Stochastic hydrodynamics and long time tails of a non-equilibrium fluid

Mauricio Martinez Guerrero
Initial Stages 2019
June 24-28, 2019
New York, USA

Work in collaboration with
Hydrodynamics has become the ‘workhorse’ of dynamical modeling of ultra-relativistic heavy-ion collisions

- Multiplicity of particles in HIC $dN/dy \sim \mathcal{O}(10^{2-4})$
  - Large enough for hydrodynamics to be applicable even in far-from-equilibrium situations
  See posters by Martinez et. al., Schlichting et. al.
  See talks by Schlichting, Berges, Mazeliauskas, Noronha, Denicol and Heller
  - Sufficiently small that one cannot neglect fluctuations
  Evolution of thermal fluctuations is fundamental for possible signals of QCD critical behavior
Brownian motion and Einstein relation

The brownian motion of a massive particle

\[ \frac{dp}{dt} = -\alpha_D \vec{p} + \vec{s}'(t) \]

Drag coefficient

\[ \langle s_i(t) s_j(t') \rangle = \kappa \delta_{ij} \delta(t-t') \]

Particle eventually thermalizes

\[ \langle p^2 \rangle = 2 m T \]

Drag coefficient and noise are related via the Einstein relation (fluctuation-dissipation theorem)

\[ \kappa = k_B T \alpha_D \]

An element missing in hydrodynamical modeling in HIC:

The dynamics of hydrodynamic fluctuations
Challenges with the stochastic hydro approach

In Equilibrium

- Fluctuations of hydro variables are related with thermodynamic properties of the system

\[ \left< \frac{\delta p \delta p}{c_s^2} \right> \sim T^2 c_p \]

- Fluctuations of conjugate hydro variables vanish

\[ \left< \delta p \delta (s/n) \right> = 0 \]
\[ \left< \delta p \delta u^i \right> = 0 \]

For rapidly expanding plasmas (out-of-equilibrium) correlations can appear
Hydrokinetics of a charged expanding fluid
Hydrokinetics: basic idea

$L \sim c_s \tau$

$\gamma_\eta = \frac{\eta}{\epsilon + p}$

$\gamma_\eta k^2$ vs. $(c_s \tau)^{-1}$

Modes equilibrate if

$k \gg k_* = \sqrt{\frac{1}{\gamma_\eta \tau}}$

Modes deviate from equilibrium for

$k \sim \frac{1}{\sqrt{\gamma_\eta \tau}}$

Effective theory for modes with $k \sim k_*$

Akamatsu, Teaney, Mazeliauskas (2016)
Martinez & Schaefer (2018)
X. An, Basar, Stephanov & Yee (2019)
Linearized hydro fluctuations

\[ T^{\mu\nu} = T_b^{\mu\nu} + \delta T^{\mu\nu} + S^{\mu\nu} \]
\[ J^\mu = J_b^\mu + \delta J^\mu + I^\mu \]

\text{Evolving background}

\[ D_\mu T_b^{\mu\nu} = 0 \]
\[ D_\mu J_b^\mu = 0 \]

\text{Fluctuations + noise sources}

\[ D_\mu (\delta T^{\mu\nu}(\delta p/v_\alpha, \delta u^i) + S^{\mu\nu}) = 0 \]
\[ D_\mu (\delta J^\mu(\delta q + I^\mu) = 0 \]

\[ \langle S^{\mu\nu} \rangle = 0, \quad \langle I^\mu \rangle = 0, \quad \langle S^{\mu\nu} I^\lambda \rangle = 0 \]
\[ \langle S^{\mu\nu}(x) S^{\lambda\delta}(x') \rangle = 2T \eta \Delta^{\mu\nu\lambda\delta} \delta^{(4)}(x - x') \]
\[ \langle I^\mu(x) I^\nu(x') \rangle = 2T \sigma_0 \Delta^{\mu\nu} \delta^{(4)}(x - x') \]
Challenges with the stochastic hydro approach

A naive discretization of the white noise correlators implies

\[ \langle SS \rangle \sim \delta(t - t') \delta^3(\vec{x} - \vec{x}') \sim (\Delta t a^3)^{-1} \]

Lattice size \(a\) limits the spatial extent of hydro fluctuating fields

\[ |S| \sim (\Delta t a^3)^{-1/2} \sim \frac{\Lambda^{3/2}}{\sqrt{\Delta t}} \]

- Noise terms have a large magnitude & numerically difficult to implement
- Instead of solving equations for hydro fluctuating fields one can alternatively solve equations for the correlations themselves (Andreev 70’s)
Hydrokinetics at finite $\mu$

Deterministic equation for 2 point matrix correlator

\[ C_{AB} = \frac{1}{2} \langle \{ \phi_A(t, k), \phi_B^\dagger(t, k) \} \rangle \quad A = \pm, d, T_{1,2} \]

\[ \partial_0 C + [A, C] + \{D, C\} = \mathcal{P} C + C \mathcal{P}^\dagger + \frac{1}{2} (\mathcal{N} + \mathcal{N}^\dagger) \]

Evolution + reactive + diffusive = sources + noise correlator

\[
C_{AB} = \begin{pmatrix}
C_{++} & C_{+-} & C_{+T_1} & C_{+T_2} & C_{+d} \\
C_{-+} & C_{--} & C_{-T_1} & C_{-T_2} & C_{-d} \\
C_{T_1+} & C_{T_1-} & C_{T_1T_1} & C_{T_1T_2} & C_{T_1d} \\
C_{T_2+} & C_{T_2-} & C_{T_2T_1} & C_{T_2T_2} & C_{T_2d} \\
C_{d+} & C_{d-} & C_{dT_1} & C_{dT_2} & C_{dd}
\end{pmatrix}
\]
Hydrokinetics at finite $\mu$

Deterministic equation for 2 point matrix correlator

\[ C_{AB} = \frac{1}{2} \langle \{ \phi_A(t, k), \phi_B^\dagger(t, k) \} \rangle \]

\[ \partial_0 C + [A, C] + \{D, C\} = \mathcal{P} C + C \mathcal{P}^\dagger + \frac{1}{2} (\mathcal{N} + \mathcal{N}^\dagger) \]

Evolution + reactive + diffusive = sources + noise correlator

\[ C_{AB} = \begin{pmatrix}
C_{++} & \cancel{C_{+-}} & \cancel{C_{+T_1}} & \cancel{C_{+T_2}} & \cancel{C_{+d}} \\
\cancel{C_{-+}} & C_{--} & \cancel{C_{-T_1}} & \cancel{C_{-T_2}} & \cancel{C_{-d}} \\
\cancel{C_{T_1+}} & \cancel{C_{T_1-}} & C_{T_1T_1} & \cancel{C_{T_1T_2}} & \cancel{C_{T_1d}} \\
\cancel{C_{T_2+}} & \cancel{C_{T_2-}} & \cancel{C_{T_2T_1}} & C_{T_2T_2} & \cancel{C_{T_2d}} \\
\cancel{C_{d+}} & \cancel{C_{d-}} & \cancel{C_{dT_1}} & \cancel{C_{dT_2}} & C_{dd}
\end{pmatrix} \]

NEW!!
Hydrokinetics at finite $\mu$

Deterministic equation for 2 point matrix correlator

$$C_{AB} = \frac{1}{2} \langle \{ \phi_A(t, k), \phi_B^\dagger(t, k) \} \rangle$$

$$\partial_0 C + [A, C] + \{D, C\} = \mathcal{P}C + C\mathcal{P}^\dagger + \frac{1}{2} \left( \mathcal{N} + \mathcal{N}^\dagger \right)$$

Evolution + reactive + diffusive = sources + noise correlator

Close to equilibrium (asymptotic regime)

$$C_{AA} = C_{eq} \left( 1 + \frac{\#}{D_{AA}k^2\tau} \right)$$

$$C_{dT_i} = \# \frac{C_{eq}}{(D_0 + \gamma\eta)k^2\tau}$$
Hydrokinetic contributions

We restrict here to the conformal case.

- The hydrodynamic fluctuating contributions to the longitudinal component of the particle current is

\[
\langle J^z \rangle = \frac{D_0}{\alpha_2} \mathcal{E}^z + \frac{1}{\epsilon_0 + p_0} \left( \langle \frac{\delta p}{v_a^2} g^z \rangle - \langle \delta q g^z \rangle \right)
\]

\[
\sim \int^\Lambda d^3k \left( C_{++}(t, \mathbf{k}) - C_{--}(t, \mathbf{k}) \right)
\frac{\#/(\gamma_\eta k^2\tau)}{\#/(D + \gamma_\eta k^2\tau)}
\]

\[
\sim \int^\Lambda d^3k \left( C_{dT_1}(t, \mathbf{k}) + C_{dT_2}(t, \mathbf{k}) \right)
\frac{\#/(\gamma_\eta k^2\tau)}{\#/(D + \gamma_\eta k^2\tau)}
\]

Linearly divergent integrals which are regularized
Martinez and Schaefer (2018)
Hydrokinetic contributions

The hydrodynamic fluctuating contributions to the particle current are

\[ \frac{\langle J^z \rangle}{\mathcal{E}^z} = D_0 + \alpha_2 \frac{T}{\bar{w}^2} \frac{\Lambda}{\pi^2} \left( \frac{1}{8\gamma \eta_0} + \frac{T c_p}{3\bar{H}(D_0 + \gamma \eta_0)} \right) \]

\[ - \frac{T \tau}{\bar{w}^2} \left( \frac{0.04282}{(\gamma \eta_0 \tau)^{3/2}} + \frac{T c_p}{\bar{H}} \left[ (D_0 + \gamma \eta_0) \tau \right]^{3/2} \right) \]

- Universal low frequency behaviour, i.e. modes with \( k < \Lambda \).
- Renormalized diffusion coefficient coincides with the static limit (diagrammatic approach).
- Non-universal high frequency behaviour, i.e. modes with \( k > \Lambda \).
- Long time tails \( \mathcal{O}(\tau^{-3/2}) \)
Hydrokinetic contributions

The hydrodynamic fluctuating contributions to the particle current are

\[
\frac{\langle J^z \rangle}{E^z} = D_0 + \alpha_2 \frac{T}{\bar{w}^2} \frac{\Lambda}{\pi^2} \left( \frac{1}{8\gamma\eta_0} + \frac{Tc_p}{3\bar{H}(D_0 + \gamma\eta_0)} \right) - \frac{T\tau}{\bar{w}^2} \left( \frac{0.04282}{(\gamma\eta_0\tau)^{3/2}} + \frac{Tc_p}{\bar{H}} \frac{0.008}{(D_0 + \gamma\eta_0)\tau^{3/2}} \right)
\]

If one run some numbers one finds that the correction associated to hydrodynamic fluctuations is \(\sim 10-15\%\) for small and intermediate values of \(\eta/s \sim (1-2)/4\pi\).
Hydrokinetic contributions

The hydrodynamic fluctuating contributions to the particle current are

\[ \langle J^\tau \rangle = \bar{n}(\Lambda) + \frac{1}{4\pi^2} \frac{T}{\bar{w}} \Lambda^3 + \frac{T}{\bar{w}} \frac{0.04808}{(\gamma_\eta \tau)^{3/2}} \]

- Low frequency behaviour, i.e. modes with \( k < \Lambda \).
- Renormalized particle density is the same as in the static case.
- Non-universal high frequency behaviour, i.e. modes with \( k > \Lambda \).
- Long time tails \( O(\tau^{-3/2}) \).
Conclusions

- We studied the role of hydrodynamic fluctuations on different energy, momentum and density correlation functions.

- Hydrokinetics has been generalized for rapidly expanding fluids at finite chemical potential.

- The mix of the mix shear-diffusive mode as well as the sound modes modify the tails of the particle current.

- We determine the universal short length behaviour of the hydrodynamic fluctuations which renormalize the particle density and diffusion coefficient.
Outlook

- Non-conformal fluid at finite chemical potential
  Martinez and Schaefer 19xx.xxxxx
- Enhancement of bulk viscosity for QCD EOS with critical behavior
  Martinez, Schaefer & Skokov 19xx.xxxxx
Backup slides
Hydrokinetics at finite $\mu$

\[ \delta \phi_a = (\delta p/c_s, g_i, \delta q) \sim \delta \left( \frac{s}{n} \right) \]

Navier-Stokes-Langevin equations

\[ \frac{d}{dt} \delta \phi_a + k A_{ab} \delta \phi_b + k^2 D_{ab} \delta \phi_b = P_{ab} \delta \phi_b + \xi_a \]

The acoustic matrix $A$ has 5 hydro modes + 5 eigenvalues

<table>
<thead>
<tr>
<th>Hydrodynamic eigenmode</th>
<th>Eigenvector</th>
<th>Eigenvalue</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sound modes $\phi_{\pm}$</td>
<td>$\frac{1}{\sqrt{2}} \left( 1, \pm \hat{k}, 0 \right)$</td>
<td>$\pm i \nu_a$</td>
</tr>
<tr>
<td>Diffusive mode $\phi_d$</td>
<td>$(0, 0, 1)$</td>
<td>0</td>
</tr>
<tr>
<td>Shear modes $\phi_{T_i}$ $i = 1, 2$</td>
<td>$(0, \hat{e}_{T_i}, 0)$</td>
<td>0</td>
</tr>
</tbody>
</table>
Beyond gradient hydro expansion

Hydrodynamical variables fluctuate (Landau & Lifshitz, 1957)

\[
\langle \delta \nu_i(t, \vec{x}) \nu_j(t, \vec{x}') \rangle = \frac{T}{\rho} \delta^{(3)}(\vec{x} - \vec{x}')
\]

Linearized hydrodynamics propagates fluctuations of different modes, e.g., shear and sound modes

\[
\langle \delta \nu^T_i \delta \nu^T_j \rangle_{\omega, k} = \frac{2T}{\rho} \left( \delta_{ij} - \hat{k}_i \hat{k}_j \right) \frac{\nu k^2}{\omega^2 + (\nu k^2)^2}
\]

\[
\text{shear}
\]

\[
\langle \delta \nu^L_i \delta \nu^L_j \rangle_{\omega, k} = \frac{2T}{\rho} \hat{k}_i \hat{k}_j \frac{\omega k^2 \Gamma}{(\omega^2 - c_s^2 k^2)^2 + (\omega k^2 \Gamma)^2}
\]

\[
\text{sound}
\]

\[
v = v_T + v_L: \quad \nabla \cdot v_T = 0, \quad \nabla \times v_L = 0 \quad \quad \quad \nu = \eta/\rho, \quad \Gamma = \frac{4}{3} \nu + \ldots
\]
Hydrokinetics

Akamatsu et. al. (2016): non-equilibrium noise contributions to the energy-momentum tensor of a fluid undergoing Bjorken flow

$$\frac{\langle T^{zz} \rangle}{\epsilon_0 + p_0} = \frac{p_0}{\epsilon_0 + p_0} - \frac{4}{3\tau} \gamma \eta + \frac{\langle G^2_z \rangle}{\epsilon_0 + p_0} + \frac{8}{9\tau^2} \left( \lambda_1 - \eta_0 \tau \pi \right)$$

$$\sim \langle \nu^2_z \rangle$$
Akamatsu et. al. (2016): non-equilibrium noise contributions to the energy-momentum tensor of a fluid undergoing Bjorken flow

\[
\frac{\langle T^{zz} \rangle}{\epsilon_0 + p_0} = \frac{p_0}{\epsilon_0 + p_0} - \frac{4}{3\tau} \gamma \eta + \frac{\langle G_z^2 \rangle}{\epsilon_0 + p_0} + \frac{8}{9\tau^2} \left( \frac{\lambda_1 - \eta_0 \tau_\pi}{\epsilon_0 + p_0} \right)
\]

For arbitrary values of \( \eta/s \)

First order  \( \geq \) 3/2 order  \( \geq \) 2nd order
Hydrokinetics at finite $\mu$: Bjorken flow

- For the Bjorken case the equations of motion of the equal time symmetric correlators are

$$\partial_0 C + [A, C] + \{D, C\} = PC + C P^\dagger + \frac{1}{2} (N + N^\dagger)$$

$$\begin{align*}
\partial_\tau \tilde{C}_{\pm \pm} + \frac{4}{3} \gamma_\eta_0 K^2 \tilde{C}_{\pm \pm} &= \tilde{N}_{\pm \pm} - \frac{(2 + v_a^2 + \cos^2 \theta_K)}{\tau} \tilde{C}_{\pm \pm} + \frac{\hat{K} \cdot \mathcal{E} (1 + v_a^2)}{v_a} \tilde{C}_{\pm \pm}, \\
\partial_\tau \tilde{C}_{T_1 T_1} + 2 \gamma_\eta_0 K^2 \tilde{C}_{T_1 T_1} &= \tilde{N}_{T_1 T_1} - \frac{2}{\tau} \tilde{C}_{T_1 T_1}, \\
\partial_\tau \tilde{C}_{T_2 T_2} + 2 \gamma_\eta_0 K^2 \tilde{C}_{T_2 T_2} &= \tilde{N}_{T_2 T_2} - \frac{2}{\tau} \tilde{C}_{T_2 T_2}, \\
\partial_\tau \tilde{C}_{dd} + 2 DK^2 \tilde{C}_{dd} &= \tilde{N}_{dd} - \frac{2}{\tau} \tilde{C}_{dd}, \\
\partial_\tau \tilde{C}_{dT_1} + (\gamma_\eta_0 + D) K^2 \tilde{C}_{dT_1} &= -\frac{2}{\tau} \tilde{C}_{dT_1} + \frac{1}{w} \hat{e}_{T_1} \cdot \mathcal{E} \left( \tilde{C}_{dd} - v_a \tilde{C}_{T_1 T_1} \right), \\
\partial_\tau \tilde{C}_{dT_2} + (\gamma_\eta_0 + D) K^2 \tilde{C}_{dT_2} &= -\frac{2 + \sin^2 \theta_K}{\tau} \tilde{C}_{dT_2} + \frac{1}{w} \hat{e}_{T_2} \cdot \mathcal{E} \left( \tilde{C}_{dd} - v_a \tilde{C}_{T_1 T_2} \right)
\end{align*}$$