Stochastic hydrodynamics and long time tails of a non-equilibrium fluid

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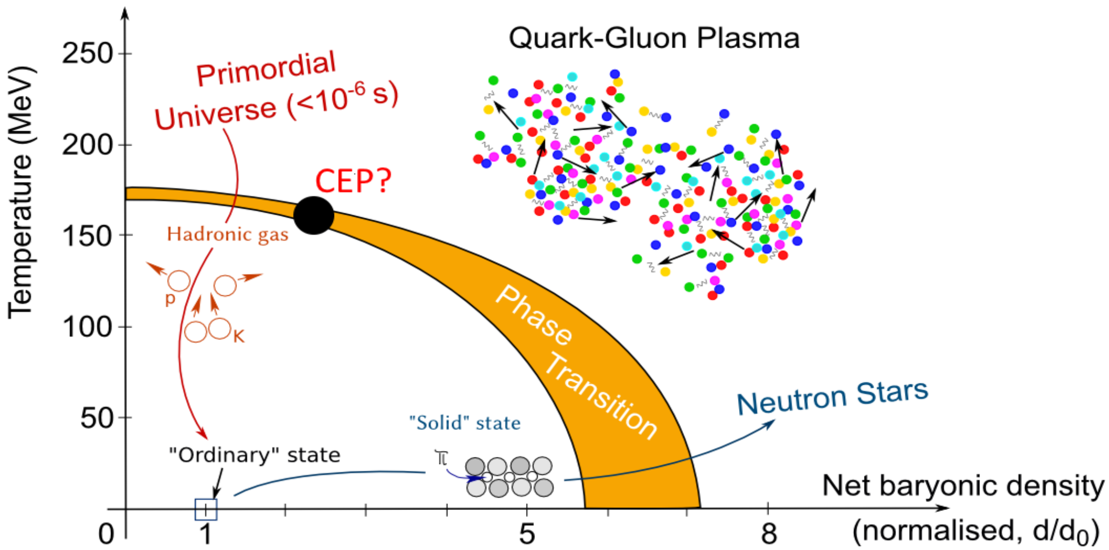
Initial Stages 2019
June 24-28, 2019
New York, USA

Work in collaboration with
T. Schäfer: PRC 99 (2019) 054902

BEST COLLABORATION

NC STATE UNIVERSITY

Motivation

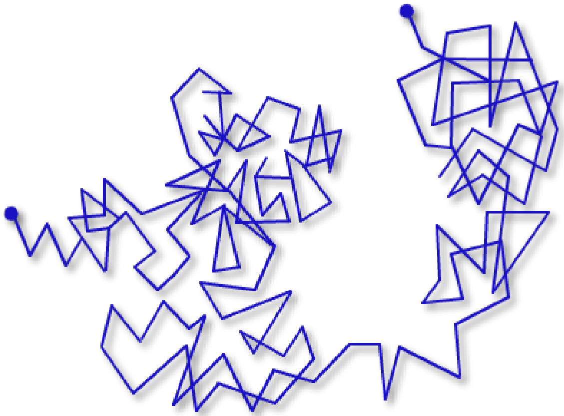


- ***Hydrodynamics*** has become the ‘workhorse’ of dynamical modeling of ultra-relativistic heavy-ion collisions

- Multiplicity of particles in HIC $dN/dy \sim \mathcal{O}(10^{2-4})$
  - Large enough for hydrodynamics to be applicable even in far-from-equilibrium situations
  See posters by Martinez et. al., Schlichting et. al.
  See talks by Schlichting, Berges, Mazeliauskas, Noronha, Denicol and Heller
  - Sufficiently small that one cannot neglect fluctuations
  Evolution of thermal fluctuations is fundamental for possible signals of QCD critical behavior

Brownian motion and Einstein relation

The brownian motion of a massive particle



$$\frac{d\vec{p}}{dt} = -\alpha_D\, \vec{p} + \vec{s}(t)$$

Drag coefficient

$$\langle\, s_i(t)\, s_j(t')\,\rangle = \kappa\, \delta_{ij}\, \delta(t - t')$$

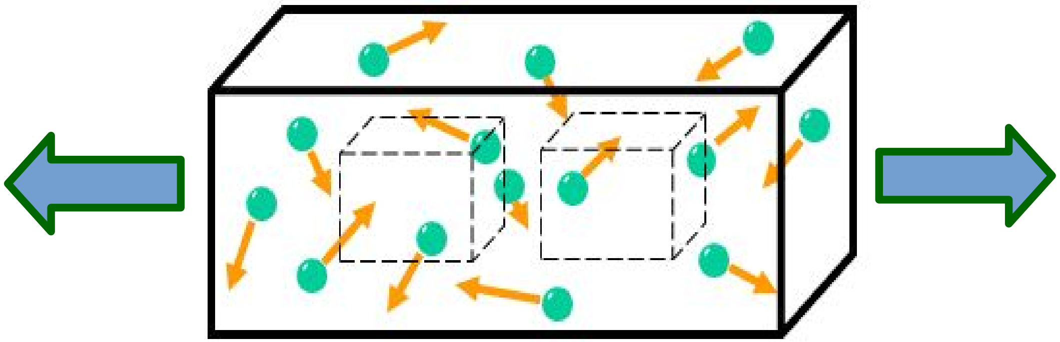
Particle eventually thermalizes $\langle\, \mathbf{p}^2 \,\rangle = 2\, m\, T$

Drag coefficient and noise are **related** via the Einstein relation (**fluctuation-dissipation** theorem)

$$\kappa = k_B\, T\, \alpha_D$$

An element missing in hydrodynamical modeling in HIC:
The dynamics of hydrodynamic fluctuations

Challenges with the stochastic hydro approach



In Equilibrium

- Fluctuations of hydro variables are related with thermodynamic properties of the system

$$\langle \frac{\delta p \, \delta p}{c_s^2} \rangle \sim T^2 \, c_p$$

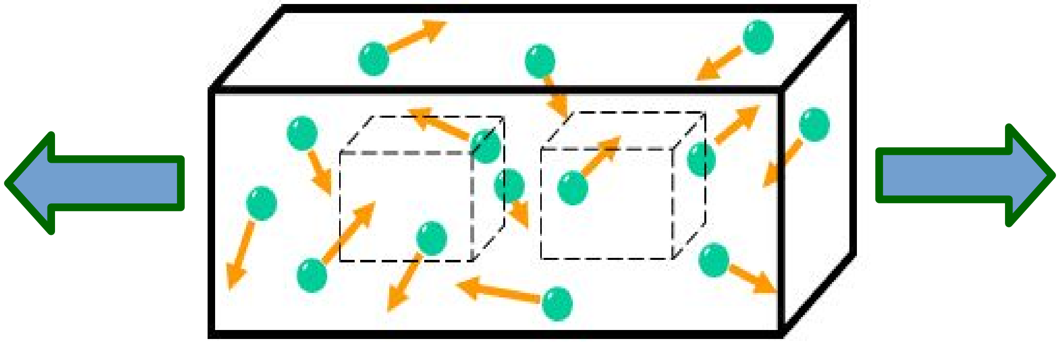
- Fluctuations of conjugate hydro variables vanish

$$\langle \delta p \, \delta(s/n) \rangle = 0$$

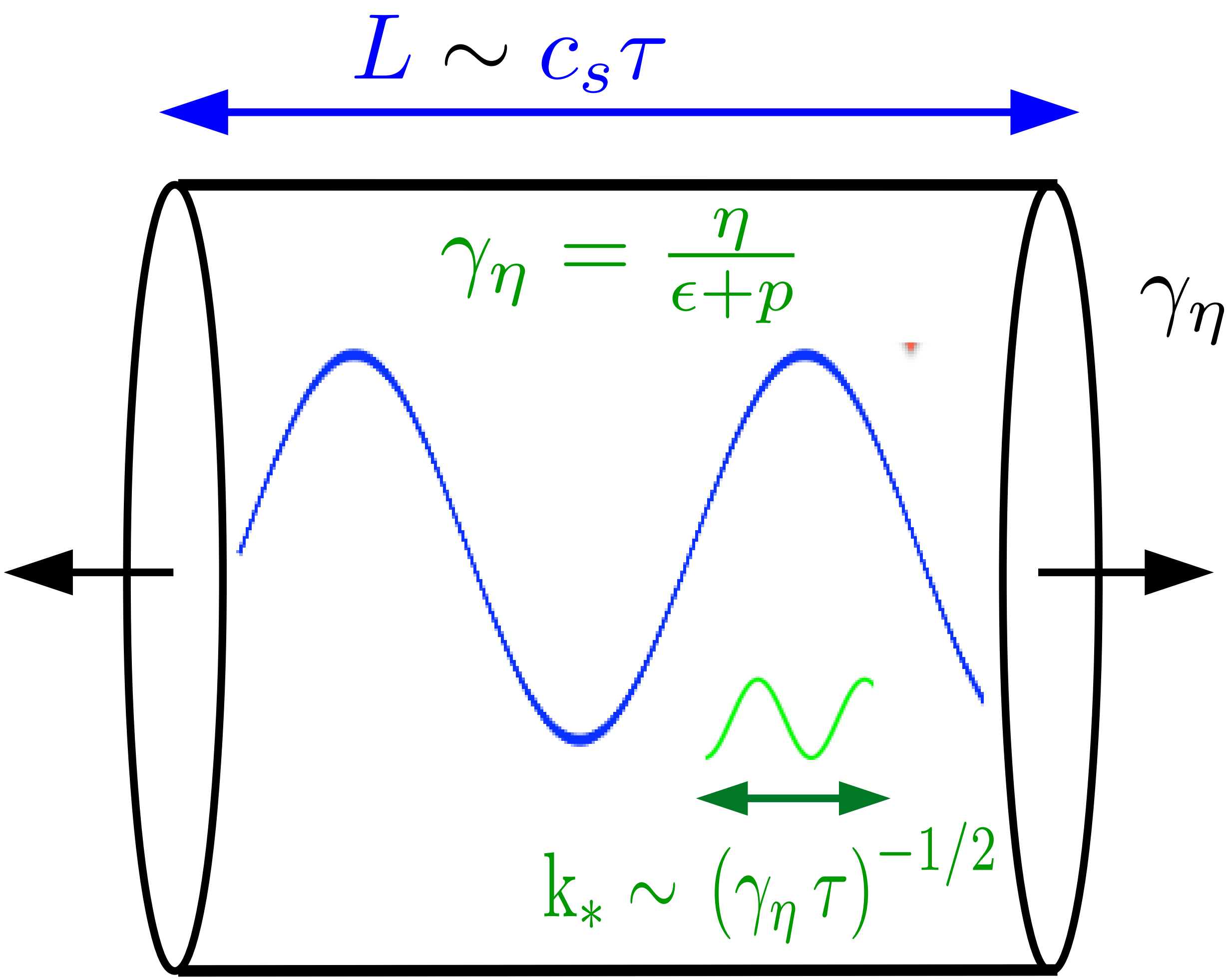
$$\langle \delta p \, \delta u^i \rangle = 0$$

For rapidly expanding plasmas (out-of-equilibrium) correlations can appear

Hydrokinetics of a charged expanding fluid



Hydrokinetics: basic idea



Competition between damping and expansion rates

$$\gamma_\eta k^2 \quad vs. \quad (c_s \tau)^{-1}$$

Modes equilibrate if

$$k \gg k_* = \sqrt{\frac{1}{\gamma_\eta \tau}}$$

Modes deviate from equilibrium for

$$k \sim \frac{1}{\sqrt{\gamma_\eta \tau}}$$

Effective theory for modes with

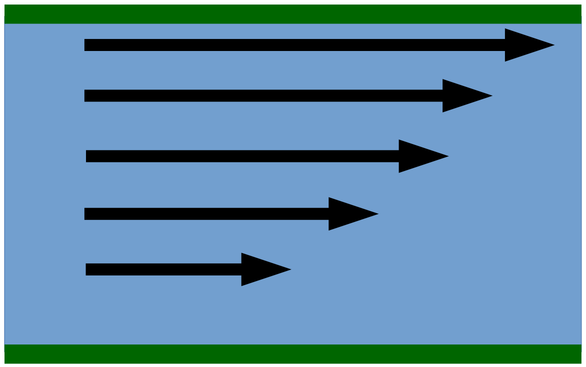
$$k \sim k_*$$

Akamatsu, Teaney, Mazeliauskas (2016)
Martinez & Schaefer (2018)
X. An, Basar, Stephanov &Yee (2019)

Linearized hydro fluctuations

$$T^{\mu\nu} = T_b^{\mu\nu} + \delta T^{\mu\nu} + S^{\mu\nu}$$

$$J^{\mu} = J_b^{\mu} + \delta J^{\mu} + I^{\mu}$$



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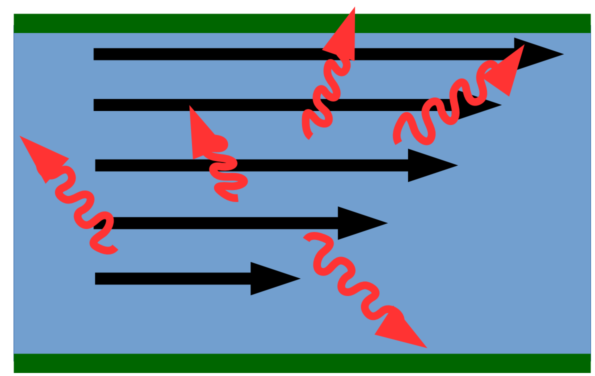

Evolving background

$$D_\mu T_b^{\mu\nu} = 0$$

$$D_\mu J_b^{\mu} = 0$$

Fluctuations + noise sources

$$\mathrm{D}_\mu \left(\delta T^{\mu\nu}(\delta p/v_a, \delta u^i) + S^{\mu\nu}\right) = 0$$

$$\mathrm{D}_\mu \left(\delta J^{\mu}(\delta q) + I^{\mu}\right) = 0$$

White random noise

$$\langle S^{\mu\nu} \rangle = 0\,, \quad \langle I^{\mu} \rangle = 0\,, \quad \langle S^{\mu\nu} I^{\lambda} \rangle = 0$$

$$\langle S^{\mu\nu}(x)\, S^{\lambda\delta}(x') \rangle = 2\,T\,\eta \Delta^{\mu\nu\lambda\delta}\, \delta^{(4)}(x-x')$$

$$\langle I^{\mu}(x)\, I^{\nu}(x') \rangle = 2\,T\,\sigma_0 \Delta^{\mu\nu}\, \delta^{(4)}(x-x')$$

Challenges with the stochastic hydro approach

A naive discretization of the white noise correlators implies

$$\langle SS \rangle \sim \delta(t-t')\,\delta^{(3)}(\vec{x}-\vec{x}') \sim (\Delta t\, a^3)^{-1}$$

Lattice size a limits the spatial extent of hydro fluctuating fields ➡ Maximum UV cutoff Λ

$$|S| \sim (\Delta t\, a^3)^{-1/2} \sim \frac{\Lambda^{3/2}}{\sqrt{\Delta t}}$$

- Noise terms have a large magnitude & numerically difficult to implement
- Instead of solving equations for hydro fluctuating fields one can alternatively solve equations for the correlations themselves (Andreev 70's)

Hydrokinetics at finite μ

Deterministic equation for 2 point matrix correlator

$$C_{AB} = \frac{1}{2}\langle\, \{\phi_A(t,\mathbf{k}), \phi^\dagger_B(t,\mathbf{k})\}\,\rangle \qquad A = \pm, d, \mathbf{T}_{1,2}$$

$$\partial_0 \mathcal{C} + [\,\mathcal{A}\,,\,\mathcal{C}] + \{\,\mathcal{D}\,,\,\mathcal{C}\,\} = \mathcal{P}\,\mathcal{C} + \mathcal{C}\,\mathcal{P}^\dagger + \frac{1}{2}\left(\mathcal{N} + \mathcal{N}^\dagger\right)$$

Evolution + reactive + diffusive = sources + noise correlator

$$C_{AB} = \begin{pmatrix} C_{++} & C_{+-} & C_{+T_1} & C_{+T_2} & C_{+d} \\ C_{-+} & C_{--} & C_{-T_1} & C_{-T_2} & C_{-d} \\ C_{T_1+} & C_{T_1-} & C_{T_1T_1} & C_{T_1T_2} & C_{T_1d} \\ C_{T_2+} & C_{T_2-} & C_{T_2T_1} & C_{T_2T_2} & C_{T_2d} \\ C_{d+} & C_{d-} & C_{dT_1} & C_{dT_2} & C_{dd} \end{pmatrix}$$

Hydrokinetics at finite μ

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$$\partial_0 \mathcal{C} + [\mathcal{A}, \mathcal{C}] + \{\mathcal{D}, \mathcal{C}\} = \mathcal{P}\mathcal{C} + \mathcal{C}\mathcal{P}^\dagger + \frac{1}{2}(\mathcal{N} + \mathcal{N}^\dagger)$$

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NEW!!

Hydrokinetics at finite μ

Deterministic equation for 2 point matrix correlator

$$C_{AB} = \frac{1}{2}\langle \{\phi_A(t,\mathbf{k}), \phi_B^\dagger(t,\mathbf{k})\} \rangle \qquad A = \pm, d, \mathbf{T}_{1,2}$$

$$\partial_0 \mathcal{C} + [\mathcal{A}, \mathcal{C}] + \{\mathcal{D}, \mathcal{C}\} = \mathcal{P}\mathcal{C} + \mathcal{C}\mathcal{P}^\dagger + \frac{1}{2}(\mathcal{N} + \mathcal{N}^\dagger)$$

Evolution + reactive + diffusive = sources + noise correlator

Close to equilibrium (asymptotic regime)

$$C_{AA} = C_{eq}\left(1 + \frac{\#}{D_{AA}k^2\tau}\right)$$

$$C_{d\mathbf{T}_i} = \# \frac{C_{eq}}{(D_0 + \gamma_\eta)k^2\tau}$$

Hydrokinetic contributions

We restrict here to the conformal case.

- The hydrodynamic fluctuating contributions to the longitudinal component of the particle current is

$$\langle J^z \rangle = \frac{D_0}{\alpha_2}\mathcal{E}^z + \frac{1}{\epsilon_0 + p_0}\left(\left\langle \frac{\delta p}{v_a^2}\, g^z \right\rangle - \langle \delta q\, g^z \rangle \right)$$

$$\sim \int^{\Lambda} d^3k\, \underbrace{\left(C_{++}(t,\mathbf{k}) - C_{--}(t,\mathbf{k})\right)}_{\#/(\gamma_\eta\, k^2 \tau)}$$

$$\sim \int^{\Lambda} d^3k\, \underbrace{\left(C_{d\mathbf{T}_1}(t,\mathbf{k}) + C_{d\mathbf{T}_2}(t,\mathbf{k})\right)}_{\#/([D+\gamma_\eta]\, k^2 \tau)}$$

Linearly divergent integrals which are regularized Martinez and Schaefer (2018)

Hydrokinetic contributions

The hydrodynamic fluctuating contributions to the particle current are

$$\frac{\langle J^z \rangle}{\mathcal{E}^z} = D_0 + \alpha_2 \frac{T}{\bar{w}^2} \frac{\Lambda}{\pi^2} \left(\frac{1}{8\gamma_{\eta_0}} + \frac{T\, c_p}{3\bar{H}(D_0 + \gamma_{\eta_0})} \right)$$

$$- \frac{T\tau}{\bar{w}^2} \left(\frac{0.04282}{(\gamma_{\eta_0}\tau)^{3/2}} + \frac{T\, c_p}{\bar{H}} \frac{0.008}{[(D_0 + \gamma_{\eta_0})\tau]^{3/2}} \right)$$

- Universal low frequency behaviour, i.e. modes with k < Λ.
- Renormalized diffusion coefficient coincides with the static limit (diagrammatic approach)

- Non-universal high frequency behaviour, i.e. modes with k > Λ.
- Long time tails $\mathcal{O}(\tau^{-3/2})$

Hydrokinetic contributions

The hydrodynamic fluctuating contributions to the particle current are

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If one run some numbers one finds that the correction associated to hydrodynamic fluctuations is ~10-15% for small and intermediate values of η/s ~ (1-2)/4π

Hydrokinetic contributions

The hydrodynamic fluctuating contributions to the particle current are

$$\langle J^\tau \rangle = \bar{n}(\Lambda) + \frac{1}{4\pi^2}\frac{T}{\bar{w}}\Lambda^3 + \frac{T}{\bar{w}}\frac{0.04808}{(\gamma_\eta \tau)^{3/2}}$$

Low frequency behaviour, i.e. modes with k < Λ.

- Renormalized particle density is the same as in the static case

Non-universal high frequency behaviour, i.e. modes with k > Λ.

- Long time tails O($\tau^{-3/2}$)

Conclusions

- We studied the role of hydrodynamic fluctuations on different energy, momentum and density correlation functions
- Hydrokinetics has been generalized for rapidly expanding fluids at finite chemical potential
- The mix of the mix shear-diffusive mode as well as the sound modes modify the tails of the particle current
- We determine the universal short length behaviour of the hydrodynamic fluctuations which renormalize the particle density and diffussion coefficient.

Outlook

- **Non-conformal fluid at finite chemical potential**
  Martinez and Schaefer 19xx.xxxxx
- **Enhancement of bulk viscosity for QCD EOS with critical behavior**
  Martinez, Schaefer & Skokov 19xx.xxxxx

Backup slides

Hydrokinetics at finite μ

$$\delta\phi_a = (\delta p/c_s, g_i, \delta q) \longrightarrow \sim \delta\left(\frac{s}{n}\right)$$

Navier-Stokes-Langevin equations

$$\frac{d}{dt}\delta\phi_a + kA_{ab}\delta\phi_b + k^2 D_{ab}\delta\phi_b = P_{ab}\delta\phi_b + \xi_a$$

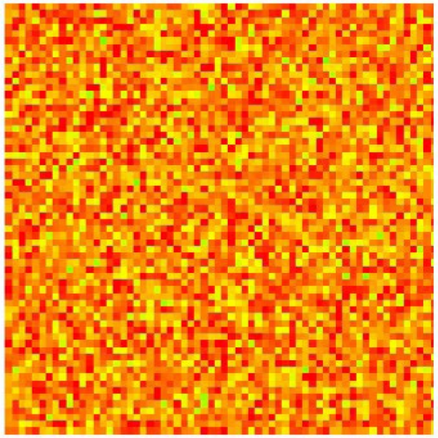
The acoustic matrix *A* has 5 hydro modes + 5 eigenvalues

Hydrodynamic eigenmode	Eigenvector	Eigenvalue
Sound modes $\phi_\pm$	$\frac{1}{\sqrt{2}}\left(1, \pm\hat{\mathbf{k}}, 0\right)$	$\pm i\, v_a$
Diffusive mode ϕ_d	$(0, \mathbf{0}, 1)$	0
Shear modes $\phi_{\mathbf{T}_i}$ $i = 1, 2$	$(0, \hat{e}_{\mathbf{T}_i}, 0)$	0

Beyond gradient hydro expansion

Hydrodynamical variables fluctuate
(Landau & Lifshitz, 1957)

$$\langle \delta v_i(t,\vec{x})\, v_j(t,\vec{x'})\,\rangle = \frac{T}{\rho}\,\delta^{(3)}(\vec{x}-\vec{x'})$$



Linearized hydrodynamics propagates fluctuations of different modes, e.g., shear and sound modes

$$\langle \delta v_i^T \delta v_j^T \rangle_{\omega,k} = \frac{2T}{\rho}(\delta_{ij} - \hat{k}_i\hat{k}_j)\frac{\nu k^2}{\omega^2 + (\nu k^2)^2} \qquad \textit{shear}$$

$$\langle \delta v_i^L \delta v_j^L \rangle_{\omega,k} = \frac{2T}{\rho}\hat{k}_i\hat{k}_j \frac{\omega k^2 \Gamma}{(\omega^2 - c_s^2 k^2)^2 + (\omega k^2 \Gamma)^2} \qquad \textit{sound}$$

$v = v_T + v_L$: $\quad \nabla \cdot v_T = 0,\ \nabla \times v_L = 0 \qquad \nu = \eta/\rho, \quad \Gamma = \frac{4}{3}\nu + \ldots$

Hydrokinetics

Akamatsu et. al. (2016) : non-equilibrium noise contributions to the energy-momentum tensor of a fluid undergoing Bjorken flow

$$\frac{\langle T^{zz}\rangle}{\epsilon_0 + p_0} = \frac{p_0}{\epsilon_0 + p_0} - \frac{4}{3\tau}\gamma_\eta + + \frac{\langle G_z^2\rangle}{\epsilon_0 + p_0} + \frac{8}{9\tau^2}\frac{(\lambda_1 - \eta_0\tau_\pi)}{\epsilon_0 + p_0}$$

$$\sim \langle v_z^2\rangle$$

Hydrokinetics

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$$= \frac{p_0}{\epsilon_0+p_0} - \frac{4}{3\tau}\gamma_\eta + \frac{1.08318}{s_0(4\pi\gamma_\eta\tau)^{3/2}} + \frac{8}{9\tau^2}\frac{(\lambda_1-\eta_0\tau_\pi)}{\epsilon_0+p_0}$$

First order   3/2 order   2nd order

For arbitrary values of η/s

First order $>$ 3/2 order $>$ 2nd order

Hydrokinetics at finite μ: Bjorken flow

- For the Bjorken case the equations of motion of the equal time symmetric correlators are

$$\partial_0 \mathcal{C} + [\mathcal{A}, \mathcal{C}] + \{\mathcal{D}, \mathcal{C}\} = \mathcal{P}\mathcal{C} + \mathcal{C}\mathcal{P}^\dagger + \frac{1}{2}\left(\mathcal{N} + \mathcal{N}^\dagger\right)$$

$$\partial_\tau \tilde{\mathcal{C}}_{\pm\pm} + \frac{4}{3}\gamma_{\eta_0}\boldsymbol{K}^2\tilde{\mathcal{C}}_{\pm\pm} = \tilde{\mathcal{N}}_{\pm\pm} - \frac{\left(2 + v_a^2 + \cos^2\theta_{\boldsymbol{K}}\right)}{\tau}\tilde{\mathcal{C}}_{\pm\pm} \mp \frac{\hat{\boldsymbol{K}}\cdot\mathcal{E}}{\bar{w}}\frac{\left(1+v_a^2\right)}{v_a}\tilde{\mathcal{C}}_{\pm\pm},$$

$$\partial_\tau \tilde{\mathcal{C}}_{\boldsymbol{T}_1\boldsymbol{T}_1} + 2\gamma_{\eta_0}\boldsymbol{K}^2\tilde{\mathcal{C}}_{\boldsymbol{T}_1\boldsymbol{T}_1} = \tilde{\mathcal{N}}_{\boldsymbol{T}_1\boldsymbol{T}_1} - \frac{2}{\tau}\tilde{\mathcal{C}}_{\boldsymbol{T}_1\boldsymbol{T}_1},$$

$$\partial_\tau \tilde{\mathcal{C}}_{\boldsymbol{T}_2\boldsymbol{T}_2} + 2\gamma_{\eta_0}\boldsymbol{K}^2\tilde{\mathcal{C}}_{\boldsymbol{T}_2\boldsymbol{T}_2} = \tilde{\mathcal{N}}_{\boldsymbol{T}_2\boldsymbol{T}_2} - \frac{2\left(1+\sin^2\theta_{\boldsymbol{K}}\right)}{\tau}\tilde{\mathcal{C}}_{\boldsymbol{T}_2\boldsymbol{T}_2},$$

$$\partial_\tau \tilde{\mathcal{C}}_{dd} + 2D\boldsymbol{K}^2\tilde{\mathcal{C}}_{dd} = \tilde{\mathcal{N}}_{dd} - \frac{2}{\tau}\tilde{\mathcal{C}}_{dd},$$

$$\partial_\tau \tilde{\mathcal{C}}_{d\boldsymbol{T}_1} + \left(\gamma_{\eta_0} + D\right)\boldsymbol{K}^2\tilde{\mathcal{C}}_{d\boldsymbol{T}_1} = -\frac{2}{\tau}\tilde{\mathcal{C}}_{d\boldsymbol{T}_1} + \frac{1}{\bar{w}}\hat{e}_{\boldsymbol{T}_1}\cdot\mathcal{E}\left(\tilde{\mathcal{C}}_{dd} - v_a\tilde{\mathcal{C}}_{\boldsymbol{T}_1\boldsymbol{T}_1}\right),$$

$$\partial_\tau \tilde{\mathcal{C}}_{d\boldsymbol{T}_2} + \left(\gamma_{\eta_0} + D\right)\boldsymbol{K}^2\tilde{\mathcal{C}}_{d\boldsymbol{T}_2} = -\frac{2+\sin^2\theta_{\boldsymbol{K}}}{\tau}\tilde{\mathcal{C}}_{d\boldsymbol{T}_2} + \frac{1}{\bar{w}}\hat{e}_{\boldsymbol{T}_2}\cdot\mathcal{E}\left(\tilde{\mathcal{C}}_{dd} - v_a\tilde{\mathcal{C}}_{\boldsymbol{T}_1\boldsymbol{T}_2}\right)$$