A complete set of splitting functions in nuclear matter to any order in opacity
A lot of credit for this work goes to my collaborators Matt Sievert and Boram Yoon

- Introduction and motivation
- Calculation of gluon emission off quarks beyond the soft approximation and to any order in opacity
- Calculation of all in-medium splitting functions / numerical evaluation
- Conclusions
In heavy ion collisions medium-induced energy loss and more generally in-medium parton showers are the cornerstone of high-$p_T$ physics.

The associated phenomena were dubbed jet quenching and established in a myriad of observables.

M. Gyulassy et al. (1992)
Full in medium splitting

- Full massless and massive in-medium splitting functions now available to first order in opacity
  
  G. Ovanesyan et al. (2011)

- SCET-based effective theories created to solve this problem
  
  F. Ringer et al. (2016)

Representative example

\[
\left. \left( \frac{dN_{\text{med}}}{dQ_0 d^2k_1} \right) \right|_{Q \to Q_g} = \frac{\alpha_s}{2\pi^2} C_F \int \frac{d\Delta z}{\lambda_g(z)} \int d^2q_1 \frac{1}{\sigma_{\text{el}}} \frac{d\sigma_{\text{med}}}{d^2q_1} \left\{ \left( \frac{1 + (1 - x)^2}{x} \right) \right. \\
\times \left. \left[ \frac{B_{\perp}}{B_{\perp}^2 + \nu^2} - \frac{C_{\perp}}{C_{\perp}^2 + \nu^2} \right] \left( 1 - \cos[(\Omega_1 - \Omega_2)\Delta z] \right) + \frac{C_{\perp}}{C_{\perp}^2 + \nu^2} \left( \frac{2}{A_{\perp}^2 + \nu^2} - \frac{A_{\perp}}{A_{\perp}^2 + \nu^2} \right) \right. \\
\left. \left. - \frac{B_{\perp}}{B_{\perp}^2 + \nu^2} \right) \left( 1 - \cos[(\Omega_1 - \Omega_3)\Delta z] \right) + \frac{B_{\perp}}{B_{\perp}^2 + \nu^2} \cdot \frac{C_{\perp}}{C_{\perp}^2 + \nu^2} \left( 1 - \cos[(\Omega_2 - \Omega_3)\Delta z] \right) \right. \\
\left. \left. + \frac{A_{\perp}}{A_{\perp}^2 + \nu^2} \cdot \left( \frac{D_{\perp}}{D_{\perp}^2 + \nu^2} - \frac{A_{\perp}}{A_{\perp}^2 + \nu^2} \right) \left( 1 - \cos[\Omega_4 \Delta z] \right) - \frac{A_{\perp}}{A_{\perp}^2 + \nu^2} \cdot \frac{D_{\perp}}{D_{\perp}^2 + \nu^2} \left( 1 - \cos[\Omega_5 \Delta z] \right) \right. \\
\left. \left. + \frac{1}{N_c^2} \frac{B_{\perp}}{B_{\perp}^2 + \nu^2} \left( \frac{A_{\perp}}{A_{\perp}^2 + \nu^2} - \frac{B_{\perp}}{B_{\perp}^2 + \nu^2} \right) \left( 1 - \cos[(\Omega_1 - \Omega_2)\Delta z] \right) \right. \\
\left. \left. + x^3 m^2 \left[ \frac{1}{B_{\perp}^2 + \nu^2} \cdot \left( \frac{1}{B_{\perp}^2 + \nu^2} - \frac{1}{C_{\perp}^2 + \nu^2} \right) \left( 1 - \cos[(\Omega_1 - \Omega_2)\Delta z] \right) + \ldots \right] \right\} \right. 
\]

- For the first time we were able to do is higher order and resummed calculations
  
  Z. Kang et al. (2015)
Higher order opacity corrections

- The very thick medium approaches – they simplify the problem to resum the interactions
  
  - Finite opacity approaches – builds expansion in the correlation between the scattering centers

Clearly the next logical steps are
  
  - Derive all in-medium splittings to any order in opacity with no approximations other than eikonal
  - Perform numerical evaluation to find the first correction to jet and heavy flavor observables

It was also done to higher orders in opacity (~9) in the soft gluon emission limit
Theoretical framework

- The theoretical framework is completely general – it is applicable for both cold nuclear matter and the QGP.
- This is achieved by isolating the medium in transport parameters and universal gluon-mediated interactions.

\[ \mathcal{L}_{\text{opac.}} = \mathcal{L}_{\text{QCD}} + \mathcal{L}_{\text{ext}}^q + \mathcal{L}_{\text{ext}}^g + \mathcal{L}_{\text{G.F.}} + \cdots \]

\[
v(q_{T}^2) \rightarrow \frac{-q_{\text{eff}}^2}{q_T^2 + \mu^2} \quad \frac{d\sigma^{el}}{dp} = \frac{1}{(2\pi)^2} \frac{C_F}{2N_c} [v(q_{T}^2)]^2
\]

In deep inelastic scattering (DIS) the lowest order processes involve prompt quark. Even at NLO the prompt gluon jet contribution is small.

- The limit we are interested in
- We neglect collisional energy losses

\[
\frac{1}{p_N^2} \ll l_f^+ \sim \lambda^+ \sim L^+
\]

\[ \mathcal{O}\left(\frac{1}{Q^2}\right) \]
**Lightcone wave functions and parton branchings**

**Example**

- The technique of lightcone wavefunctions

\[ \psi(x, k - xp) = \frac{1}{2p^+} \frac{1}{(p-k)^- - k^-} U_\sigma(p-k) \left[-g f_\lambda^*(k)\right] U_{\sigma'}(p) \]

\[ = \frac{g x (1-x)}{(k-xp)^2 + x^2 m^2} \left\{ \frac{2-x}{x \sqrt{1-x}} \left( \epsilon_\lambda^* \cdot (k - xp) \right) \left[ \frac{1}{\sigma'} + \frac{\lambda}{\sqrt{1-x}} \left( \epsilon_\lambda^* \cdot (k - xp) \right) \right] \right\}_{\sigma'} \]

\[ + \frac{i m x}{\sqrt{1-x}} \epsilon_\lambda^* \times \left[ \tau^a \right]_{\sigma'} \]  

Useful to express in Pauli matrixes

\[ \langle \psi(x, \kappa) \psi^*(x, \kappa') \rangle = \sum_{\lambda = \pm 1} \frac{1}{2} \text{tr} \left[ \psi(x, \kappa) \psi^*(x, \kappa') \right] \]

\[ = \frac{8 \pi \alpha_s (1-x)}{[\kappa_T^2 + x^2 m^2][\kappa'_T^2 + x^2 m^2]} \left[ (\kappa \cdot \kappa') \left[ 1 + (1-x)^2 \right] + x^4 m^2 \right] \]

Branchings depending on the intrinsic momentum of the splitting

**Certain advantages** – can provide in “one shot” both massive and massless splitting functions

**Have checked that results agree for massless and massive DGLAP**

\[ \frac{dN}{d^2 k \, dx \, d^2 p \, dp^+} \bigg|_{\sigma(x^0)} = \frac{\alpha_s c_F}{2 \pi^2} \frac{(k-xp)^2}{[(k-xp)^2 + x^2 m^2]^2} \left[ 1 + (1-x)^2 \right] + x^4 m^2 \]

\[ \times \left( p^+ \frac{dN_0}{d^2 p \, dp^+} \right) \]
Interaction in the amplitude and the conjugate amplitude (Direct or single Born diagrams)

- Propagators hide in the wavefunctions

\[
D_4 = \left[ \frac{-1}{2N_c C_F} \right] e^{i\Delta E^{-}(k-xp) - \Delta E^{-}(k-xp+xq)z^+} \psi(x, k-xp) \left[ 0 - e^{-i\Delta E^{-}(k-xp)z^+} \right] \times \left[ e^{i\Delta E^{-}(k-xp+xq)z^+} - e^{i\Delta E^{-}(k-xp+xq)x_0^+} \right] \psi^*(x, k-xp+xq),
\]

- Vitruallity changes enter the interference phases and are related to the propagators

\[
p^- - k^- - (p-k)^- = \Delta E^{-}(k-xp)
\]
Opacity expansion building blocks – virtual terms

- Interaction in the amplitude or the conjugate amplitude (Virtual or double Born diagrams)

Agree with the full splitting functions of
- G. Ovanesyan et al. (2011)
- F. Ringer et al. (2016)

And energy loss of
- M. Gyulassy et al. (2001)

- A more interesting diagram - Double born can contribute to virtuality changes

\[ V_8 = \left[ \frac{N_c}{2C_F} e^{i\Delta E^- (k-xp) - \Delta E^- (k-xp-q)} z^+ \right] \psi(x, k-xp) \left[ 0 - e^{-i\Delta E^- (k-xp)x_0^+} \right] \]

\[ \times \left[ e^{+i\Delta E^- (k-xp-q)} z^+ - e^{+i\Delta E^- (k-xp-q)x_0^+} \right] \psi^*(x, k-xp-q) \]
Parton branching to any order in opacity

- Treating color (one complication in QCD).

- Color is not entangled, homogeneous structure and multiplicative factors that can be algebraically treated.

- Finally, relative to the splitting vertex we classify the as
  - Initial/Initial, Initial/Final, Final/Initial and Final/Final.

\[
\begin{align*}
C_1 &= \frac{1}{N_c C_F} \text{tr}[t^b t^b t^a M^a] = C_0, \\
C_2 &= \frac{1}{N_c C_F} f^{abc} f^{cde} \text{tr}[t^a M^d] = -\frac{N_c}{C_F} C_0, \\
C_3 &= \frac{1}{N_c C_F} f^{abc} \text{tr}[t^b t^a M^c] = \frac{i N_c}{2 C_F} C_0, \\
C_4 &= \frac{1}{N_c C_F} \text{tr}[t^a t^b t^a M^a] = C_0, \\
C_5 &= \frac{1}{N_c C_F} \text{tr}[t^b t^a t^b M^a] = -\frac{1}{2 N_c C_F} C_0, \\
C_6 &= \frac{1}{N_c C_F} f^{abc} \text{tr}[t^a t^b M^c] = \frac{i N_c}{2 C_F} C_0.
\end{align*}
\]
Upper triangular structure. Suggests specific strategy how to solve it. Calculated: initial conditions, kernels, and wrote a Mathematica code to solve it.

Simplest kernel

\[ \mathcal{K}_9 = \left[ e^{-q \cdot \nabla_p} e^{+(z^+-x^+)} \partial_{x^+} + e^{+(z^+-y^+)} \partial_{y^+} \right] + \left[ -\frac{1}{2} \right] \left[ e^{+(z^+-x^+)} \partial_{x^+} + e^{+(z^+-y^+)} \partial_{y^+} \right] \]

Most complicated kernel

\[ \mathcal{K}_1 = \left[ e^{i[\Delta E-(k-xp+xq)-\Delta E-(k-xp)]} e^{i[\Delta E-(k'-xp)-\Delta E-(k'-xp+xq)]} \right] e^{-q \cdot \nabla_p} e^{+(z^+-x^+)} \partial_{x^+} + e^{+(z^+-y^+)} \partial_{y^+} \]

+ \left[ \frac{N_c}{C_F} e^{i[\Delta E-(k-xp-(1-x)q)-\Delta E-(k-xp)]} e^{i[\Delta E-(k'-xp)-\Delta E-(k'-xp-(1-x)q)]} \right] \times \left[ e^{-q \cdot \nabla_k} e^{-q \cdot \nabla_{k'}} e^{-q \cdot \nabla_p} e^{+(z^+-x^+)} \partial_{x^+} + e^{+(z^+-y^+)} \partial_{y^+} \right] + 8 \text{ more lines} \]
Explicit solution to second order in opacity

Present the first exact result to this order (including the ability to discuss broad or narrow sources)

\[
\begin{align*}
\left. \frac{dN}{d^2k dx d^2p dp^+} \right|_{\mathcal{O}(\chi^2)} &= \frac{C_F}{2(2\pi)^3(1-x)} \int_{x_0^+}^{R^+} \frac{dz_2^+}{z_2^+} \int_{x_0^+}^{x_2^+} \frac{dz_1^+}{z_1^+} \int \frac{d^2q_1}{\lambda^+} \frac{d^2q_2}{\lambda^+} \frac{d\sigma_{el}}{d^2q_1} \frac{d\sigma_{el}}{d^2q_2} \times \left\{ \left( p^+ \frac{dN_0}{d^2p dp^+} \right) N_1 \right. \\
&\left. + \left( p^+ \frac{dN_0}{d^2(p-q_1) dp^+} \right) N_2 + \left( p^+ \frac{dN_0}{d^2(p-q_2) dp^+} \right) N_3 + \left( p^+ \frac{dN_0}{d^2(p-q_1-q_2) dp^+} \right) N_4 \right\}
\end{align*}
\]

\[\mathcal{N}_1 = \]

\[
\left| \psi(k-xp) \right|^2 \left[ \frac{(C_F + N_c)^2}{C_F^2} - \frac{N_c(C_F + N_c)}{C_F^2} \cos(\delta z_1 \Delta E^-(k-xp)) + \frac{N_c^2}{2C_F^2} \cos(\delta z_2 \Delta E^-(k-xp)) \right.
\]

\[
\left. - \frac{N_c(2C_F + N_c)}{2C_F^2} \cos((\delta z_1 + \delta z_2) \Delta E^-(k-xp)) \right]
\]

For broad sources and in the soft gluon limit we have checked that the result reduces to the GLV second order in opacity
Note – all splittings have the same topology.

Same - structure, interference phases, propagators

Different - mass dependence, wavefunctions, color (which also affects transport coefficients)

\[
\langle \psi(x, \kappa) \psi^*(x, \kappa') \rangle = \frac{8\pi\alpha_s}{[\kappa_T^2 + \nu^2m^2][\kappa_T'^2 + \nu^2m^2]} \left[ g(x) (\kappa \cdot \kappa') + \nu^4m^2 \right]
\]

\[
\Delta E^- (\kappa) = -\frac{\kappa_T^2 + \nu^2m^2}{2x(1-x)p^+}
\]

Master table that gives all ingredients

| \( G/q \) | \( q/q \) | \( q/G \) | \( G/G \) | \( d_1 \) | \( d_2 \) | \( d_3 \) | \( d_4 \) | \( d_5 \) | \( d_6 \) | \( v_1 \) | \( v_2 \) | \( v_3 \) | \( v_4 \) | \( \lambda_R^+ \) | \( \lambda_q^+ \) | \( \lambda_G^+ \) | \( \nu \) | \( f(x) \) | \( g(x) \) |
| 1 | 1 | 1 | 1 | \( N_c \) | \( C_F \) | \( 1 \) | \( 1/2 \) | \( 1/2 \) | \( -1/2 \) | \( -1 \) | \( -2 \) | \( -2 \) | \( 2 \) | \( 2 \) | \( 2N_c^2 \) | \( 1 \) | \( x(1-x) \) | \( x^2 + (1-x)^2 \) |
| 1 | 1 | 1 | \( C_F \) | \( 2N_cC_F \) | \( 2N_c \) | \( 2 \) | \( 2N_c^2 \) | \( 2 \) | \( -2N_c^2 \) | \( -1 \) | \( -2 \) | \( 2 \) | \( 2 \) | \( 2N_c^2 \) | \( 1 \) | \( x \) | \( 1 + x^2 \) |
| 1 | 1 | \( 1/2 \) | \( 1/2 \) | \( 1/2 \) | \( 1/2 \) | \( 1/2 \) | \( 1/2 \) | \( 1/2 \) | \( 1/2 \) | \( 1/2 \) | \( 1/2 \) | \( 1/2 \) | \( 1/2 \) | \( 1/2 \) | \( 1/2 \) | \( 1 \) | \( 1 + x^4 + (1-x)^4 \) | 1 |

We have now solved the problem for all splitting functions

M. Sievert et al. (2019)
Evolution of $T$ in $\text{Au+Au}$

- Code is **restructured** (in C++) and shortened ($24K \rightarrow 8K$ lines). **20x speed improvement**

**Effective incorporation of simulated QGP medium**

- Reduced overhead for calling QGP medium grid function. **2x speed improvement**

**Efficient on-node parallelization**

- New parallelization shows much better scaling. **10x speed improvement**

**Overall improvement:**

18 days $\rightarrow$ 1 hour

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C. Shen et al. (2014)
Medium-induced splitting intensity

Porting to code

- Results are directly exported from Mathematica to C++

Challenges

- Arise from larger number of evaluations

\[ \mathcal{I}^{x_{\text{max}}}_{x_{\text{min}}} = \int_{x_{\text{min}}}^{x_{\text{max}}} \, dx \int d^2 k \, x \frac{dN}{d^2 k \, dx} \]

Energy loss – not a well defined concept for parton shower processes - define splitting intensity

- The main result is a change in the energy dependence of the splitting intensity – smoother, or more slowly varying with E (understand jet modification with p_T)
Reduction of small-\(x\) and large-\(x\) probabilities (asymptotics modulated by thermal mass)

Enhancement of democratic branching (\(x \sim 0.5\))

- Broder angular enhancement region
- Oscillating series – the average of 1\(^{\text{st}}\) and 1\(^{\text{st}} + 2\(^{\text{nd}}\) order- candidate for pheno.
Application of the new formalism & code to jets

- First application to inclusive heavy flavor jets - using the technique of semi-inclusive jet functions
- Application to jet fragmentation – inclusive and photon tagged jets – captures the qualitative differences needs improvement (higher orders)

Note: new code but to 1st order in opacity

H. Li et al. (2018)

M. Aaboud et al. (2019)
Conclusions

- Consistent progress over the past 2 decades, from energy loss to in medium parton showers to first order in opacity.

- For the first time consistent NLO and resummed calculations in the QGP environment - same language, techniques as in HEP.

- Calculated the quark splitting function to any order in opacity and presented explicit results to 2\textsuperscript{nd} order. Cross checked the result against the known soft gluon emission limit.

- We have now extended the derivation to all orders in opacity for all 4 splitting functions. Demonstrated numerical evaluation in realistic media.

- Applications of new code/grids to jet physics problems, applications to second order in opacity in the near future.
Motivation for numerical evaluation

Approaches to the theoretical problem

- The very thick medium approaches – they simplify the problem to resum the interactions
- Finite opacity approaches – builds expansion in the correlation between the scattering centers

Realistic QGP - q ~ 2.5 scatterings, g ~5. Also problems with BDMPS not recovering known asymptotic limits

- Any order in opacity result

\[ \frac{dN^{(n)}}{dx d^2k} = \frac{C_R \alpha_s}{\pi^2} \frac{1}{n!} \left( \frac{L}{\lambda_g(1)} \right)^n \prod_{i=1}^{n} \left\{ dq_i \left( \frac{\lambda_g(1)}{\lambda_g(i)} \right) \left( \frac{\bar{\nu}^2(q_i) - \delta(q_i)}{\delta(q_i)} \right) \right\} \]

Medium properties

propagators

Interference phases
Numerical implementation

- Numerics can be challenging due to lengthy equations and multi-dimensional integration.
- Implementation for the case of QGP (simplified Bjorken expansion).
  Lashoff-Regas et al. (2014)
- Still, code needed 3 days for a set of splittings.

**iEBE-VISHNU package**
- Hydro + hadron cascade simulator for relativistic heavy-ion collisions
- Developed by Chun Shen and collaborators