Off-equilibrium infrared structure of self-interacting scalar fields

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Outline

- Introduction: non-thermal fixed points, ultra-violet & infrared fixed points, examples: turbulence in kinetic processes, vortices as topological defects
- Map: from relativistic real scalar $\phi$ to non-relativistic complex scalar $\psi$, derive GP theory from that of $\phi$
- Simulations of relativistic real scalar $\phi$ in 2D which give the dynamics of vortices and scaling exponents appearing in GP theory for $\psi$
Physical systems far from equilibrium at different scales

universe  Heavy-ion collisions  Ultracold atoms

Classical and quantum turbulence

Question:

Do they have common properties?

Talks by Berges, Mazeliauskas, Heller, Noronha, Martinez, Denicol

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Example: IS in HIC far from equilibrium

- IS in HIC at early times is a weak coupling system with $g<<1$

- Highly correlated system (color glass condensate)

- The system can take detour to NTFP before thermalization

$f(p \lesssim Q) \sim 1/g^2$  \hspace{1cm} (or $\langle AA \rangle \sim Q^2/g^2$)

Berges, Epelbaum, Gasenzer, Gelis, Mcclerran, Moore, Schlichting, Sexty, Venugopalan, ……

Non-thermal fixed-point

Nowak, Scholle, Gasenzer
Non-thermal fixed-points & universality classes

• At a non-thermal fixed point
  → Memory loss of the details of the initial conditions;
  → Self-similar evolution of distribution function $f$
    (critical slowing down)
  → Scaling behavior in time $f \sim t^\alpha$, $p \sim t^{-\beta}$
  \[ f(p, t) = t^\alpha f_S(t^\beta p) \]
  → Several fixed-points in different momentum regimes
    (inertial range)
  → Often connected to kinetic processes (turbulence) or
    topological defects (vortices)

• Classification: universality classes for system far from equilibrium
  → by NTFP with dynamical exponents $\alpha$, $\beta$ and the scaling
    function $f_S(x)$
Dual cascade in scalar field theory

Self-similarity

\[ f(p, t) = t^\alpha f_S(t^\beta p) \]

Perturbative
Micha & Tkachev,

Non-perturbative
Berges, Rothkopf & Schmidt,
PRL (2008)
Nowak, Sexty & Gasenzer
PRB(2011)

Bose condensation
Berges & Sexty, PRL(2012)

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• Start with fluctuating initial conditions at initial time $t_0$
• Solve initial value problem on the lattice. For example, self-interacting real scalars:

$$
\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 - \frac{1}{4!} \lambda \phi^4
$$

$$(\partial_\mu \partial^\mu + m^2)\phi + \frac{1}{6} \lambda \phi^3 = 0 \quad \Rightarrow \quad \phi(t)
$$

• We know $\phi(t) \to 0[\phi(t)]$ at $t$ for a specific IC, ensemble average over initial conditions at time $t$ to obtain the observable $\langle O[\phi(t)] \rangle$

[Aarts & Berges, PRL 88, 041603 (2002); Jeon, PRC 72, 014907 (2005)]
Relativistic vs non-relativistic fields

Relativistic real scalar
(high energy physics)

\[ \mathcal{L} = \frac{1}{2} (\partial_{\mu} \phi)(\partial^{\mu} \phi) - \frac{1}{2} m^2 \phi^2 - \frac{1}{4!} \lambda \phi^4 \]

\[ (\partial_{\mu} \partial^{\mu} + m^2)\phi + \frac{1}{6} \lambda \phi^3 = 0 \]

Non-relativistic complex scalar
(Superfluid Helium, cold atoms)

\[ \mathcal{H} = -\frac{1}{2m} \psi^* \nabla^2 \psi + V|\psi|^2 + \frac{1}{2} g |\psi|^4 \]

\[ i \frac{\partial \psi}{\partial t} = \left( -\frac{1}{2m} \nabla^2 + V + g |\psi|^2 \right) \psi \]

Low-momentum universality class of scalar field theory: dynamical scaling exponents of the infrared fixed-point

<table>
<thead>
<tr>
<th>exponent</th>
<th>relativistic</th>
<th>non-relativistic</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha )</td>
<td>3</td>
<td>3/2</td>
</tr>
<tr>
<td>( \beta )</td>
<td>1</td>
<td>1/2</td>
</tr>
</tbody>
</table>

What is the connection between relativistic real scalar and non-relativistic complex scalar field theory?

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Mapping: from $\Phi$ to $\Psi$

- Hamiltonian density, scalar field and its canonical momentum

$$\mathcal{H} = \frac{1}{2} [\pi^2 + (\nabla \phi)^2 + m^2 \phi^2] + \frac{1}{24} \lambda \phi^4$$

$$\phi(x) = \int [d^3k] \frac{1}{\sqrt{2E_k}} (a_k e^{ik \cdot x} + a_k^* e^{-ik \cdot x})$$

$$\pi(x) = -i \int [d^3k] \sqrt{\frac{E_k}{2}} (a_k e^{ik \cdot x} - a_k^* e^{-ik \cdot x})$$

- Hamiltonian in $a_k$ and $a_k^*$

$$H(a, a^*) = \int d^3x \mathcal{H} = \int [d^3k] E_k a_k a_k^*$$

$$+ \frac{1}{24} \lambda \int [d^3k][d^3k_1][d^3k_2][d^3k_3] \frac{(2\pi)^3}{\sqrt{16E_k E_{k_1} E_{k_2} E_{k_3}}}$$

$$[a_k a_{k_1} a_{k_2} a_{k_3} \delta(k + k_1 + k_2 + k_3) + 4a_k a_{k_1} a_{k_2} a_{k_3}^* \delta(k + k_1 + k_2 - k_3)$$

$$+ 6a_k a_{k_1} a_{k_2} a_{k_3}^* \delta(k + k_1 - k_2 + k_3) + 4a_k^* a_{k_1}^* a_{k_2}^* a_{k_3}^* \delta(k - k_1 - k_2 - k_3)$$

$$+ a_k^* a_{k_1}^* a_{k_2}^* a_{k_3}^* \delta(-k - k_1 - k_2 - k_3)]$$

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EOM and Poisson brackets

- EOM in Poisson brackets

\[ i \frac{da_p}{dt} = \{a_p, H\}_a, \quad i \frac{da^*_p}{dt} = \{a^*_p, H\}_a \]

- Poisson brackets in the $a_p$ basis

\[
\{F(p), G(p_1)\}_a = \int [d^3k] \left[ \frac{\partial F(p)}{\partial a_k} \frac{\partial G(p_1)}{\partial a^*_k} - \frac{\partial F(p)}{\partial a^*_k} \frac{\partial G(p_1)}{\partial a_k} \right] \\
\{a_p, a^*_{p_1}\}_a = (2\pi)^3 \delta(p - p_1) \\
\{a_p, a_{p_1}\}_a = \{a^*_p, a^*_{p_1}\}_a = 0
\]

- Generally we can use any canonical variables $(a_k, a^*_k) \to (b_p, b^*_p)$, if $(b_p, b^*_p)$ have the same Poisson brackets in the $a_p$-basis

\[
\{b_p, b^*_{p_1}\}_a = (2\pi)^3 \delta(p - p_1) \\
\{b_p, b_{p_1}\}_a = \{b^*_p, b^*_{p_1}\}_a = 0
\]

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• Poisson brackets are invariant under the change of canonical basis, for example

\[ i \frac{\partial b_p}{\partial t} = \{ b_p, H \}_b = \{ b_p, H \}_a \]

• Canonical variables can be defined by time shift through \( H_{\text{aux}}(b_p, b_{\ast p}) \):

\[
\begin{align*}
b_p(z) & = b_p + z \frac{db_p(z)}{dz} \bigg|_{z=0} + \frac{1}{2} z^2 \frac{d^2 b_p(z)}{dz^2} \bigg|_{z=0} + \frac{1}{6} z^3 \frac{d^3 b_p(z)}{dz^3} \bigg|_{z=0} + \cdots \\
& = b_p - iz\{b_p, H_{\text{aux}}\}_b + \frac{1}{2}(-iz)^2\{\{b_p, H_{\text{aux}}\}_b, H_{\text{aux}}\}_b \\
& \quad + \frac{1}{6}(-iz)^3\{\{\{b_p, H_{\text{aux}}\}_b, H_{\text{aux}}\}_b, H_{\text{aux}}\}_b + \cdots
\end{align*}
\]

\[ b_p \equiv b_p(0) \]
\[ b_{\ast p} \equiv b_{\ast p}(0) \]

• One can prove that \( b_p(z), b_{\ast p}(z) \) satisfy canonical relations in \( b_p \) basis

\[
\begin{align*}
\{a_p, a_{p1}\}_b &= (2\pi)^3 \delta(p - p_1) \\
\{a_p, a_p\}_b &= \{a_{\ast p}, a_{\ast p1}\}_b = 0
\end{align*}
\]
New canonical variable through $H_{\text{aux}}$

- **Auxiliary Hamiltonian**

\[
H_{\text{aux}}(b, b^*) = \int [d^3k_1][d^3k_2][d^3k_3][d^3k_4] \times \left\{ \frac{1}{24} [B_1(k_1, k_2, k_3, k_4)] b_{k_1} b_{k_2} b_{k_3} b_{k_4} + \text{c.c.} \right. \\
\times \delta(k_1 + k_2 + k_3 + k_4) \\
\left. + \frac{1}{6} [B_2(k_1, k_2, k_3, k_4)] b_{k_1} b_{k_2} b_{k_3} b_{k_4}^* + \text{c.c.} \right. \\
\times \delta(k_1 + k_2 + k_3 - k_4) \right\}
\]

- $B_1$ and $B_2$ will be determined by removing non-resonant terms in the original Hamiltonian
Canonical transformation: derive $H_{GP}$

- New canonical variable by time shift through $H_{aux}$

\[
H'(b, b^*) = H(b, b^*) + (-iz)\{H(b, b^*), H_{aux}\}_b + \frac{1}{2}(-iz)^2\{\{H(b, b^*), H_{aux}\}_b, H_{aux}\}_b + \frac{1}{6}(-iz)^3\{\{\{H(b, b^*), H_{aux}\}_b, H_{aux}\}_b, H_{aux}\}_b + \ldots
\]

Initial auxiliary time $z=0$

- $b_p \equiv b_p(0)$
- $b^*_p \equiv b^*_p(0)$

Auxiliary time $z$

- $a_p \equiv b_p(z)$
- $a^*_p \equiv b^*_p(z)$

Substitute

\[
a_p = b_p - iz\{b_p, H_{aux}\}_b + \frac{1}{2}(-iz)^2\{\{b_p, H_{aux}\}_b, H_{aux}\}_b + \ldots
\]

With only resonant terms, global $U(1)$

Vanishing of non-resonant terms

Fix $B_1$ and $B_2$ in $H_{aux}$
Gross-Pitaevskii Hamiltonian

The main result: (1) derivation of GP Hamiltonian from $\phi$
Express $b_p$ as function of $a_p$

- Change the basis from $b_p$ to $a_p$ for all Poisson brackets

\[
b_p = a_p + iz \{b_p, H_{aux}(b)\}_a - \frac{1}{2} (iz)^2 \{\{b_p, H_{aux}(b)\}_a, H_{aux}(b)\}_a + \cdots
\]

- $b_p$ can be solved as a function of $a_p$ in perturbation expansion parameter:

\[
\frac{\lambda \phi^2}{m^2}
\]

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Express \((\Psi, \Psi^*)\) as function of \((\phi, \pi)\)

- Transform to coordinate space

\[
\psi(x) = \int [d^3k] b_k e^{ik \cdot x}
\]

- Express \((\Psi, \Psi^*)\) as function of \((\phi, \pi)\). In the large mass limit, the zeroth order

\[
\psi(0)(x) = \int \frac{d^3k}{(2\pi)^3} a_k e^{ik \cdot x}
\]

\[
= \sqrt{\frac{m}{2}} \left( 1 - \frac{1}{4m^2} \nabla_x^2 \right) \phi(x) + i \frac{1}{\sqrt{2m}} \left( 1 + \frac{1}{4m^2} \nabla_x^2 \right) \pi(x)
\]

\[
\equiv \Phi + \Pi - \frac{1}{4} \Phi'' + \frac{1}{4} \Pi''
\]

\[
\Phi \equiv \sqrt{\frac{m}{2}} \phi(x) \quad \Pi \equiv i \frac{1}{\sqrt{2m}} \pi(x) \quad O'' \equiv \frac{1}{m^2} \nabla_x^2 O
\]

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Express \((\Psi, \Psi^*)\) as function of \((\Phi, \pi)\)

- The 1\(^{st}\) order contribution at large mass limit \((\lambda \Phi^2 / m^2\) is small)

\[
\psi(1)(x) = iz \int [d^3p] e^{ip \cdot x} \{a_p, H_{\text{aux}}(a, a^*)\}_a \\
\psi(1) \sim \frac{\lambda \Phi^2}{m^2} \psi(0)
\]

- The main result: (2) a rigorous map from \(\phi\) to \(\psi = \psi(0) + \psi(1) + \ldots\) in non-relativistic limit

- An important application of our method is to study axion as dark matter candidate.

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Comparison with previous approaches

• Compared with previous approaches (Guth, Braaten, …):

• Our approach: a rigorous map between relativistic real scalar field to non-relativistic complex scalar field based on classical canonical transformation. The mapped relation can be derived order by order in large mass limit.

• It is a non-relativistic effective field theory (NREFT) built in a classical way!
Numerical simulation of massive relativistic scalar fields in (2+1)D

- Solve classical EOM for $\phi$ in real time lattice simulation

$$\partial_t \phi(t,x) = \pi(t,x),$$

$$\partial_t \pi(t,x) = \partial_i \partial^i \phi(t,x) - m^2 \phi(t,x) - \frac{\lambda}{6} \phi(t,x)^3$$

- With fluctuating initial condition

$$\phi_0(x) = \frac{1}{(N_s a_s)^2} \sum_p \frac{1}{\sqrt{2E_p}} [\alpha_p e^{+ip \cdot x} + \alpha^*_p e^{-ip \cdot x}],$$

$$\pi_0(x) = \frac{(-i)}{(N_s a_s)^2} \sum_p \sqrt{\frac{E_p}{2}} [\alpha_p e^{+ip \cdot x} - \alpha^*_p e^{-ip \cdot x}],$$

- Single particle distribution

$$f(t,p) = \frac{1}{(N_s a_s)^2} \sqrt{\langle |\tilde{\phi}(t,p)|^2 \rangle \langle |\tilde{\pi}(t,p)|^2 \rangle},$$

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Single-particle spectrum from real scalar $\phi$

- Non-thermal fixed-point in IR associated with self-similar cascade
- Different exponents from 2PI 1/N prediction ($\alpha=1, \beta=1/2$)

$$f(t,p) = \frac{1}{(N_s a_s)^2} \sqrt{\langle |\tilde{\phi}(t,p)|^2 \rangle \langle |\tilde{\pi}(t,p)|^2 \rangle},$$

$$f(p,t) = t^\alpha f_S(t^\beta p)$$

$\beta = 0.24 \pm 0.08$

$\alpha = 2\beta$

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Numerical results

- Non-relativistic complex scalar $\Psi$ constructed from $\phi$. We can draw density $\rho = |\psi|^2$ and phase $\theta(x)$

$$\psi(x) = \sqrt{\rho(x)} e^{i\theta(x)}$$

$$\psi(0)(x) \equiv \Phi + \Pi - \frac{1}{4}\Phi'' + \frac{1}{4}\Pi''$$

- Vortices $\rho \to 0, \theta \to \pm 2\pi$

$$\psi(1)(x) = -\frac{\lambda}{16m^3} \left[ -\frac{5}{6} \Phi^3 - \frac{7}{4} \Phi^2 \Phi'' - \frac{27}{8} \Phi \Phi' \Phi' - \frac{5}{2} \Phi^2 \Pi \\
+ \frac{11}{8} \Pi \Phi' \Phi' + \frac{3}{2} \Phi \Pi \Phi'' + \frac{11}{4} \Phi \Phi' \Pi' + 3 \Phi^2 \Pi'' \\
+ \frac{3}{2} \Phi \Pi^2 + \frac{5}{4} \Pi^2 \Phi'' + \frac{5}{4} \Pi \Phi' \Pi' + \frac{5}{8} \Phi \Pi' \Pi' \\
+ 2 \Phi \Pi \Pi'' - \frac{1}{2} \Pi^3 - \frac{21}{8} \Pi \Pi' \Pi - 2 \Pi^2 \Pi'' \right]$$

$$\psi(1) \sim \frac{\lambda \phi^2}{m^2} \psi(0)$$

expansion parameter: $\lambda \phi^2 / m^2$
Numerical results: density and phase of $\psi$

$|\psi|^2 \to 0$

$|\theta| \to 2\pi$
Numerical results: density and phase of $\psi$
Defect structure of relativistic fields

- Vortices of non-rel. fields correspond to intersections of domain walls of $\phi$ and $\pi$ fields

Scalar field profile at $Q_t=2000$

Momentum field profile at $Q_t=2000$

$\Phi$ field

$\Pi$ field
Numerical results: evolution of vortex and anti-vortex

- Striking similarities with previous observations in simulations of Gross-Pitaevski equation
- Non-equilibrium realization of topological phase transition (KT transition)
- Density of vortices decreases due to annihilation of vortex and anti-vortex

- vortex
- anti-vortex

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Numerical results: vortex density

Evolution of the vortex density as functions of time $t$ in the unit of the healing length

Vortex density decays in power law of $t$

\[ n_V \sim t^{-\zeta} \]
Extraction of the scaling exponent characterizing the power law decay
\[ n_V \sim t^{-\zeta} = t^{-2\beta} \]
or the mean distance of two vortices increases in the power law as
\[ l_V \sim t^\beta \]
We constructed a formal map between the infrared structure of an N=1 relativistic self-interacting scalar field theory and the Gross-Pitaevskii (GP) theory for nonrelativistic fields.

This map is constructed by classical canonical transformation in a perturbation scheme in non-relativistic limit. In this way, we build up a non-relativistic effective field theory (NREFT) in a classical way!

Many applications:
(1) axion as dark matter candidate
(2) superfluids + superconductors
(3) turbulence
(4) polarization in fluids
Backup
QCD axion: real scalar field

• QCD axion is a solution for strong CP problem
  [Peccei, Quinn (1977)]

• QCD axion: a real pseudoscalar, Goldstone boson, candidate for cold dark matter
  [Weinberg (1978), Wilczek(1978)]

• Lagrangian for $\phi$

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \mathcal{V}(\phi)$$

• Instanton potential

$$\mathcal{V}(\phi) = m_a^2 f_a^2 [1 - \cos(\phi/f_a)]$$

• For QCD axion, $m_a$ and $f_a$ are not independent
• For axion-like particles, $m_a$ and $f_a$ are independent
QCD axion: real scalar field

- For QCD axion, we have constraints from astrophysics and cosmology

\[ 10^8 \text{ GeV} < f_a < 10^{13} \text{ GeV} \quad \Rightarrow \quad 10^{-6} \text{ eV} < m_a < 10^{-2} \text{ eV} \]

Very weak self-interaction \quad Tiny mass (\lambda \sim 10^{-3} m)

- Axion self-interaction may be too weak to thermalize themselves, but gravitational interaction can thermalize axions
  1. Bring initially incoherent axions into coherent ones
  2. Keep the axion field in BEC

- QCD axions are almost static: non-relativistic particles
- Non-relativistic EFT: effective Hamiltonian is built up by symmetry and matching procedure to fix coefficients

\[ \phi \rightarrow \psi \]

Guth et al. (2014, 2017)
Braaten, Zhang, et al. (2016)

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