Off-equilibrium infrared structure of self-interacting scalar fields

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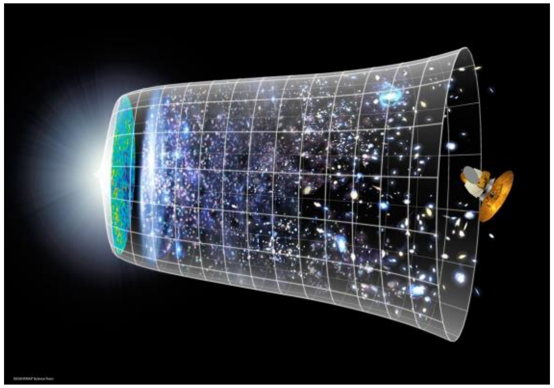
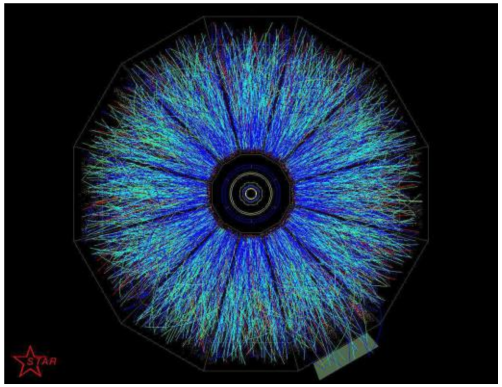
Outline

- **Introduction: non-thermal fixed points, ultra-violet & infrared fixed points, examples: turbulence in kinetic processes, vortices as topological defects**
- **Map: from relativistic real scalar ϕ to non-relativistic complex scalar ψ, derive GP theory from that of ϕ**
- **Simulations of relativistic real scalar ϕ in 2D which give the dynamics of vortices and scaling exponents appearing in GP theory for ψ**

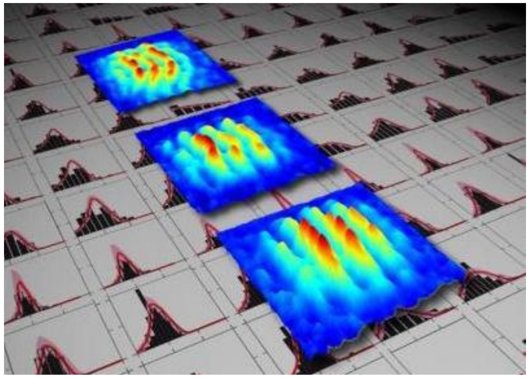
Physical systems far from equilibrium at different scales



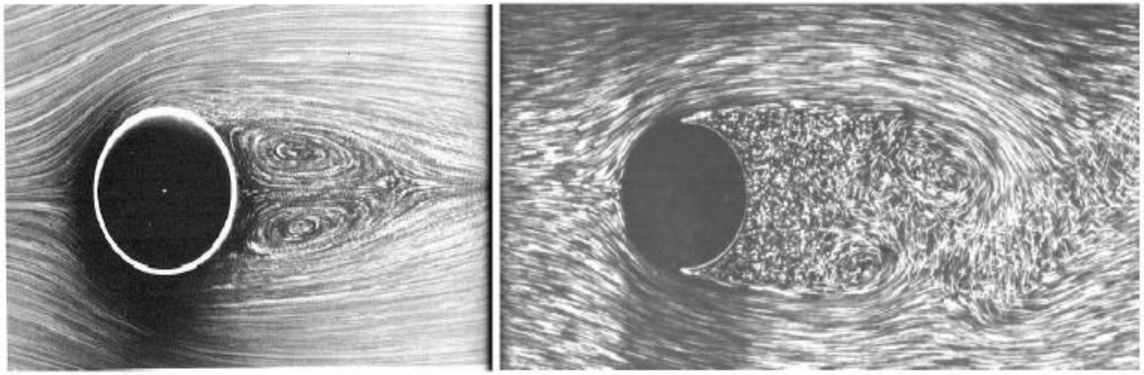
universe



Heavy-ion collisions



Ultracold atoms



Classical and quantum turbulence

Question:

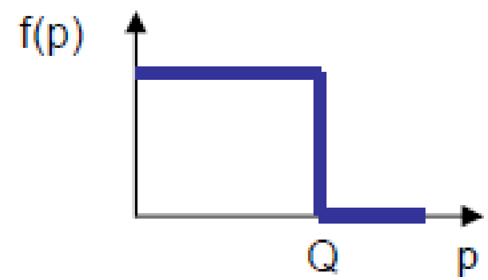
Do they have common properties?

Talks by Berges, Mazeliauskas, Heller, Noronha, Martinez, Denicol

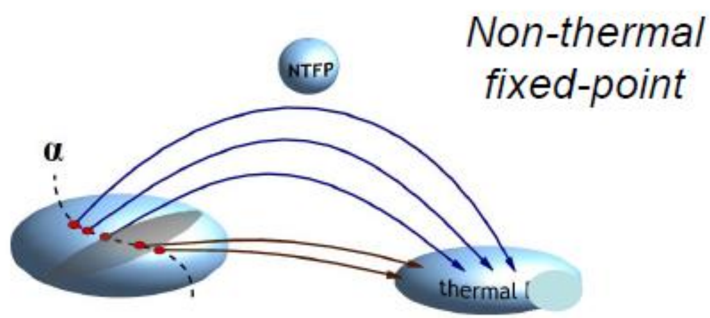
Example: IS in HIC far from equilibrium

- **IS in HIC at early times is a weak coupling system with g<<1**
- **Highly correlated system (color glass condensate)**
- **The system can take detour to NTFP before thermalization**

  Berges, Epelbaum, Gasenzer, Gelis, Mclerran, Moore, Schlichting, Sexty, Venugopalan, ……



$f(p \lesssim Q) \sim 1/g^2$ (or $\langle A\, A \rangle \sim Q^2/g^2$)



Nowak, Scholle, Gasenzer

Non-thermal fixed-points & universality classes

- **At a non-thermal fixed point**
  $\rightarrow$ **Memory loss of the details of the initial conditions;**
  $\rightarrow$ **Self-similar evolution of distribution function f (critical slowing down)**
  $\rightarrow$ **Scaling behavior in time** $f \sim t^{\alpha}$, $p \sim t^{-\beta}$

$$f(p,t) = t^{\alpha} f_S(t^{\beta} p)$$

  $\rightarrow$ **Several fixed-points in different momentum regimes (inertial range)**
  $\rightarrow$ **Often connected to kinetic processes (turbulence) or topological defects (vortices)**

- **Classification: universality classes for system far from equilibrium**
  $\rightarrow$ **by NTFP with dynamical exponents α, β and the scaling function $f_S(x)$**

Dual cascade in scalar field theory

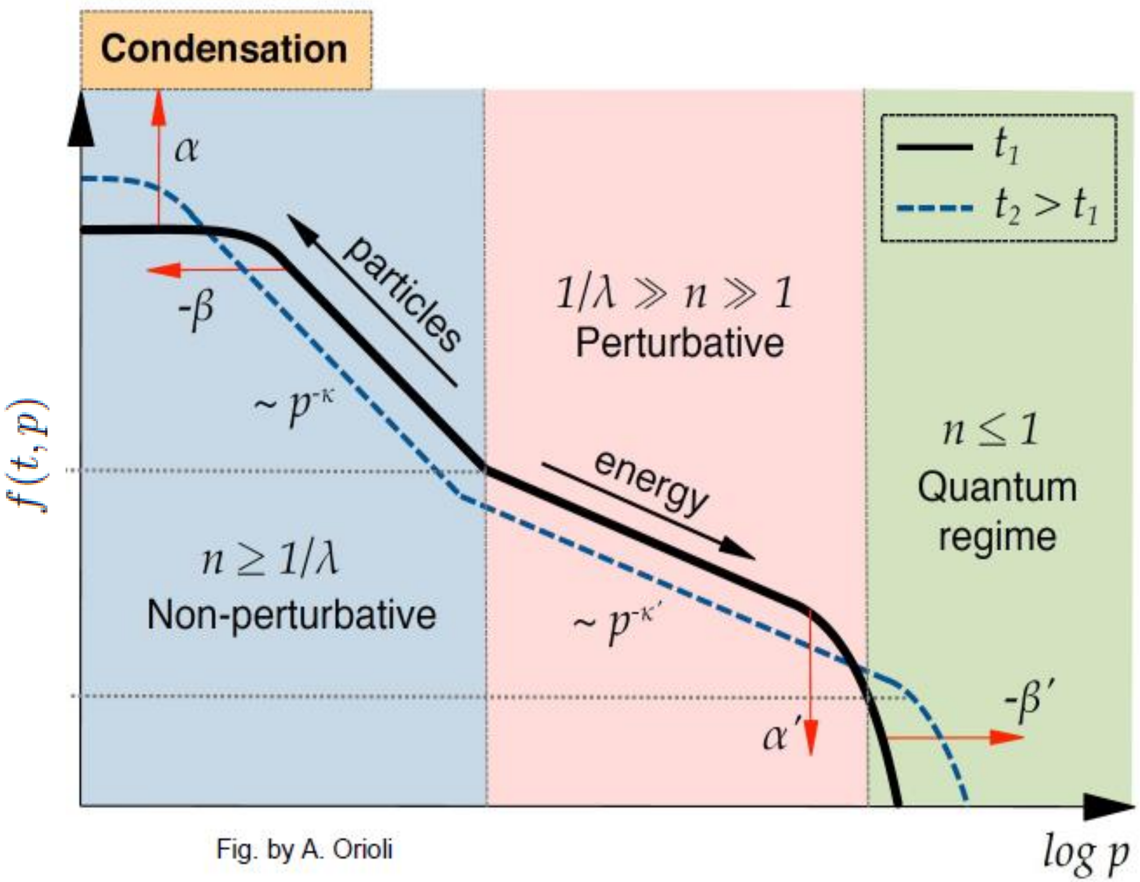


Fig. by A. Orioli

Self-similarity

$$f(p,t) = t^{\alpha} f_S(t^{\beta} p)$$

Perturbative

Micha & Tkachev, PRL(2003), PRD(2004)

Non-perturbative

Berges, Rothkopf & Schmidt, PRL (2008)

Nowak, Sexty & Gasenzer PRB(2011)

Bose condensation

Berges & Sexty, PRL(2012)

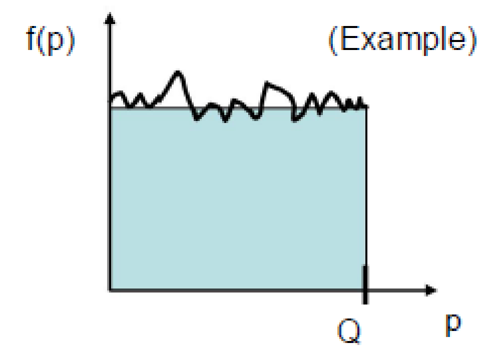
Classical-statistical lattice simulation

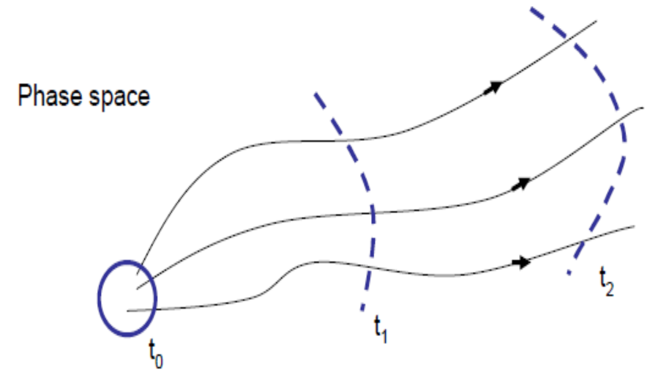
- **Start with fluctuating initial conditions at initial time t_0**
- **Solve initial value problem on the lattice. For example, self-interacting real scalars:**

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4$$

$$(\partial_\mu\partial^\mu + m^2)\phi + \frac{1}{6}\lambda\phi^3 = 0 \Longrightarrow \boldsymbol{\phi(t)}$$

- **We know $\boldsymbol{\phi(t) \to O[\phi(t)]}$ at $\boldsymbol{t}$ for a specific IC, ensemble average over initial conditions at time $\boldsymbol{t}$ to obtain the observable $\langle \boldsymbol{O[\phi(t)]} \rangle$**





[Aarts & Berges, PRL 88, 041603 (2002); Jeon, PRC 72, 014907 (2005)]

Relativistic vs non-relativistic fields

Relativistic real scalar

(high energy physics)

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4$$

$$(\partial_\mu\partial^\mu + m^2)\phi + \frac{1}{6}\lambda\phi^3 = 0$$

~~global U(1)~~

Non-relativistic complex scalar

(Superfluid Helium, cold atoms)

$$\mathcal{H} = -\frac{1}{2m}\psi^*\nabla^2\psi + V|\psi|^2 + \frac{1}{2}g|\psi|^4$$

$$i\frac{\partial\psi}{\partial t} = \left(-\frac{1}{2m}\nabla^2 + V + g|\psi|^2\right)\psi$$

global U(1)

Low-momentum universality class of scalar field theory: dynamical scaling exponents of the infrared fixed-point

exponent	relativistic	non-relativistic
α	3	3/2
β	1	1/2

Self-similar distribution

$$\boxed{f(p,t) = t^{\alpha} f_S(t^{\beta}p)}$$

What is the connection between relativistic real scalar and non-relativistic complex scalar field theory?

Mapping: from Φ to Ψ

- **Hamiltonian density, scalar field and its canonical momentum**

$$
\begin{aligned}
\mathcal{H} &= \frac{1}{2}[\pi^2 + (\nabla\phi)^2 + m^2\phi^2] + \frac{1}{24}\lambda\phi^4 \\
\phi(\mathbf{x}) &= \int [d^3\mathbf{k}] \frac{1}{\sqrt{2E_k}} \left(\underline{a_{\mathbf{k}}} e^{i\mathbf{k}\cdot\mathbf{x}} + \underline{a^*_{\mathbf{k}}} e^{-i\mathbf{k}\cdot\mathbf{x}} \right) \\
\pi(\mathbf{x}) &= -i \int [d^3\mathbf{k}] \sqrt{\frac{E_k}{2}} \left(\underline{a_{\mathbf{k}}} e^{i\mathbf{k}\cdot\mathbf{x}} - \underline{a^*_{\mathbf{k}}} e^{-i\mathbf{k}\cdot\mathbf{x}} \right)
\end{aligned}
$$

all complex classical variables

- **Hamiltonian in a_k and a^*_k**

$$
\begin{aligned}
H(a, a^*) &= \int d^3\mathbf{x}\mathcal{H} = \int [d^3\mathbf{k}] E_k a_{\mathbf{k}} a^*_{\mathbf{k}} \\
&+ \frac{1}{24}\lambda \int [d^3\mathbf{k}][d^3\mathbf{k}_1][d^3\mathbf{k}_2][d^3\mathbf{k}_3] \frac{(2\pi)^3}{\sqrt{16E_k E_{k1} E_{k2} E_{k3}}} \\
&\left[\underline{a_{\mathbf{k}} a_{\mathbf{k}1} a_{\mathbf{k}2} a_{\mathbf{k}3}} \delta(\mathbf{k} + \mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) + 4\underline{a_{\mathbf{k}} a_{\mathbf{k}1} a_{\mathbf{k}2} a^*_{\mathbf{k}3}} \delta(\mathbf{k} + \mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3) \right. \\
&+ 6\underline{a_{\mathbf{k}} a_{\mathbf{k}1} a^*_{\mathbf{k}2} a^*_{\mathbf{k}3}} \delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) + 4\underline{a_{\mathbf{k}} a^*_{\mathbf{k}1} a^*_{\mathbf{k}2} a^*_{\mathbf{k}3}} \delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) \\
&\left. + \underline{a^*_{\mathbf{k}} a^*_{\mathbf{k}1} a^*_{\mathbf{k}2} a^*_{\mathbf{k}3}} \delta(-\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) \right]
\end{aligned}
$$

resonant term: equal number of a_k and a^*_k

non-resonant terms: un-equal number of a_k and a^*_k , correspond to particle number changing processes

EOM and Poisson brackets

- **EOM in Poisson brackets**

$$i\frac{da_{\mathbf{p}}}{dt} = \{a_{\mathbf{p}}, H\}_a , \qquad i\frac{da^*_{\mathbf{p}}}{dt} = \{a^*_{\mathbf{p}}, H\}_a$$

- **Poisson brackets in the a_p basis**

$$\{F(\mathbf{p}), G(\mathbf{p}_1)\}_a = \int [d^3\mathbf{k}] \left[\frac{\partial F(\mathbf{p})}{\partial a_{\mathbf{k}}}\frac{\partial G(\mathbf{p}_1)}{\partial a^*_{\mathbf{k}}} - \frac{\partial F(\mathbf{p})}{\partial a^*_{\mathbf{k}}}\frac{\partial G(\mathbf{p}_1)}{\partial a_{\mathbf{k}}}\right]$$

$$\begin{aligned}\{a_{\mathbf{p}}, a^*_{\mathbf{p}1}\}_a &= (2\pi)^3\delta(\mathbf{p}-\mathbf{p}_1) \\ \{a_{\mathbf{p}}, a_{\mathbf{p}1}\}_a &= \{a^*_{\mathbf{p}}, a^*_{\mathbf{p}1}\}_a = 0\end{aligned}$$

- **Generally we can use any canonical variables** $(a_{\mathbf{k}}, a^*_{\mathbf{k}}) \rightarrow (b_{\mathbf{p}}, b^*_{\mathbf{p}})$ **,**

  if $(b_{\mathbf{p}}, b^*_{\mathbf{p}})$ **have the same Poisson brackets in the a_p-basis**

$$\begin{aligned}\{b_{\mathbf{p}}, b^*_{\mathbf{p}1}\}_a &= (2\pi)^3\delta(\mathbf{p}-\mathbf{p}_1) \\ \{b_{\mathbf{p}}, b_{\mathbf{p}1}\}_a &= \{b^*_{\mathbf{p}}, b^*_{\mathbf{p}1}\}_a = 0\end{aligned}$$

Canonical transformation

- **Poisson brackets are invariant under the change of canonical basis, for example**

$$i\frac{\partial b_{\mathbf{p}}}{\partial t}=\{b_{\mathbf{p}},H\}_{\underline{b}}=\{b_{\mathbf{p}},H\}_{\underline{a}}$$

- **Canonical variables can be defined by time shift through H_{aux} (b_p,b^*_p):**

z is auxiliary time

$$\begin{aligned} b_{\mathbf{p}}(z) &= b_{\mathbf{p}}+z\left.\frac{db_{\mathbf{p}}(z)}{dz}\right|_{z=0}+\frac{1}{2}z^2\left.\frac{d^2b_{\mathbf{p}}(z)}{d^2z}\right|_{z=0}+\frac{1}{6}z^3\left.\frac{d^3b_{\mathbf{p}}(z)}{d^3z}\right|_{z=0}+\cdots \\ &= b_{\mathbf{p}}-iz\{b_{\mathbf{p}},H_{\text{aux}}\}_b+\frac{1}{2}(-iz)^2\{\{b_{\mathbf{p}},H_{\text{aux}}\}_b,H_{\text{aux}}\}_b \\ &\quad+\frac{1}{6}(-iz)^3\{\{\{b_{\mathbf{p}},H_{\text{aux}}\}_b,H_{\text{aux}}\}_b,H_{\text{aux}}\}_b+\cdots \end{aligned}$$

$$\boxed{a_{\mathbf{p}}\equiv b_{\mathbf{p}}(z)}$$

$$\boxed{\begin{aligned} b_{\mathbf{p}} &\equiv b_{\mathbf{p}}(0) \\ b^*_{\mathbf{p}} &\equiv b^*_{\mathbf{p}}(0) \end{aligned}}$$

- **One can prove that** $b_{\mathbf{p}}(z), b^*_{\mathbf{p}}(z)$ **satisfy canonical relations in b_p basis**

$$\begin{aligned} \{a_{\mathbf{p}},a^*_{\mathbf{p}1}\}_b &= (2\pi)^3\delta(\mathbf{p}-\mathbf{p}_1) \\ \{a_{\mathbf{p}},a_{\mathbf{p}1}\}_b &= \{a^*_{\mathbf{p}},a^*_{\mathbf{p}1}\}_b=0 \end{aligned}$$

New canonical variable through H_{aux}

- **Auxiliary Hamiltonian**

$$
\begin{aligned}
H_{\rm aux}(b,b^*) \quad = \quad & \int [d^3\mathbf{k}_1][d^3\mathbf{k}_2][d^3\mathbf{k}_3][d^3\mathbf{k}_4] \\
& \times \left\{ \frac{1}{24}[\underline{B_1(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)}b_{\mathbf{k}1}b_{\mathbf{k}2}b_{\mathbf{k}3}b_{\mathbf{k}4} + \mathrm{c.c.}] \right. \\
& \times \delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4) \\
& + \frac{1}{6}[\underline{B_2(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3;\mathbf{k}_4)}b_{\mathbf{k}1}b_{\mathbf{k}2}b_{\mathbf{k}3}b^*_{\mathbf{k}4} + \mathrm{c.c.}] \\
& \left. \times \delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3-\mathbf{k}_4) \right\}
\end{aligned}
$$

- **B_1 and B_2 will be determined by removing non-resonant terms in the original Hamiltonian**

Canonical transformation: derive H_{GP}

- **New canonical variable by time shift through H_{aux}**

initial auxiliary time z=0

$$b_{\mathbf{p}} \equiv b_{\mathbf{p}}(0)$$
$$b^*_{\mathbf{p}} \equiv b^*_{\mathbf{p}}(0)$$

$$H_{\mathrm{aux}}(b, b^*)$$

auxiliary time z

$$a_{\mathbf{p}} \equiv b_{\mathbf{p}}(z)$$
$$a^*_{\mathbf{p}} \equiv b^*_{\mathbf{p}}(z)$$

$$\begin{aligned} a_{\mathrm{p}} &= b_{\mathrm{p}} - iz\{b_{\mathrm{p}}, H_{\mathrm{aux}}\}_b \\ &\quad + \frac{1}{2}(-iz)^2\{\{b_{\mathrm{p}}, H_{\mathrm{aux}}\}_b, H_{\mathrm{aux}}\}_b \\ &\quad + \cdots \end{aligned}$$

substitute

$$H(a, a^*)$$

$$H'(b, b^*)$$

$$\begin{aligned} H'(b,b^*) &= H(b,b^*) + (-iz)\{H(b,b^*), H_{\mathrm{aux}}\}_b \\ &\quad + \frac{1}{2}(-iz)^2\{\{H(b,b^*), H_{\mathrm{aux}}\}_b, H_{\mathrm{aux}}\}_b \\ &\quad + \frac{1}{6}(-iz)^3\{\{\{H(b,b^*), H_{\mathrm{aux}}\}_b, H_{\mathrm{aux}}\}_b, H_{\mathrm{aux}}\}_b \\ &\quad + \cdots \end{aligned}$$

with only resonant terms, global U(1)

$$H_{\mathrm{GP}}(b, b^*)$$

vanishing of non-resonant terms fix B_1 and B_2 in H_{aux}

Gross-Pitaevskii Hamiltonian

$$H_{\mathrm{GP}}(b,b^*) \approx \int [d^3\mathbf{p}] E_p b_{\mathbf{p}} b_{\mathbf{p}}^* + \frac{1}{16}\lambda \int \prod_{i=1}^{4} [d^3\mathbf{k}_i] \frac{1}{\sqrt{E_{k1}E_{k2}E_{k3}E_{k4}}} \times b_{\mathbf{k}1} b_{\mathbf{k}2} b_{\mathbf{k}3}^* b_{\mathbf{k}4}^* (2\pi)^3 \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4)$$

global U(1) symmetry

$$E_k \approx m + \frac{\mathbf{k}^2}{2m}$$

$$\psi(\mathbf{x}) = \int [d^3\mathbf{k}] b_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}$$

$$H_{\mathrm{GP}}(\psi,\psi^*) \approx m \int d^3\mathbf{x} |\psi(t,\mathbf{x})|^2 + \int d^3\mathbf{x} \psi^*(t,\mathbf{x}) \left(-\frac{1}{2m}\nabla^2\right) \psi(t,\mathbf{x}) + \frac{\lambda}{16m^2} \int d^3\mathbf{x} |\psi(t,\mathbf{x})|^4$$

The main result: (1) derivation of GP Hamiltonian from ϕ

Express b_p as function of a_p

- **Change the basis from b_p to a_p for all Poisson brackets**

$$\begin{aligned} b_{\mathbf{p}} &= a_{\mathbf{p}} + iz\{b_{\mathbf{p}}, H_{\mathrm{aux}}(b)\}_a \\ &\quad -\frac{1}{2}(iz)^2\{\{b_{\mathbf{p}}, H_{\mathrm{aux}}(b)\}_a, H_{\mathrm{aux}}(b)\}_a + \cdots \end{aligned}$$

- **b_p can be solved as a function of a_p in perturbation**

$$\begin{aligned} b_{\mathbf{p}}^{(0)} &= a_{\mathbf{p}} \\ b_{\mathbf{p}}^{(1)} &= iz\ \{b_{\mathbf{p}}, H_{\mathrm{aux}}(b)\}_a\big|_{b\to a} \\ b_{\mathbf{p}}^{(2)} &= -\frac{1}{2}(iz)^2\ \{\{b_{\mathbf{p}}, H_{\mathrm{aux}}(b)\}_a, H_{\mathrm{aux}}(b)\}_a\big|_{b\to a} \\ &\quad + iz\ \{b_{\mathbf{p}}, H_{\mathrm{aux}}(b)\}_a\big|_{b\to a+b^{(1)}} \\ \cdots &\quad \cdots \quad \cdots \end{aligned}$$

expansion parameter:

$$\frac{\lambda\phi^2}{m^2}$$

Express (Ψ, Ψ*) as function of (ϕ,π)

- **Transform to coordinate space**

$$\psi(\mathbf{x}) = \int [d^3\mathbf{k}] b_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}$$

- **Express (Ψ, Ψ*) as function of (ϕ,π). In the large mass limit, the zeroth order**

$$\boxed{\begin{aligned} a_{\mathbf{k}} &= \sqrt{\frac{E_k}{2}}\phi_{\mathbf{k}} + i\sqrt{\frac{1}{2E_k}}\pi_{\mathbf{k}} \\ &= \sqrt{\frac{E_k}{2}}\int d^3x\phi(x)e^{-i\mathbf{k}\cdot\mathbf{x}} + i\sqrt{\frac{1}{2E_k}}\int d^3x\pi(x)e^{-i\mathbf{k}\cdot\mathbf{x}} \end{aligned}}$$

$$\begin{aligned} \psi_{(0)}(\mathbf{x}) &= \int \frac{d^3k}{(2\pi)^3} a_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} \\ &= \sqrt{\frac{m}{2}}\left(1 - \frac{1}{4m^2}\nabla_x^2\right)\phi(x) + i\frac{1}{\sqrt{2m}}\left(1 + \frac{1}{4m^2}\nabla_x^2\right)\pi(x) \\ &\equiv \Phi + \Pi - \frac{1}{4}\Phi'' + \frac{1}{4}\Pi'' \end{aligned}$$

$$\Phi \equiv \sqrt{\frac{m}{2}}\phi(\mathbf{x}) \qquad \Pi \equiv i\frac{1}{\sqrt{2m}}\pi(\mathbf{x}) \qquad O'' \equiv \frac{1}{m^2}\nabla_{\mathbf{x}}^2 O$$

Express (Ψ, Ψ*) as function of (Φ,π)

- **The 1st order contribution at large mass limit ($\lambda\phi^2/m^2$ is small)**

$$\psi_{(1)}\sim\frac{\lambda\phi^2}{m^2}\psi_{(0)}$$

expansion parameter: $\lambda\phi^2/m^2$

$$\begin{aligned}
\psi_{(1)}(\mathbf{x}) &= iz\int[d^3\mathbf{p}]e^{i\mathbf{p}\cdot\mathbf{x}}\left\{a_{\mathbf{p}},H_{\mathrm{aux}}(a,a^*)\right\}_a \\
&= -\frac{\lambda}{16m^3}\left[-\frac{5}{6}\Phi^3-\frac{7}{4}\Phi^2\Phi''-\frac{27}{8}\Phi\Phi'\Phi'+\frac{5}{2}\Phi^2\Pi\right. \\
&\quad+\frac{11}{8}\Pi\Phi'\Phi'+\frac{3}{2}\Phi\Pi\Phi''+\frac{11}{4}\Phi\Phi'\Pi'+3\Phi^2\Pi'' \\
&\quad+\frac{3}{2}\Phi\Pi^2+\frac{5}{4}\Pi^2\Phi''+\frac{5}{4}\Pi\Phi'\Pi'+\frac{5}{8}\Phi\Pi'\Pi' \\
&\quad\left.+\,2\Phi\Pi\Pi''-\frac{1}{2}\Pi^3-\frac{21}{8}\Pi\Pi'\Pi'-2\Pi^2\Pi''\right]
\end{aligned}$$

- **The main result: (2) a rigorous map from ϕ to $\psi=\psi_{(0)}+\psi_{(1)}+\cdots$ in non-relativistic limit**
- **An important application of our method is to study axion as dark matter candidate.**

Comparison with previous approaches

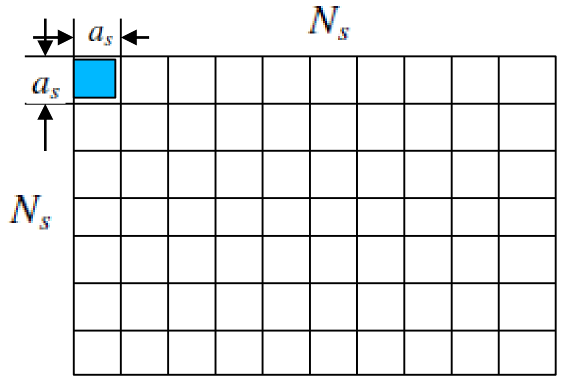
- **Compared with previous approaches (Guth, Braaten, …):**
- **Our approach: a rigorous map between relativistic real scalar field to non-relativistic complex scalar field based on classical canonical transformation. The mapped relation can be derived order by order in large mass limit.**
- **It is a non-relativistic effective field theory (NREFT) built in a classical way !**

Numerical simulation of massive relativistic scalar fields in (2+1)D

- **Solve classical EOM for ϕ in real time lattice simulation**

$$\partial_t \phi(t,\mathbf{x}) = \pi(t,\mathbf{x}) \ ,$$

$$\partial_t \pi(t,\mathbf{x}) = \partial_i \partial^i \phi(t,\mathbf{x}) - m^2 \phi(t,\mathbf{x}) - \frac{\lambda}{6}\phi(t,\mathbf{x})^3$$



- **With fluctuating initial condition**

$$\phi_0(\mathbf{x}) = \frac{1}{(N_s a_s)^2} \sum_{\mathbf{p}} \frac{1}{\sqrt{2E_p}} [\alpha_{\mathbf{p}} e^{+i\mathbf{p}\cdot\mathbf{x}} + \alpha^*_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}}],$$

$$\pi_0(\mathbf{x}) = \frac{(-i)}{(N_s a_s)^2} \sum_{\mathbf{p}} \sqrt{\frac{E_p}{2}} [\alpha_{\mathbf{p}} e^{+i\mathbf{p}\cdot\mathbf{x}} - \alpha^*_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}}],$$

Sampled with Gaussian magnitude and uniform random-phase distribution

$$\langle \alpha_{\mathbf{p}} \alpha^*_{\mathbf{q}} \rangle = (N_s a_s)^2 \delta_{\mathbf{p},\mathbf{q}} f(t=0,p)$$

- **Single particle distribution**

$$f(t,p) = \frac{1}{(N_s a_s)^2} \sqrt{\langle |\tilde{\phi}(t,\mathbf{p})|^2 \rangle \langle |\tilde{\pi}(t,\mathbf{p})|^2 \rangle},$$

$$f(t=0,p) = \frac{6 n_0 Q}{\lambda} \theta(Q-p).$$

Fourier transform of ϕ(t,x) and π(t,x)

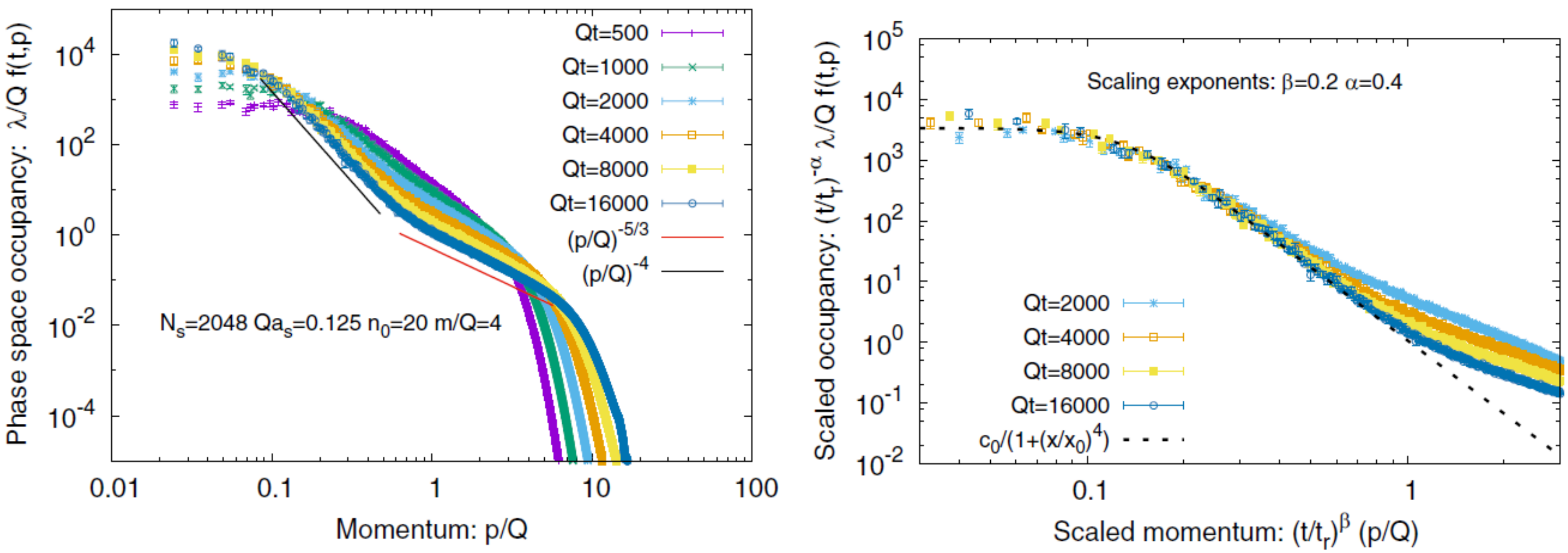
Single-particle spectrum from real scalar ϕ

- **Non-thermal fixed-point in IR associated with self-similar cascade**
- **Different exponents from 2PI 1/N prediction (α=1, β=1/2)**

$$\boxed{f(t,p) = \frac{1}{(N_s a_s)^2}\sqrt{\langle|\tilde{\phi}(t,\mathbf{p})|^2\rangle\langle|\tilde{\pi}(t,\mathbf{p})|^2\rangle},}$$

$$\boxed{f(p,t) = t^{\alpha} f_S(t^{\beta} p)}$$

$$\boxed{\begin{aligned}\beta &= 0.24 \pm 0.08\\ \alpha &= 2\beta\end{aligned}}$$

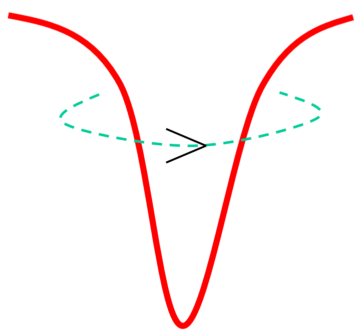


Numerical results

- **Non-relativistic complex scalar Ψ constructed from ϕ. We can draw density $\rho = |\psi|^2$ and phase $\theta(x)$**

$$\psi(\mathbf{x}) = \sqrt{\rho(\mathbf{x})}e^{i\theta(\mathbf{x})}$$

- **Vortices $\rho \to 0, \theta \to \pm 2\pi$**



$$\psi_{(0)}(\mathbf{x}) \equiv \Phi + \Pi - \frac{1}{4}\Phi'' + \frac{1}{4}\Pi''$$

$$\begin{aligned}\psi_{(1)}(\mathbf{x}) \quad = \quad & -\frac{\lambda}{16m^3}\Big[-\frac{5}{6}\Phi^3 - \frac{7}{4}\Phi^2\Phi'' - \frac{27}{8}\Phi\Phi'\Phi' + \frac{5}{2}\Phi^2\Pi \\ & + \frac{11}{8}\Pi\Phi'\Phi' + \frac{3}{2}\Phi\Pi\Phi'' + \frac{11}{4}\Phi\Phi'\Pi' + 3\Phi^2\Pi'' \\ & + \frac{3}{2}\Phi\Pi^2 + \frac{5}{4}\Pi^2\Phi'' + \frac{5}{4}\Pi\Phi'\Pi' + \frac{5}{8}\Phi\Pi'\Pi' \\ & + 2\Phi\Pi\Pi'' - \frac{1}{2}\Pi^3 - \frac{21}{8}\Pi\Pi'\Pi' - 2\Pi^2\Pi''\Big]\end{aligned}$$

$$\Phi \equiv \sqrt{\frac{m}{2}}\phi(\mathbf{x})$$

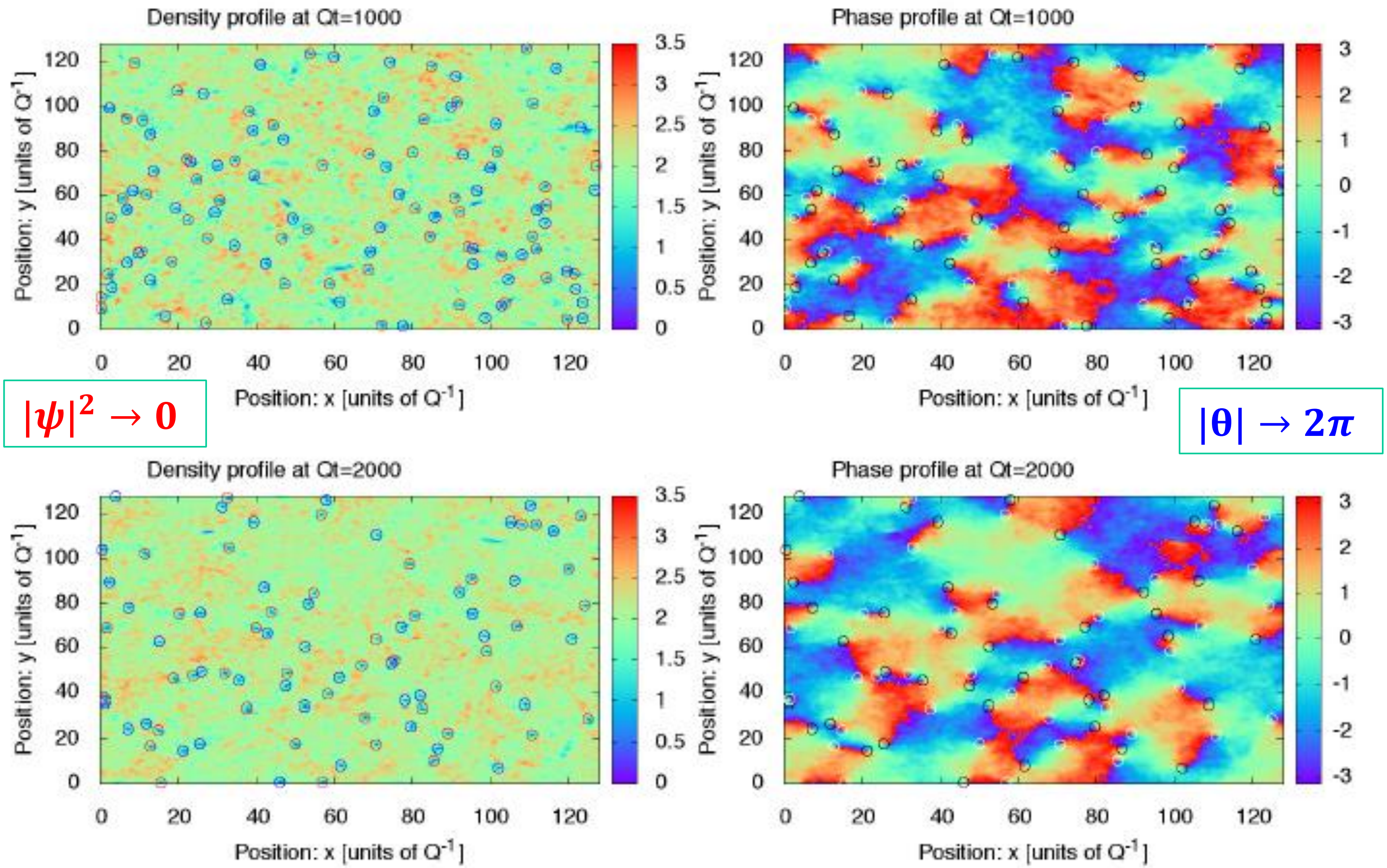
$$\Pi \equiv i\frac{1}{\sqrt{2m}}\pi(\mathbf{x})$$

$$O'' \equiv \frac{1}{m^2}\nabla^2_{\mathbf{x}} O$$

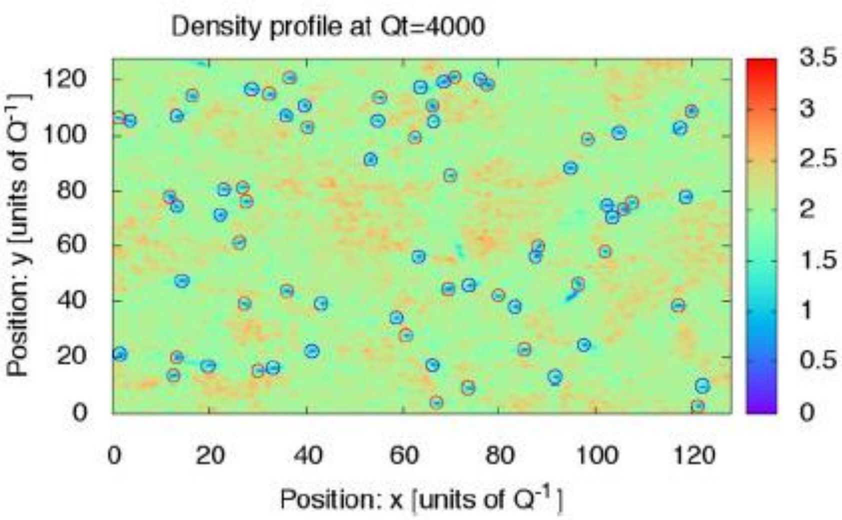
$$\psi_{(1)} \sim \frac{\lambda\phi^2}{m^2}\psi_{(0)}$$

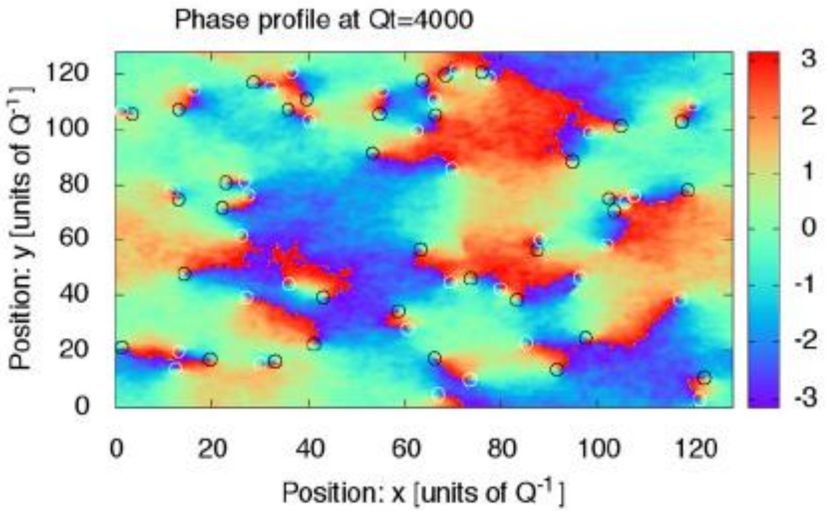
expansion parameter: $\lambda\phi^2/m^2$

Numerical results: density and phase of ψ



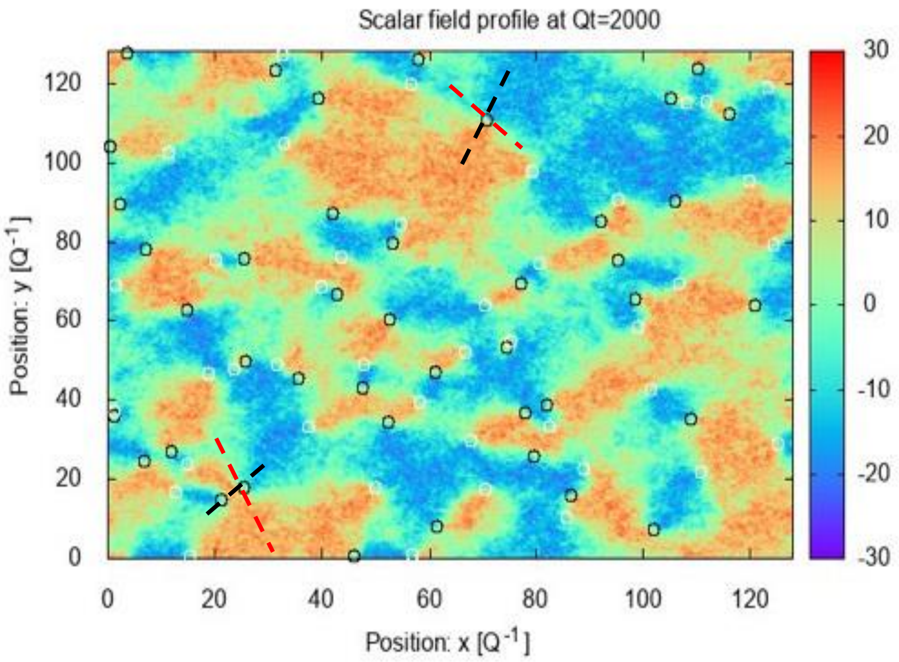
Numerical results: density and phase of ψ



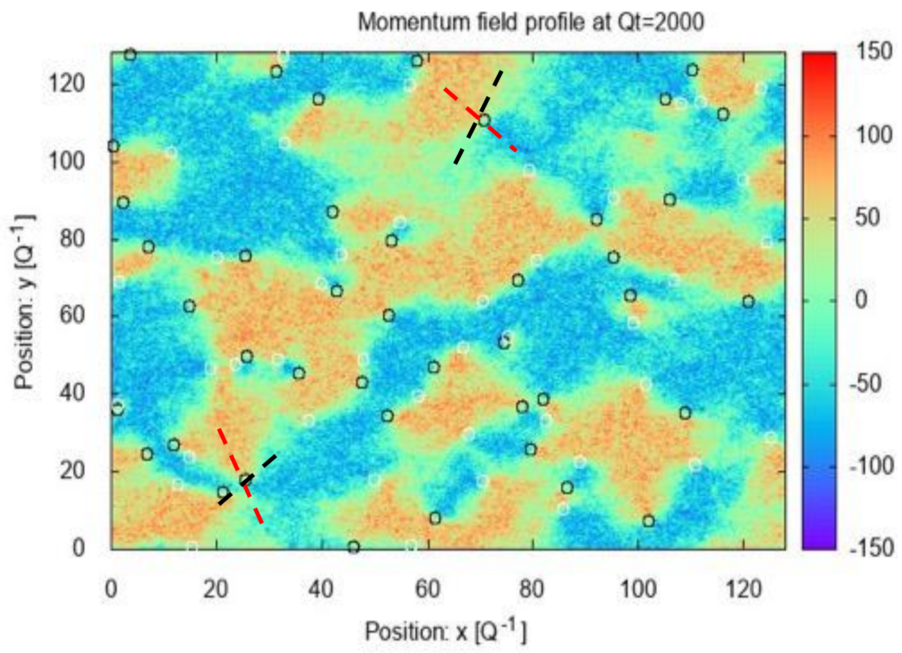


Defect structure of relativistic fields

- **Vortices of non-rel. fields correspond to intersections of domain walls of ϕ and π fields**

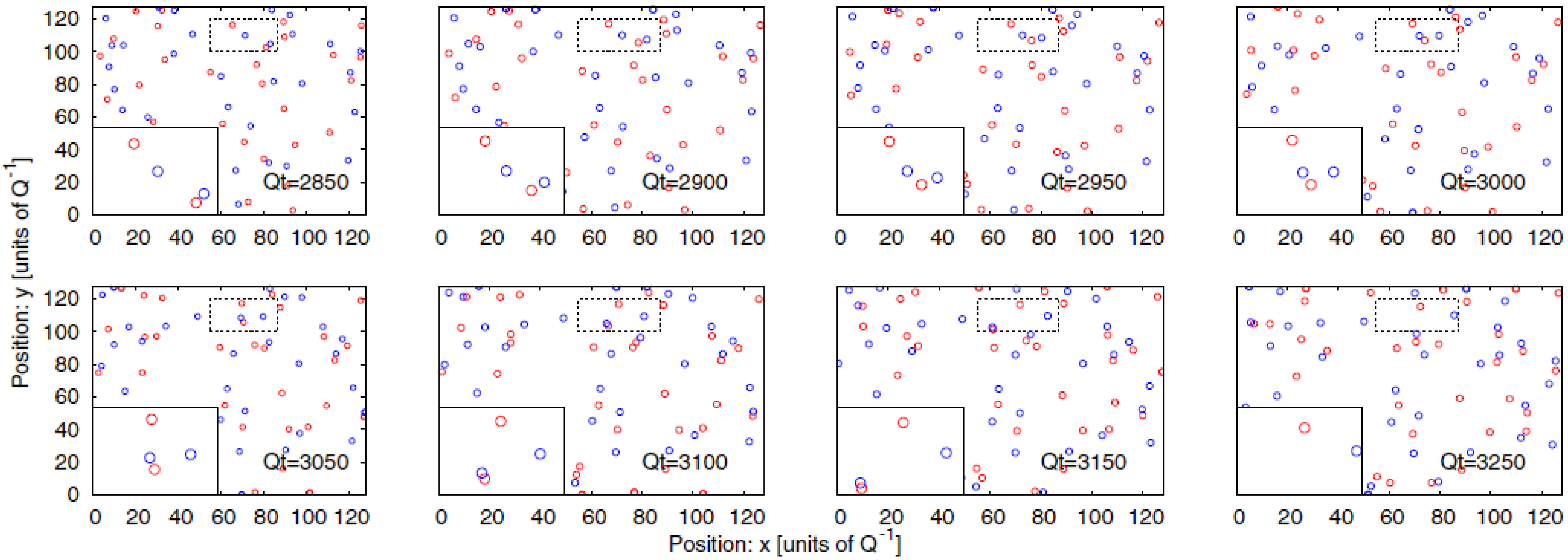


Φ field



Π field

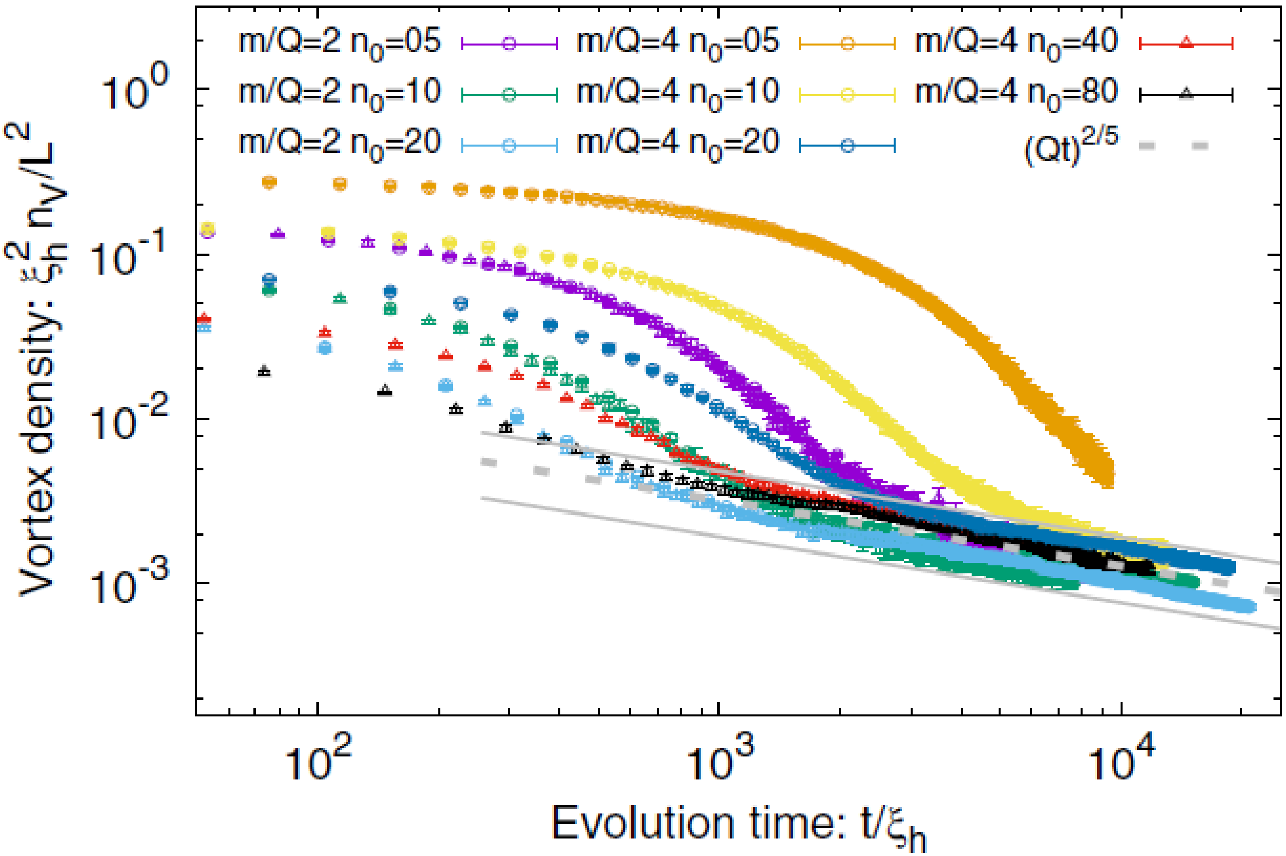
Numerical results: evolution of vortex and anti-vortex



- **Striking similarities with previous observations in simulations of Gross-Pitaevski equation**
- **Non-equilibrium realization of topological phase transition (KT transition)**
- **Density of vortices decreases due to annihilation of vortex and anti-vortex**

O vortex **O anti-vortex**

Numerical results: vortex density

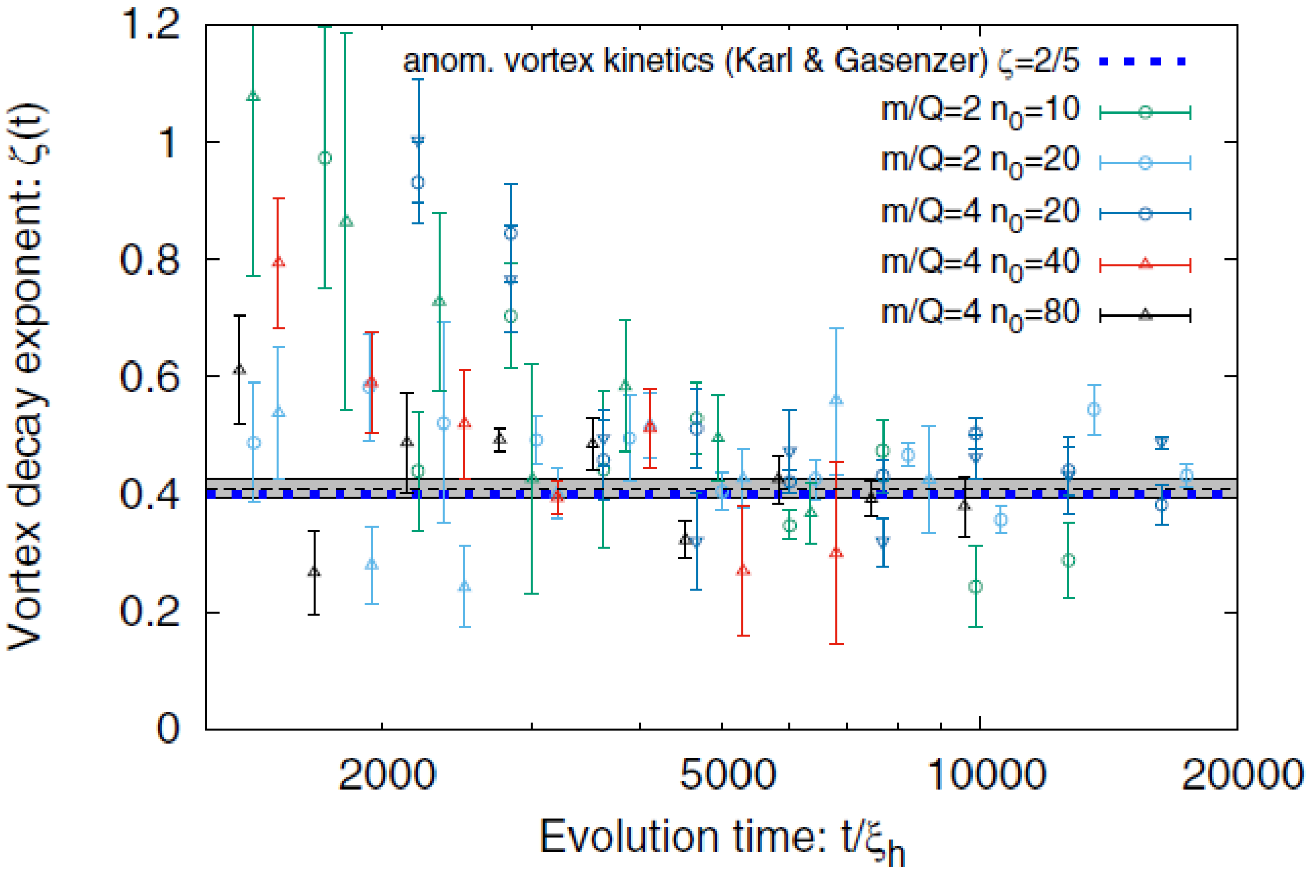


Evolution of the vortex density as functions of time t in the unit of the healing length

Vortex density decays in power law of t

$n_V \sim t^{-\zeta}$

Numerical results: vortex density



Extraction of the scaling exponent characterizing the power law decay

$n_V \sim t^{-\zeta} = t^{-2\beta}$

or the mean distance of two vortices increases In the power law as $l_V \sim t^{\beta}$

Summary

- **We constructed a formal map between the infrared structure of an N=1 relativistic self-interacting scalar field theory and the Gross-Pitaevskii (GP) theory for nonrelativistic fields.**
- **This map is constructed by classical canonical transformation in a perturbation scheme in non-relativistic limit. In this way, we build up a non-relativistic effective field theory (NREFT) in a classical way!**
- **Many applications:**

(1) axion as dark matter candidate

(2) superfluids + superconductors

(3) turbulence

(4) polarization in fluids

Backup

QCD axion: real scalar field

- QCD axion is a solution for strong CP problem

  [Peccei, Quinn (1977)]

- QCD axion: a real pseudoscalar, Goldstone boson, candidate for cold dark matter

  [Weinberg (1978), Wilczek(1978)]

- Lagrangian for ϕ

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \mathcal{V}(\phi)$$

- Instanton potential

m_a axion mass

f_a decay constant

$$\mathcal{V}(\phi) = m_a^2 f_a^2 [1 - \cos(\phi/f_a)]$$

- For QCD axion, m_a and f_a are not independent
- For axion-like particles, m_a and f_a are independent

QCD axion: real scalar field

- **For QCD axion, we have constraints from astrophysics and cosmology**

$$10^8 \text{ GeV} < f_a < 10^{13} \text{ GeV} \Longrightarrow 10^{-6} \text{ eV} < m_a < 10^{-2} \text{ eV}$$

Very weak self-interaction **Tiny mass ($\lambda \sim 10^{-3}\, m$)**

- **Axion self-interaction may be too weak to thermalize themselves, but gravitational interaction can thermalize axions**

  (1) Bring initially incoherent axions into coherent ones

  (2) Keep the axion field in BEC

- **QCD axions are almost static: non-relativistic particles**
- **Non-relativistic EFT: effective Hamiltonian is built up by symmetry and matching procedure to fix coefficients**

$$\phi \to \psi$$

Guth et al. (2014,2017)
Braaten, Zhang, et al. (2016)