\rangle}P + \eta_6\nabla_\perp^{\langle\mu}\nabla_\perp^{\nu\rangle}P + \mathcal{O}\left[K_N^3\right],$$

Now increase K_N "adiabatically" towards K_N ~ 1:



What can happen to $\pi^{\mu\nu}$?

- Large rearrangement of the series.
- 3rd order terms ~ $\sigma^{\mu\nu}\sigma_{\alpha\beta}\sigma^{\alpha\beta}$ may be grouped with $2\eta\sigma^{\mu\nu}$ and etc.
- Resummation:

Resummed Knudsen number

$$\eta \to \eta^R = \eta^R(K_N)$$

$$K_N^R \sim \left(\frac{\eta}{s}\right)^R \frac{\sqrt{\sigma^{\mu\nu}\sigma_{\mu\nu}}}{T}$$

Grouping all terms, the symmetries impose that

$$\begin{aligned}\pi^{\mu\nu} &= 2\eta^R\sigma^{\mu\nu} + \eta_1^R\sigma_\lambda^{\langle\mu}\sigma^{\nu\rangle\lambda} + \eta_2^R\sigma_\lambda^{\langle\mu}\omega^{\nu\rangle\lambda} + \eta_3^R\omega_\lambda^{\langle\mu}\omega^{\nu\rangle\lambda} + \eta_5^R\nabla_\perp^{\langle\mu}P\nabla_\perp^{\nu\rangle}P \\ &+ \eta_6^R\nabla_\perp^{\langle\mu}\nabla_\perp^{\nu\rangle}P + \mathcal{O}\left[(K_N^R)^3\right],\end{aligned}$$

Far-from-equilibrium hydrodynamics

Example: Israel-Stewart's equations undergoing general flow

G. Denicol, JN, to appear (this week?)

Inverse Reynolds number: $\mathcal{X}^{\mu\nu} = \dfrac{\pi^{\mu\nu}}{\varepsilon + P}$

DNMR, PRD 2012

+ conservation laws

$$\Rightarrow \quad \tau_R D\mathcal{X}^{\langle\mu\nu\rangle} = -\mathcal{X}^{\mu\nu} + \frac{2}{5}\tau_R\sigma^{\mu\nu} - \frac{4}{3}\tau_R\mathcal{X}^{\mu\nu}\mathcal{X}^{\alpha\beta}\sigma_{\alpha\beta}$$

Slow-roll series → $D\chi^{\langle\mu\nu\rangle} \sim 0 \;\rightarrow\; \pi^{\mu\nu} = 2\eta^R(K_N)\sigma^{\mu\nu}$

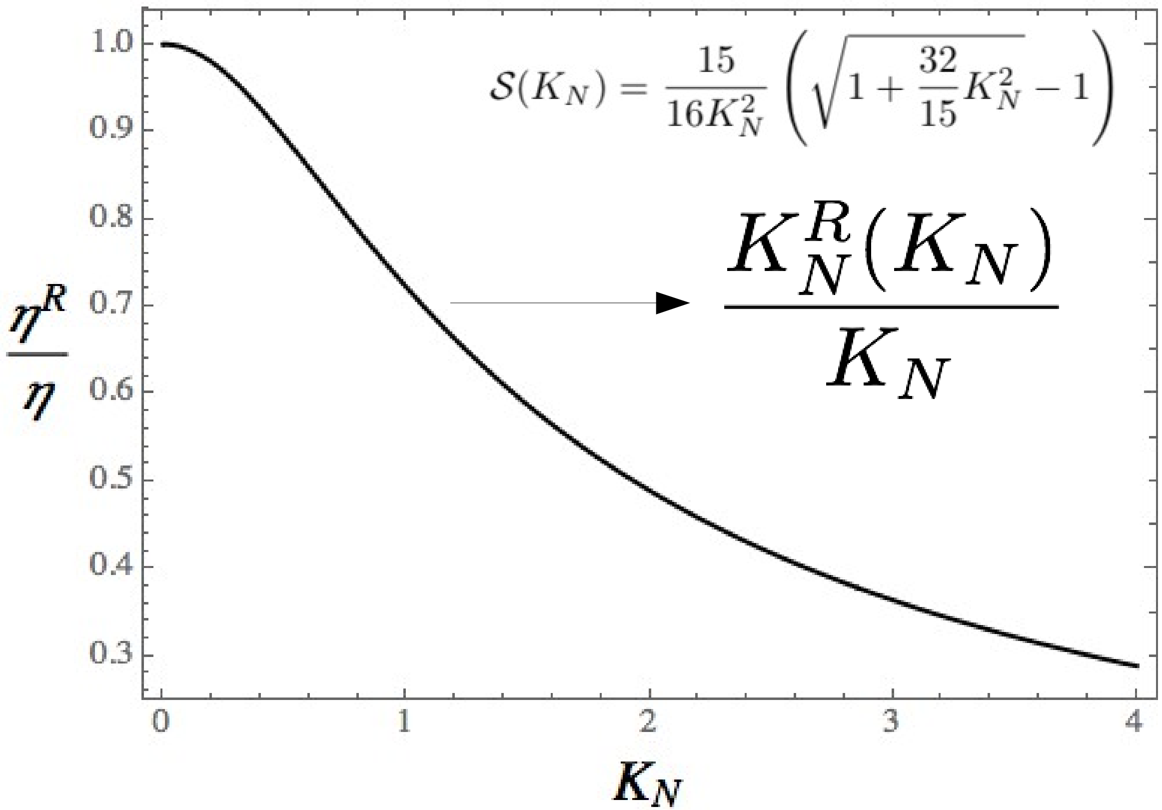
Heller, Spalinski (2015)

Resummed shear viscosity

$$\eta^R = \eta\,\mathcal{S}(K_N)$$

+ other coefficients at higher orders

Resummed shear viscosity



1[st] time for general flow

- Effective Knudsen number can be small even if gradients are large.
- Entropy production remains bounded as gradients grow large.
- See also results from Romatschke, Blaizot+Yan, Behtash+Martinez + Camacho in models with highly symmetrical flow profiles.

Conclusions & Outlook

- Viscous hydrodynamics simulations in heavy ions have always involved a resummation in gradients (Israel-Stewart-like).

- Hydrodynamic attractor → decoupling of the hydro variables from other degrees of freedom (even at large gradients).

- Novel resummed transport coefficients far-from-equilibrium (effective Knudsen number << “naive” Knudsen).

- This may help explain the (un)reasonable effectiveness of hydrodynamics in heavy ion collisions.

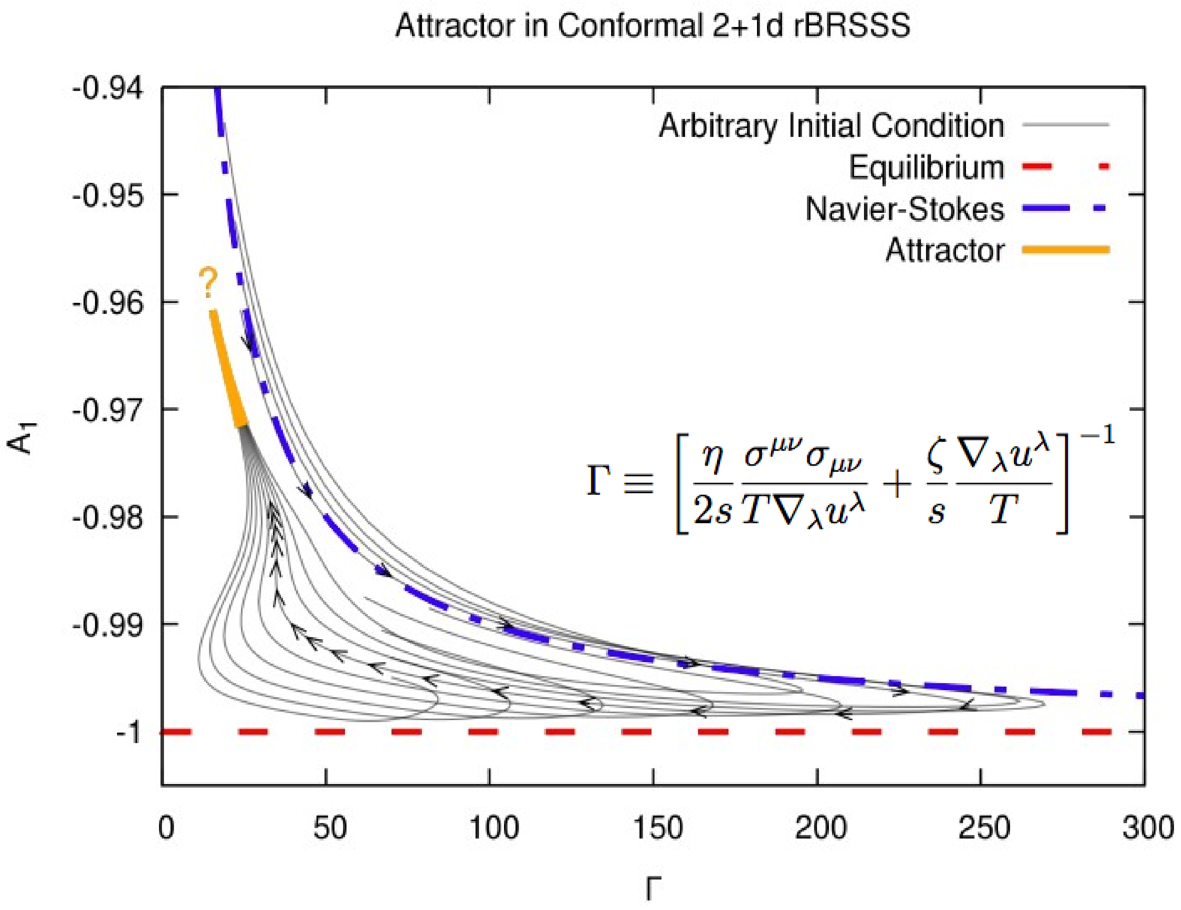
ADDITIONAL SLIDES

P. Romatschke, JHEP 2017

Relativistic Navier-Stokes

$$\frac{D\epsilon}{(\epsilon+P)\nabla_\lambda u^\lambda} = \frac{D\ln s}{\nabla_\lambda u^\lambda} = -1 + \frac{\eta}{2s}\frac{\sigma^{\mu\nu}\sigma_{\mu\nu}}{T\nabla_\lambda u^\lambda} + \frac{\zeta}{s}\frac{\nabla_\lambda u^\lambda}{T}$$

$$A_1 = \frac{D\ln s}{\nabla_\lambda u^\lambda}$$

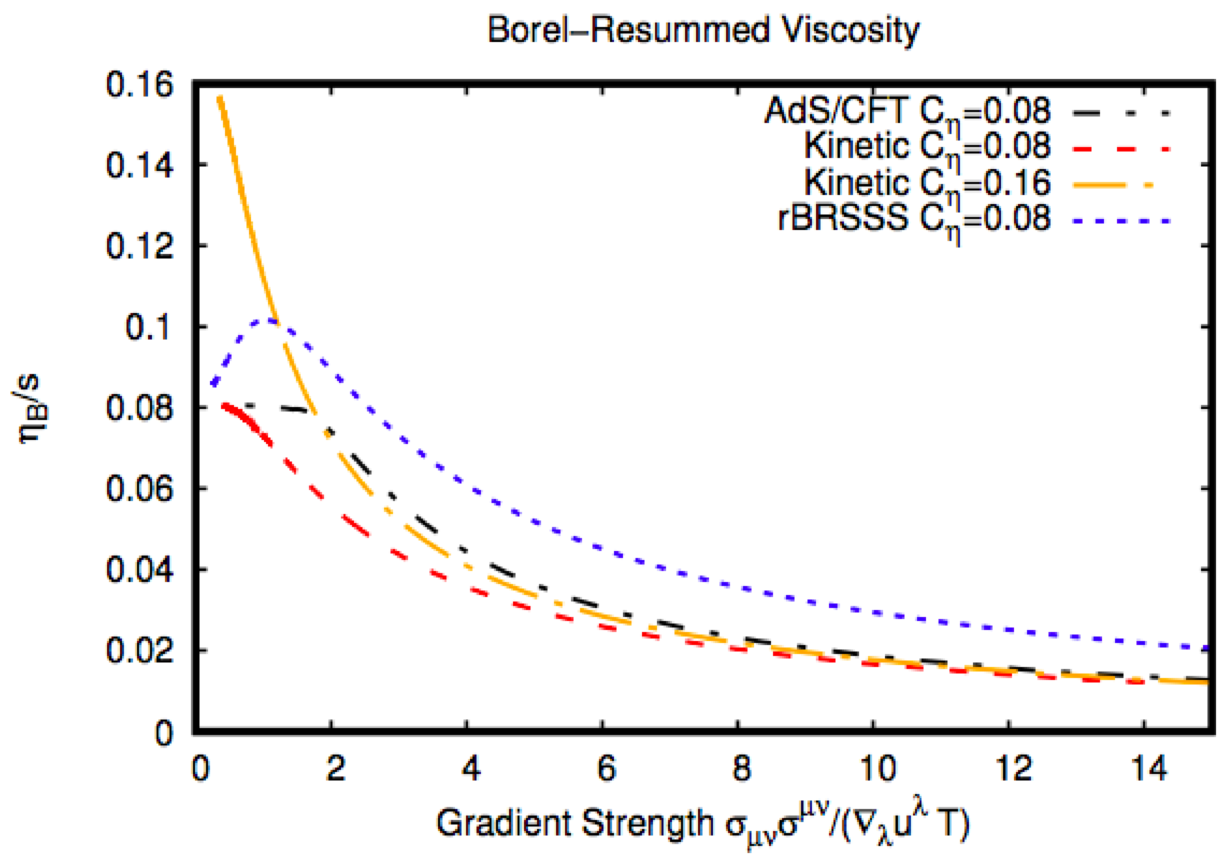


P. Romatschke, PRL 2018

$$T^{\mu\nu}_{\text{hydro}} = (\epsilon + P_B)u^\mu u^\nu + P_B g^{\mu\nu} - \eta_B \sigma^{\mu\nu}$$

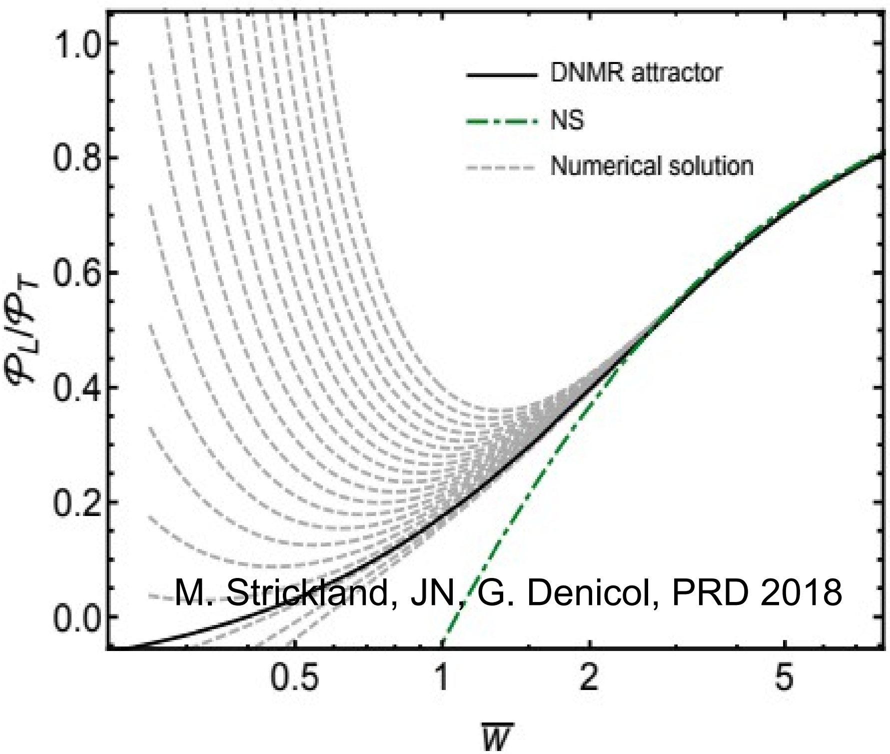
Bjorken flow

$$\partial_\tau \ln \epsilon = -\frac{4}{3} + \frac{16 C_\eta}{9\tau T}\frac{\eta_B}{\eta}$$

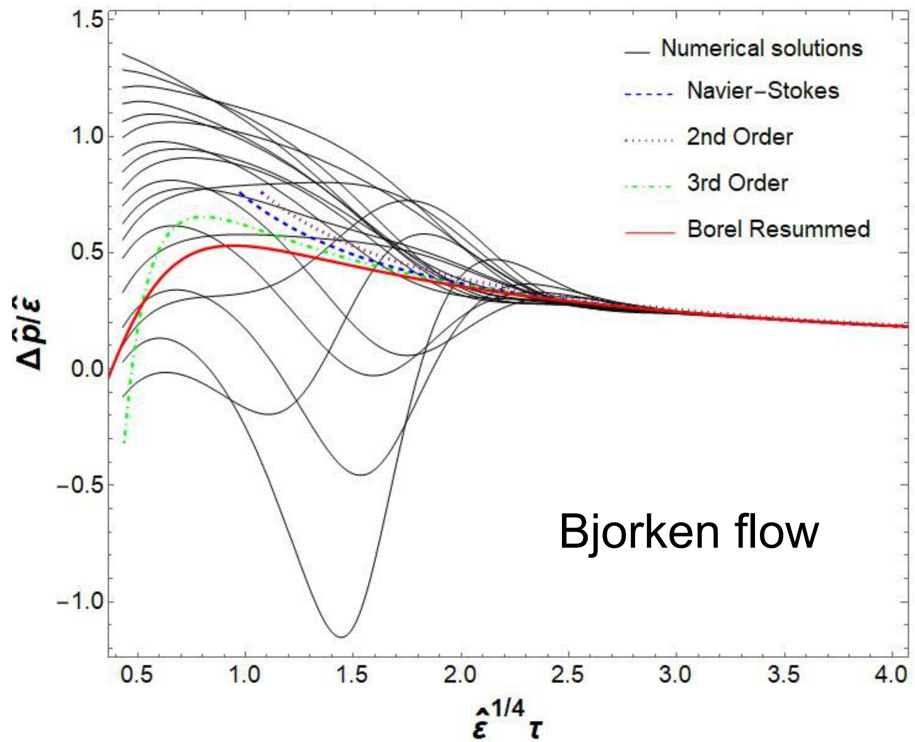


Far-from-equilibrium hydro – Attractor solutions

Boltzmann → Israel-Stewart equations



N = 4 SYM at strong coupling

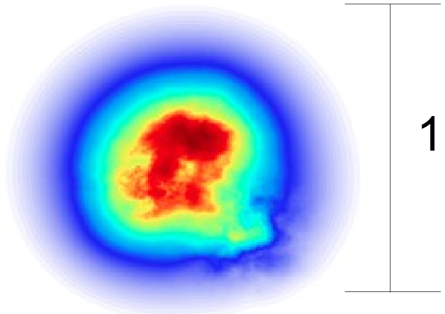


- Very different transient behavior at weak vs. strong coupling.

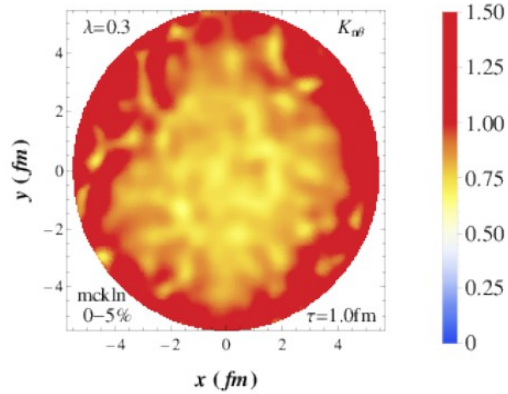
- Presence of far-from-equilibrium attractor in both cases.

Challenges to the foundations of fluid dynamics

1) “Hydro”-like behavior in small systems?



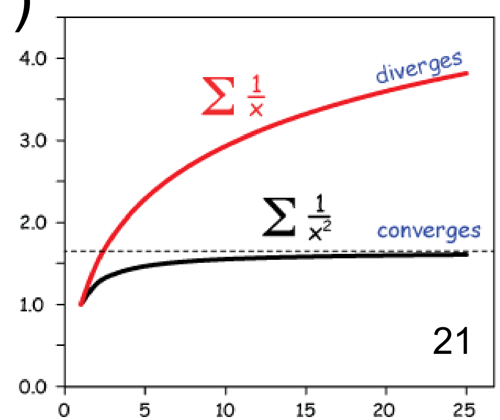
2) What happens when $K_N \sim 1$?



3) Gradient series diverges (even if Kn << 1)

- AdS/CFT: Heller, Janik, Witaszczyk PRL (2013);
  Heller, Buchel, JN, PRD (2016)

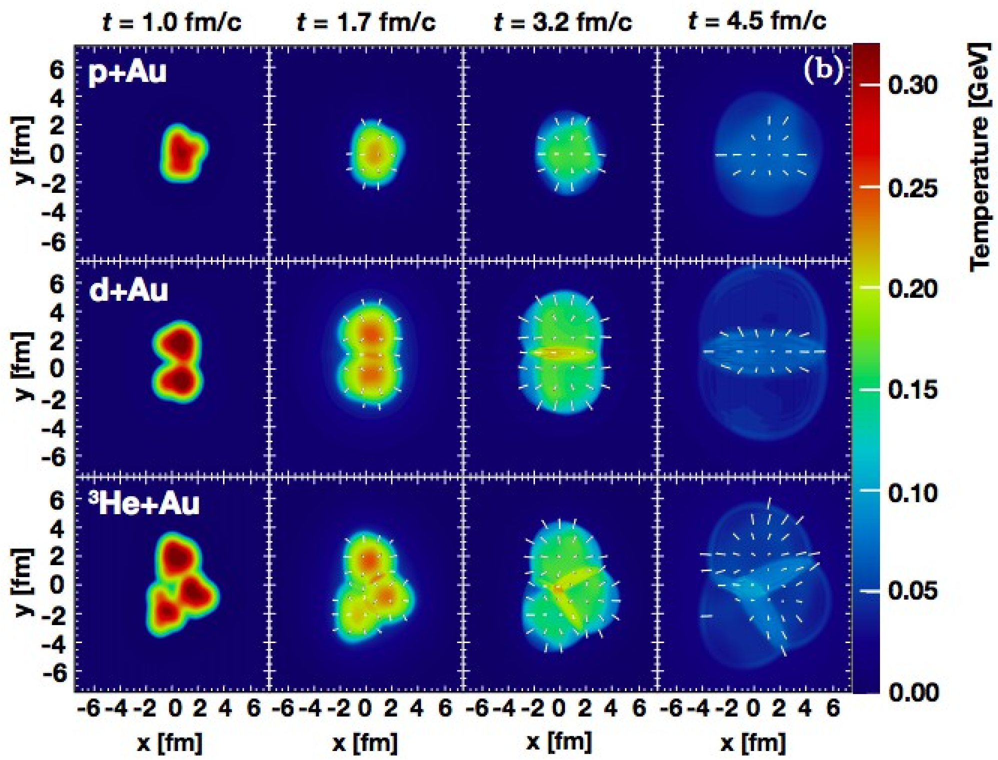
- Kinetic theory: Denicol, JN, 1608.07869;
  Heller, Kurkela, Spalinski, Svensson, 1609.04803



Truncation? Systematic improvement?

Interpretation of "vn" data in small systems?

PHENIX, 1805.02973, Nature Physics



HYDRO → $K_N \ll 1$

"macro" $\partial\varepsilon/\varepsilon_0 \sim 1/L$

micro $\ell \sim 1/T$

(Naive) Knudsen number

$$K_N \sim \ell\partial\varepsilon \gtrsim 1$$

THIS IS NOT CONVENTIONAL HYDRODYNAMICS !!!!