Hydrodynamics far from equilibrium: a concrete example

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What you will see:

✔ Motivation: why fluid-dynamical descriptions work?

✔ Derivation of fluid dynamics using method of moments

✔ Can we have hydrodynamic behavior far from equilibrium?
**Empirical:** fluid-dynamical models of heavy ion collisions work well at RHIC and LHC energies

Main assumption: fluid dynamics is applied on very small time scales ~1 fm

Does this make sense?
Validity of fluid dynamics *traditionally* associated with:

- proximity to (local) equilibrium
- “small” gradients

Separation of scales $\rightarrow$ macroscopic: $\ell$ microscopic: $\ell$

Knudsen number: $K_N \sim \frac{\ell}{L} \ll 1$

Do these things occur early in Heavy Ion Collisions? *No reason to believe that they do.*

Then why does hydro work? What assumptions are really required?
We can study this problem in Kinetic theory

\[ k^\mu \partial_\mu f_k = C[f] \]

Boltzmann eq.

\[ \tau_\Pi \dot{\Pi} + \Pi = -\zeta \theta + \ldots \]

\[ \tau_\pi \dot{\pi} \langle \mu \nu \rangle + \pi^{\mu \nu} = 2\eta \sigma^{\mu \nu} + \ldots \]

2\textsuperscript{nd} - order hydro

In particular, we can use Israel-Stewart´s approach
Israel-Stewart theory: basic ideas
Israel-Stewart theory: 14-moment approximation

\[ f_k = f_{0k} + f_{0k} (1 - \alpha f_{0k}) \phi_k \]

\( \phi_k = \varepsilon + \varepsilon_{\mu} k^\mu + \varepsilon_{\mu\nu} k^\mu k^\nu \)

- degrees of freedom reduced by the explicit truncation of expansion!
- 14 fields left

Israel-Stewart theory: \textit{14-moment approximation}

\[
f_k = f_{0k} + f_{0k} (1 - \alpha f_{0k}) \phi_k
\]

\begin{align*}
\text{equilibrium} & : \quad u_{\mu} N^\mu = n_0 \\
\text{non-equilibrium} & : \quad u_{\mu} T^{\mu\nu} = \varepsilon_0 u^\nu
\end{align*}

2 - Expansion coefficients mapped to conserved currents via \textit{matching conditions}

\begin{align*}
\pi^{\mu\nu} & = \Delta_{\alpha\beta}^{\mu\nu} T^{\alpha\beta} \\
n^{\mu} & = \Delta_{\alpha}^{\mu} N^\alpha \\
\Pi & = -\frac{1}{3} \Delta_{\mu\nu} T^{\mu\nu}
\end{align*}

Israel-Stewart theory: *14-moment approximation*

\[ f_k = f_{0k} + f_{0k} (1 - a f_{0k}) \phi_k \]

**Equilibrium**

**Non-equilibrium**

3 – Equations of motion taken from the *second moment* of the Boltzmann equation

\[ \Delta_{\mu\nu}^{\lambda\rho} \left( \partial_\alpha \int_K k^\alpha k^\mu k^\nu f_k = \int_K C[f] k^\mu k^\nu \right) \]

\[ u_\nu \Delta_{\mu}^{\lambda} \left( \partial_\alpha \int_K k^\alpha k^\mu k^\nu f_k = \int_K C[f] k^\mu k^\nu \right) \]

\[ u_\mu u_\nu \left( \partial_\alpha \int_K k^\alpha k^\mu k^\nu f_k = \int_K C[f] k^\mu k^\nu \right) \]

shear

diffusion

bulk

Final Equations of motion

GSD et al, PRD 85, 114047 (2012)

\[
\dot{\Pi} = -\frac{\Pi}{\tau_\Pi} - \beta_\Pi \theta - \ell_\Pi n \partial \cdot n - \tau_\Pi n \cdot \dot{u} - \delta_\Pi \Pi \Pi \theta \\
- \lambda_\Pi n \cdot \nabla \alpha_0 + \lambda_\Pi \pi^{\mu \nu} \sigma_{\mu \nu}
\]

\[
\dot{n}^{(\mu)} = -\frac{n^{(\mu)}}{\tau_n} + \beta_n \nabla^{(\mu)} \alpha_0 - n_\nu \omega^{(\nu \mu)} - \delta_{nn} n^{(\mu) \theta} - \ell_{n \Pi} \nabla^{(\mu)} \Pi \\
+ \ell_{n \pi} \Delta^{(\mu \nu)} \partial_\lambda \pi^{(\lambda)}_\nu + \tau_{n \Pi} \Pi \Pi \Pi \dot{u}^{(\mu)} - \tau_{n \pi} \pi^{(\mu)}_\nu \dot{u}^{(\nu)} \\
- \lambda_{nn} n^{(\nu)} \sigma^{(\mu)}_\nu + \lambda_{n \Pi} \Pi \nabla^{(\mu)} \alpha_0 - \lambda_{n \pi} \pi^{(\mu \nu)} \nabla^{(\nu)} \alpha_0
\]

\[
\dot{\pi}^{(\mu \nu)} = -\frac{\pi^{(\mu \nu)}}{\tau_\pi} + 2\beta_\pi \sigma^{(\mu \nu)} + 2\pi^{(\mu \omega^{(\nu)} \alpha - \tau_{n \pi} n^{(\mu \dot{u}^{(\nu)}} \\
+ \ell_{n \pi} \nabla^{(\mu \cdot n}^{(\nu)} - \delta_{n \pi} \pi^{(\mu \nu) \theta - \tau_{n \pi} \pi^{(\mu \nu)} \partial_\alpha \sigma^{(\nu)}_\alpha} \\
+ \lambda_{n \pi} n^{(\mu \cdot \nabla^{(\nu)} \alpha_0 + \lambda_{n \Pi} \Pi \sigma^{(\mu \nu)}
\]

Many terms originally omitted by Israel and Stewart.

• Nontrivial assumption: application of matching conditions

• This step does not require proximity to eq.

• All previous steps can be applied assuming the form: $f_{0k} \left( \lambda, u_{\mu} k^{\mu} / \Lambda \right)$

\[
f_k = f_{0k} + \delta f_k
\]

- isotropic, non-equilibrium
- correction, anisotropic
Matching conditions

\[ f_k = f_{0k} \left( \lambda, u_\mu k^\mu / \Lambda \right) + \delta f_k \]

5 parameters – can be associated with velocity, energy density and particle density

\[
\begin{align*}
\lambda &= \lambda (n, \varepsilon) \\
\Lambda &= \Lambda (n, \varepsilon)
\end{align*}
\]

14-moment approx.: shear term only

\[ f_k = f_{0k} + \frac{1}{2I_{42}} \pi^{\mu\nu} k_\mu k_\nu \]

\[ I_{42} = \frac{1}{15} \int \frac{d^3 k}{(2\pi)^3 k^0} |k|^4 f_{0k} \]
Equations of motion: *ultrarelativistic gas of hard spheres*

We recover the usual equation for the shear stress:

\[
\frac{\dot{\pi}}{\pi} + \frac{\pi_{\mu\nu}}{\tau_\pi} - \frac{2I_{40}}{3I_{50}}\sigma_{\pi}^{\lambda\mu\nu\lambda} = 2\frac{\eta}{\tau_\pi}\sigma_{\mu\nu} - 2\sigma_{\lambda}^{\mu\nu}\lambda - \frac{4}{3}\pi_{\mu\nu}\theta
\]

Transport coefficients: *functional dependence on \( f_{0k} \)*

\[
\frac{1}{\tau_\pi} = \left(1 + 4\frac{P_0I_{40}}{n_0I_{50}}\right)\frac{1}{3\ell_{\text{mfp}}}
\]

\[
\eta = \frac{4I_{40}\varepsilon^2}{3n_0I_{50} + 12P_0I_{40}}\ell_{\text{mfp}}
\]

Thermodynamic integrals:

\[
I_{nq} = \frac{(-1)^q}{(2q + 1)!!} \int dKE_{k}^{n-2q} (\Delta^{\alpha\beta} k_{\alpha} k_{\beta})^{q} f_{0k}
\]
Equations of motion: ultrarelativistic gas of hard spheres

We recover the usual equation for the shear stress:

\[
\frac{\dot{\pi}}{\pi} + \frac{\pi_{\mu\nu}}{\tau_\pi} - \frac{2I_{40}}{3I_{50}} \sigma \pi \lambda^{\mu\pi\nu} = 2 \frac{\eta}{\tau_\pi} \sigma^{\mu\nu} - 2 \sigma^{\mu\pi\nu} \lambda^{\pi} \lambda - \frac{4}{3} \pi^{\mu\nu} \theta
\]

“Equilibrium” Transport coefficients:

\[
\tau_\pi = \frac{9}{5} \ell_{\text{mfp}}
\]

\[
\eta = \frac{6}{5} \frac{T}{\sigma}
\]

Coefficients derived by Israel-Stewart
Example of non-equilibrium state: “over-occupied” state

\[ f_{0k} = \lambda \Theta (\Lambda - E_k) \]

matching

\[ \Lambda = 4T \]

\[ \lambda \sim \exp(\alpha) \]
Equations of motion: ultrarelativistic gas of hard spheres

We recover the usual equation for the shear stress:

\[
\dot{\pi} \langle \mu \nu \rangle + \frac{\pi^{\mu \nu}}{\tau_{\pi}} - \frac{2I_{40}}{3I_{50}} \sigma \pi \chi \langle \mu \pi \chi \rangle = 2 \frac{\eta}{\tau_{\pi}} \sigma^{\mu \nu} - 2 \sigma \langle \mu \pi \nu \rangle \lambda - \frac{4}{3} \pi^{\mu \nu} \theta
\]

Over-occupied Transport coefficients:

\[
\begin{align*}
\tau_{\pi} &= \frac{18}{13} \ell_{\text{mfp}} \\
\eta &= \frac{84}{65} \frac{T}{\sigma_T}
\end{align*}
\]

- qualitatively the same
- appears to be slightly more viscous
Equations of motion: ultrarelativistic gas of hard spheres

We recover the usual equation for the shear stress:

\[
\dot{\pi}^\mu\nu + \frac{\pi^{\mu\nu}}{\tau_\pi} - \frac{2 I_{40}}{3 I_{50}} \sigma_\pi^\lambda \langle \mu \pi \nu \lambda \rangle = 2 \frac{\eta}{\tau_\pi} \sigma^{\mu\nu} - 2 \sigma_\lambda^\mu \pi^\lambda \nu \lambda - \frac{4}{3} \pi^{\mu\nu} \theta
\]

<table>
<thead>
<tr>
<th>$f_{0k}$</th>
<th>$\lambda \exp(-E_k/\Lambda)$</th>
<th>$\lambda \Theta (\Lambda - E_k)$</th>
<th>$\lambda \delta (E_k - \Lambda)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau_\pi$</td>
<td>$\frac{9}{5} \ell_{mfp}$</td>
<td>$\frac{18}{13} \ell_{mfp}$</td>
<td>$\frac{9}{7} \ell_{mfp}$</td>
</tr>
<tr>
<td>$\eta$</td>
<td>$\frac{6}{5} \frac{T}{\sigma_T}$</td>
<td>$\frac{84}{65} \frac{T}{\sigma_T}$</td>
<td>$\frac{9}{7} \frac{T}{\sigma_T}$</td>
</tr>
</tbody>
</table>

Coefficients do not change much with $f_{0k}$. Can we see this?
Boltzmann eq. + Bjorken flow: 
*ultrarelativistic gas of hard spheres*

Initial conditions, fixed energy
shear viscosity $\frac{\eta}{n} \approx 6$
Evolution of shear stress does not see this non-equilibrium effect

$\eta$ in equilibrium!
Conclusions

• The applicability of fluid-dynamical models of heavy ion collisions cannot be easily justified.

• The derivation of hydrodynamics using the method of moments is more general than previously considered: hydrodynamic equations can be obtained even far from equilibrium.