Continuation: QCD + QED∞ studies

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RBC-UKQCD Collaborations

2019 USQCD All-Hands Collaboration Meeting

BNL - Physics Department

April 26-27, 2019

Outline

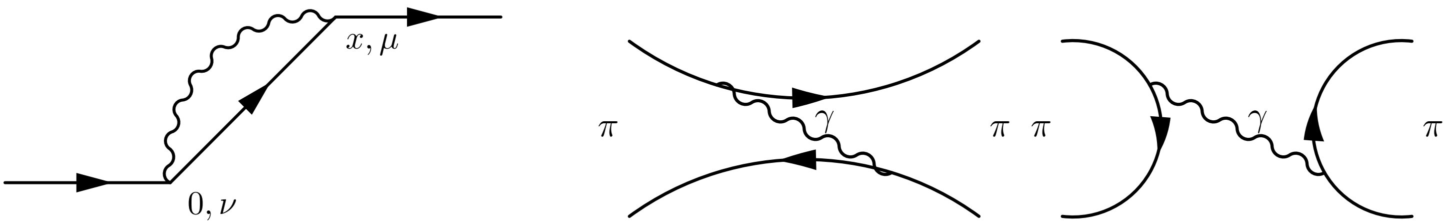
- **Physical goals and previous results**
- Lattice QED and $\mathrm{QED_L}$
- QED_∞

  (no power-law finite volume effects for HVP, HLbL, $\pi^0 \to e^+e^-$, etc)
- The infinite volume reconstruction method

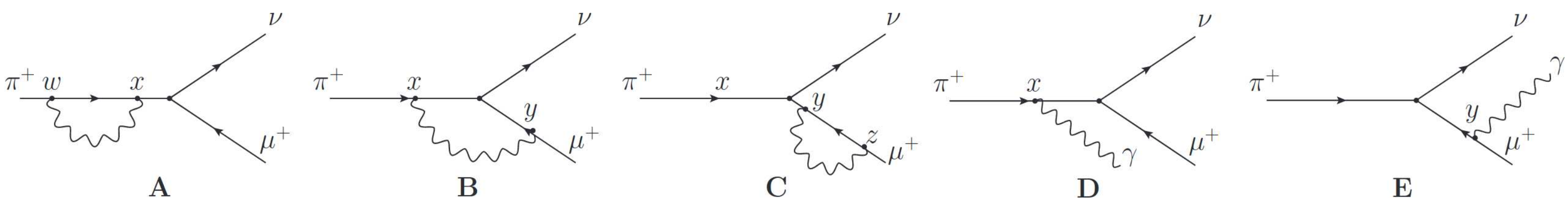
  (no power-law finite volume effects for mass, leptonic decay amplitudes, etc)
- Summary and Outlook

Physical Goals

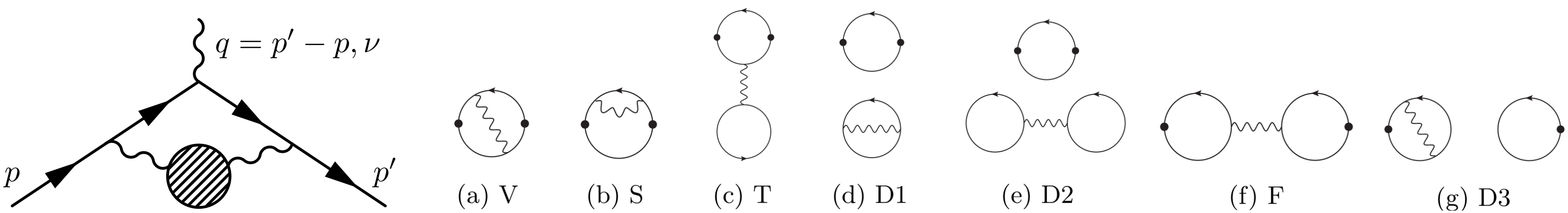
- QED corrections to mesonic masses, in particular the pion mass splitting.



- QED corrections to the pion decay constant including control of the associated infrared divergences.



- QED corrections to the hadronic vacuum polarization contribution to the muon $g-2$ including isospin corrections needed to include τ decay data.



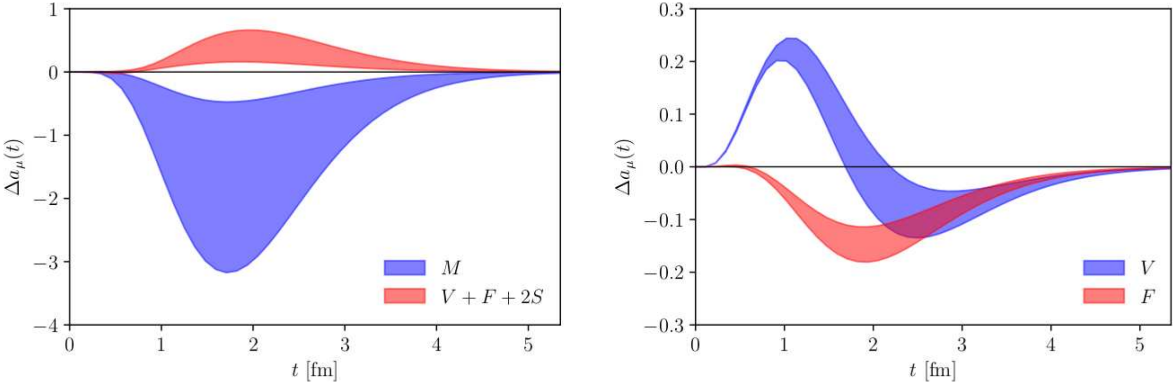
- QED corrections to HVP (**C. Lehner** et al, **Phys.Rev.Lett. 121, no. 2, 022003 (2018)**)

$$a_\mu^{\text{udsc,isospin}} = 705.9(14.6)_S(2.9)_C(3.7)_V(1.8)_A(0.4)_Z(2.3)_L \times 10^{-10}$$
$$a_\mu^{\text{QED,SIB}} = 9.5(7.4)_S(0.7)_C(6.9)_V(1.7)_E(1.3)_{E48} \times 10^{-10}$$
$$a_\mu = 715.4(16.3)_S(3.0)_C(7.8)_V(1.9)_A(0.4)_Z(1.7)_E(2.3)_L(1.5)_{E48} \times 10^{-10}$$

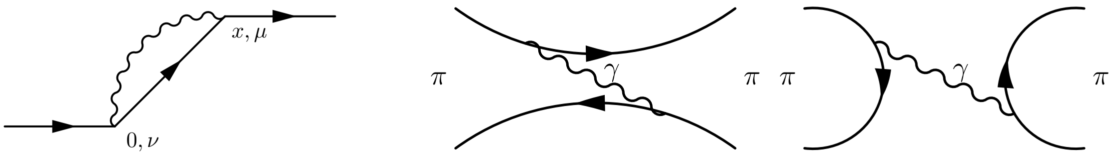
S(statistics), C(discretization), V(finite volume), A(lattice spacing), Z(renormalization), E(energy of $\pi\gamma$), L(long distance for quark disconnected diagram), E48(energy of $\pi\pi$ for 48I)

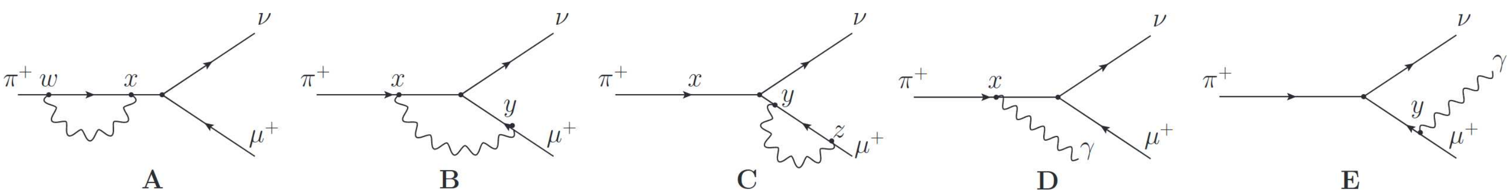
QED,SIB error is small compare with leading contribution error. But with many recent algorithmic improvements for $a_\mu^{\text{udsc,isospin}}$ (to be published), reducing uncertainty for $a_\mu^{\text{QED,SIB}}$ is also important (on-going).

- Hadronic τ decay study (**M. Bruno** et al, **PoS LATTICE 2018, 135 (2018)**)

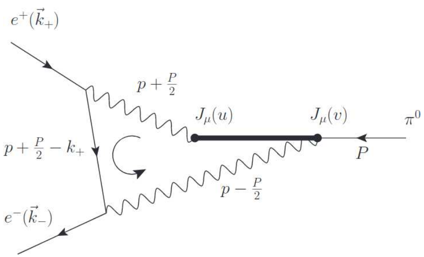


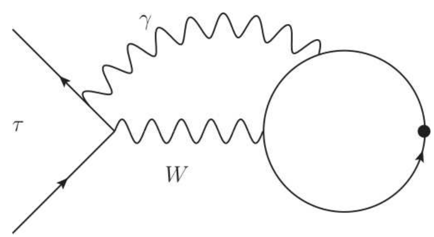
- **Infinite volume reconstruction method: Feng and Jin (2018)** apply to mass corrections and charged meson decays.





- $\pi^0 \to e^+e^-$ as the first step towards $K_L \to \mu^+\mu^-$ (**Yidi Zhao, Norman Christ, etc**).
- Initial-state radiation contributions to hadronic τ decays (**Mattia Bruno, etc**).





- Continue QED corrections to the hadronic vacuum polarization contribution to the muon $g-2$.
- Request 12.2 Mio KNL core-hours on the BNL (preferred) or JLAB KNLs.

  Plus 50 TB temporary disk space or 0.375 Mio Sky-core-hours.

- Physical goals and previous results
- **Lattice QED and $\mathrm{QED_L}$**
- QED_∞

  (no power-law finite volume effects for HVP, HLbL, $\pi^0 \to e^+e^-$, etc)
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Compare with QCD, lattice QED has two major new features:

- QED interaction has a small coupling constant and can be treated perturbatively:

  i.e. its contribution can be calculated order by order in $\alpha_{\mathrm{QED}} = 1/137$.

  At present, usually the first non-trivial order is good enough.

$$S^{\gamma}_{\mu,\nu}(x-y) \;=\; \int \frac{d^4k}{(2\pi)^4}\frac{\delta_{\mu,\nu}}{k^2}e^{ik\cdot(x-y)} = \frac{\delta_{\mu,\nu}}{4\pi^2|x-y|^2} \tag{1}$$

- QED introduces a zero mass particle: photon, while pure QCD has no massless particle.

  This leads to various treatments and different finite volume effects.

  → $\mathrm{QED_{TL}}$: Duncan, Eichten, and Thacker (1996). S. Borsanyi et al (2015).

  → $\mathrm{QED_{L}}$: Hayakawa and Uno (2008).

  → Massive photon: M. G. Endres et al (2016).

  → C^* boundary condition: B. Lucini et al (2016).

  → $\mathrm{QED_{\infty}}$ and the Infinite volume reconstruction method: **Feng and Jin (2018)**.

Infinite continuum propagator:

$$S^{\gamma}_{\mu,\nu}(x-y) \;=\; \int \frac{d^4k}{(2\pi)^4}\,\frac{\delta_{\mu,\nu}}{k^2}\,e^{i\,k\cdot(x-y)} \tag{2}$$

In finite volume, we shall have:

$$S^{\gamma}_{\mu,\nu}(x-y) \;=?\; \frac{1}{V}\sum_{k}\,\frac{\delta_{\mu,\nu}}{k^2}\,e^{i\,k\cdot(x-y)} \tag{3}$$

There is a singularity when $k=0$! Not a problem in the infinite volume, but in the finite volume, we need some special treatment to avoid this straight forward divergence.

Uniform treatment for all time slice is required if proper transfer matrix is desired.

$\Rightarrow$ remove all $\vec{k}=0$ modes $\mathrm{QED_L}$ scheme (**Hayakawa and Uno (2008)**)

$$\begin{aligned} S^{\gamma,L}_{\mu,\nu}(x-y) \;&=\; \frac{1}{L^3T}\sum_{k\,(\vec{k}\neq 0)}\,\frac{\delta_{\mu,\nu}}{k^2}\,e^{i\,k\cdot(x-y)} & (4)\\ &=\; \frac{1}{L^3}\sum_{\vec{k}\,(\vec{k}\neq 0)}\,\frac{\delta_{\mu,\nu}}{2|\vec{k}|}\,e^{i\,\vec{k}\cdot(\vec{x}-\vec{y})}\,e^{-|\vec{k}||x_t-y_t|} & (5)\end{aligned}$$

- This treatment – simply remove some certain modes in the integral – usually leads to power-law suppressed finite volume errors: $\mathcal{O}(1/L^n)$.
- Here n depends on problem, and usually the first 1 or 2 terms are universal in the sense that it only depends on total charge of the particle.
- e.g. QED corrections to particle mass. S. Borsanyi et al (2015)

$$\begin{aligned} \Delta m(L) &= \Delta m(\infty) - \frac{q^2}{4\pi}\frac{\kappa}{2L}\left(1 + \frac{2}{mL}\right) \\ \kappa &= 2.837297\cdots \end{aligned}$$

- Other formulations suffer similar power-law suppressed finite volume errors:
  - → Massive photon: M. G. Endres et al (2016).
  - → C^* boundary condition: B. Lucini et al (2016).
- The problem of the power-law suppressed finite volume error is that:

  The error does not decrease much when the size of the volume is increased.
- So QED_∞?

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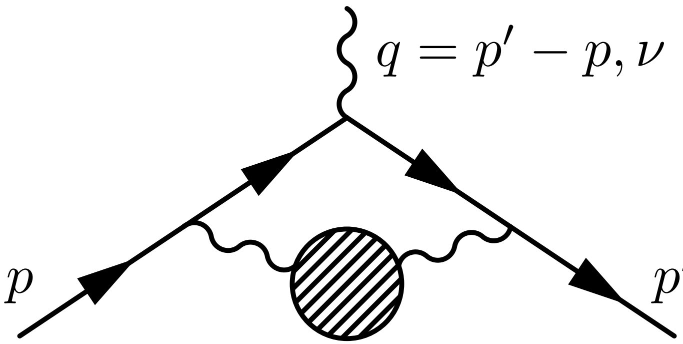
- Muon anomalous magnetic moment - hadronic vacuum polarization contribution



- The non-hadronic part – including photon and muon propagators – can be calculated in the infinite volume and continuum! **T. Blum (2003)**

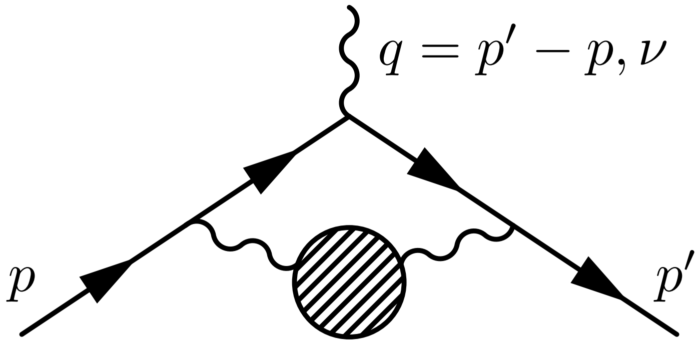
$$a_\mu^{\mathrm{HVP}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty f(K^2)\, \hat{\Pi}(K^2)\, dK^2$$

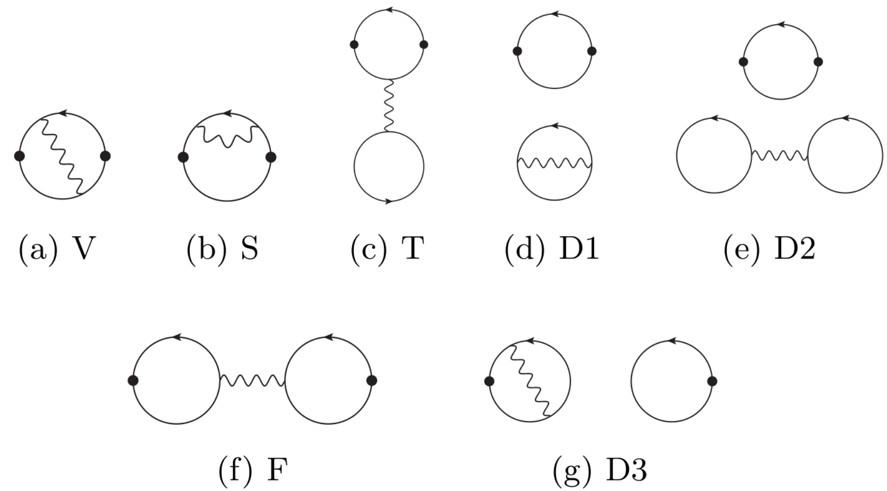
$$\Pi_{\mu,\nu}(q) = i \int d^4x\, e^{i q\cdot(x-y)} \langle J_\mu(x)\, J_\nu(y)\rangle = (q_\mu q_\nu - q^2\, \delta_{\mu,\nu})\hat{\Pi}(q^2)$$

- Convert to coordinate space. One obtain an explicit formula with only exponentially suppressed finite volume error. **Bernecker and Meyer (2012)**

$$a_\mu^{\mathrm{HVP}} = \left(\frac{\alpha}{\pi}\right)^2 \int W(x-y)\, \langle J_\mu(x)\, J_\mu(y)\rangle\, d^4x$$

- Muon anomalous magnetic moment - hadronic vacuum polarization contribution



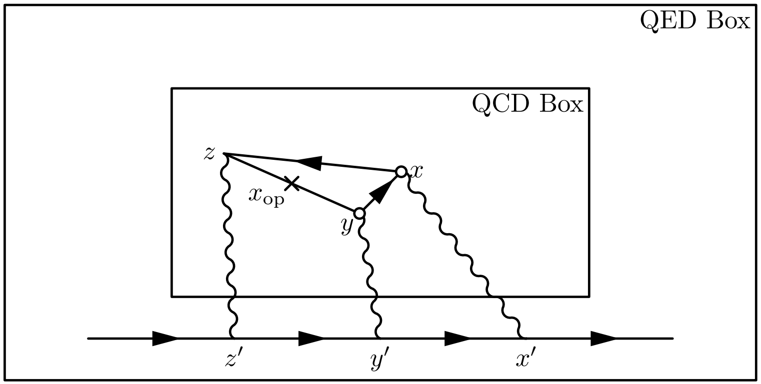


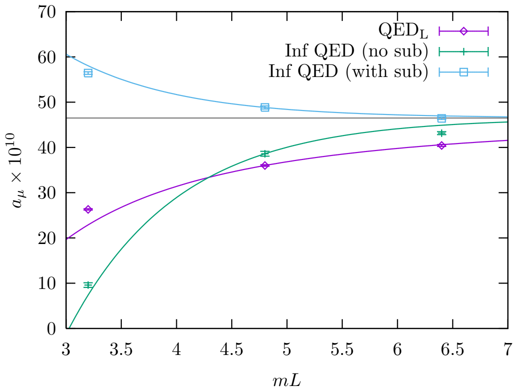
- QED (and SIB) corrections to the hadronic blob are need to be included to achieve the desired precision.
- QED$_\infty$: One can use the (continuum) infinite volume photon propagator in the calculation. Then, the finite volume error is exponentially suppressed! **C. Lehner et al (2018)**

$$G_{\mu,\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{\delta_{\mu,\nu}}{k^2} e^{ik\cdot(x-y)} = \frac{\delta_{\mu,\nu}}{4\pi^2|x-y|^2} \tag{6}$$

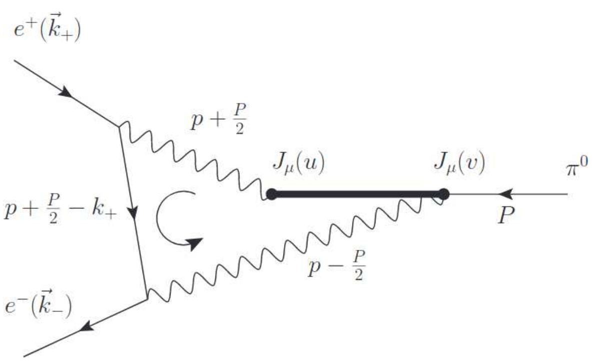
- We have V,S,F under full control and preliminary numbers for T and D1. At this point it is premature to say that we will have a first determination of the disconnected QED diagrams, but we are hopeful.

- Treat the external photon with the moment method and evaluate the photon and muon propagator in the infinite volume. **N. Asmussen et al (2016) T. Blum et al (2017)**

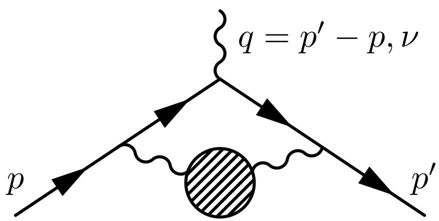




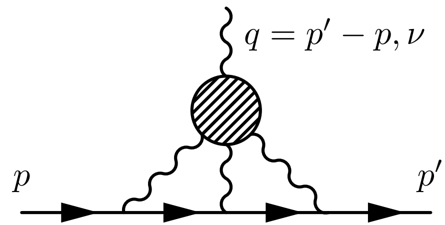
- Finite volume error for QED$_{\rm L}$ in a finite box is $\mathcal{O}(1/L^2)$, while for QED$_\infty$, the error is exponentially suppressed by the lattice size.
- $\pi^0 \to e^+e^-$ as the first step towards $K_L \to \mu^+\mu^-$ (**Yidi Zhao, Norman Christ, etc**).



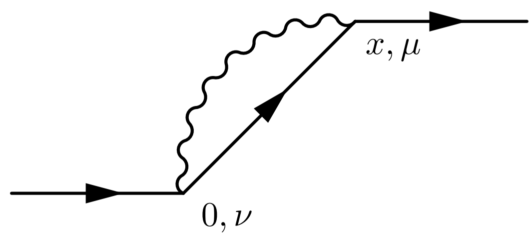
- Write down the problem of interest in the infinite volume in coordinate space.
- Evaluate the non-hadronic part in the (continuum) infinite volume.
- Express the result as an integral over the vertices of the hadronic part that are connecting to the non-hadronic part.
- Since the hadronic part vanishes exponentially at long distance (QCD mass gap), the integral only suffers exponentially suppressed finite volume error if evaluated in a finite volume.



HVP



HLbL



- Not applicable to problems involving hadron initial/final state! e.g. QED corrections to hadron masses.

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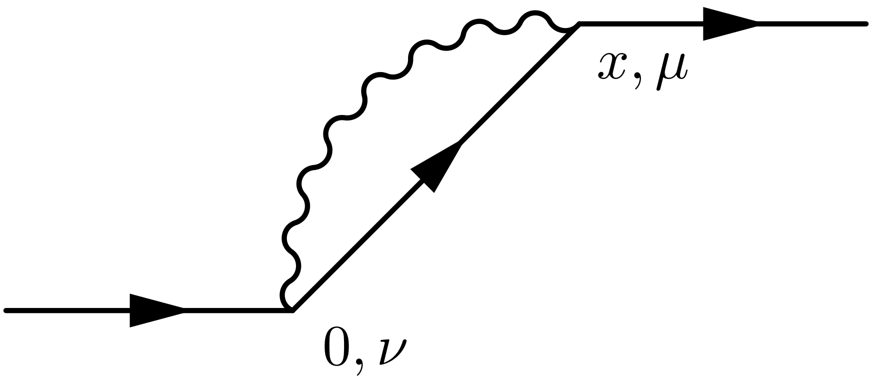
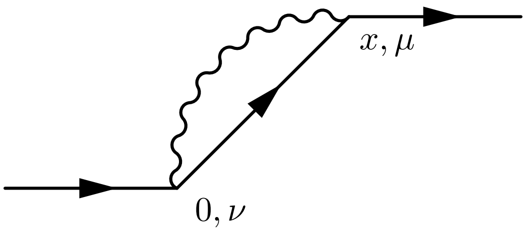
The infinite volume reconstruction method

- Write down the problem of interest in the infinite volume in coordinate space.
- Evaluate the non-hadronic part in the (continuum) infinite volume.
- Reconstruct the hadronic part – also in continuum infinite volume – from some correlation functions that can be calculated in finite volume.
- If the reconstruction step and the calculation step are both absence of power-law suppressed finite volume effects, the final result will be as well.





- Write down the problem of interest in the infinite volume: QED correction to hadron mass:

$$\Delta M \ = \ \mathcal{I} = \frac{1}{2}\int d^4x\, \mathcal{H}_{\mu,\nu}(x)\, S^{\gamma}_{\mu,\nu}(x)$$

$$\mathcal{H}_{\mu,\nu}(x) \ = \ \frac{1}{2\,M}\langle N|T J_\mu(x) J_\nu(0)|N\rangle$$

  Note that if disconnected diagrams are included, the results is not quenched!

- Evaluate the non-hadronic part in the (continuum) infinite volume.

$$S^{\gamma}_{\mu,\nu}(x) \ = \ \frac{\delta_{\mu,\nu}}{4\pi^2|x-y|^2}$$

- Reconstruct the hadronic part, $\mathcal{H}_{\mu,\nu}(x)$ – also in continuum infinite volume – from some correlation functions that can be calculated in finite volume.

The infinite volume reconstruction method

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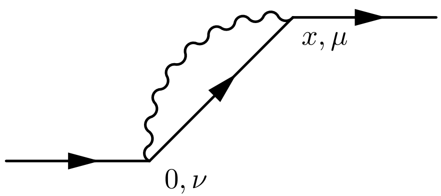


$$\mathcal{H}_{\mu,\nu}(x) = \frac{1}{2M}\langle N|TJ_\mu(x)J_\nu(0)|N\rangle$$

We consider two situations (assuming $x_t \geqslant 0$ and introducing $t_s \lesssim L$):

- $x_t \leqslant t_s$: $\mathcal{H}_{\mu,\nu}(x)$ can be evaluated directly in a finite volume lattice.
- $x_t > t_s$: The intermediate states between the two currents are mostly $N(\vec{p})$ state:

$$\mathcal{H}_{\mu,\nu}(x) \approx \int \frac{d^3p}{(2\pi)^3}\left[\frac{1}{2E_{\vec{p}}}\frac{1}{2M}\langle N|J_\mu(0)|N(\vec{p})\rangle\langle N(\vec{p})|J_\nu(0)|N\rangle\right]e^{i\vec{p}\cdot\vec{x}-(E_{\vec{p}}-M)t}$$

→ We only need to calculate the form factors: $\langle N(\vec{p})|J_\nu(0)|N\rangle$!

→ But information from all the $\vec{p}$ is needed.

$$\int d^3x\, \mathcal{H}_{\mu,\nu}(t_s,\vec{x})e^{-i\vec{p}\cdot\vec{x}+(E_{\vec{p}}-M)t_s} \approx \frac{1}{2E_{\vec{p}}}\frac{1}{2M}\langle N|J_\mu(0)|N(\vec{p})\rangle\langle N(\vec{p})|J_\nu(0)|N\rangle$$

The infinite volume reconstruction method

- The final expression for QED correction to hadron mass is split into two parts:

$$\Delta M \;=\; \mathcal{I} = \mathcal{I}^{(s,L)} + \mathcal{I}^{(l,L)}$$

- The short distance part:

$$\mathcal{I}^{(s,L)} \;=\; \frac{1}{2}\int_{-t_s}^{t_s} dt \int_{-L/2}^{L/2} d^3x\, \mathcal{H}^L_{\mu,\nu}(x)\, S^\gamma_{\mu,\nu}(x)$$

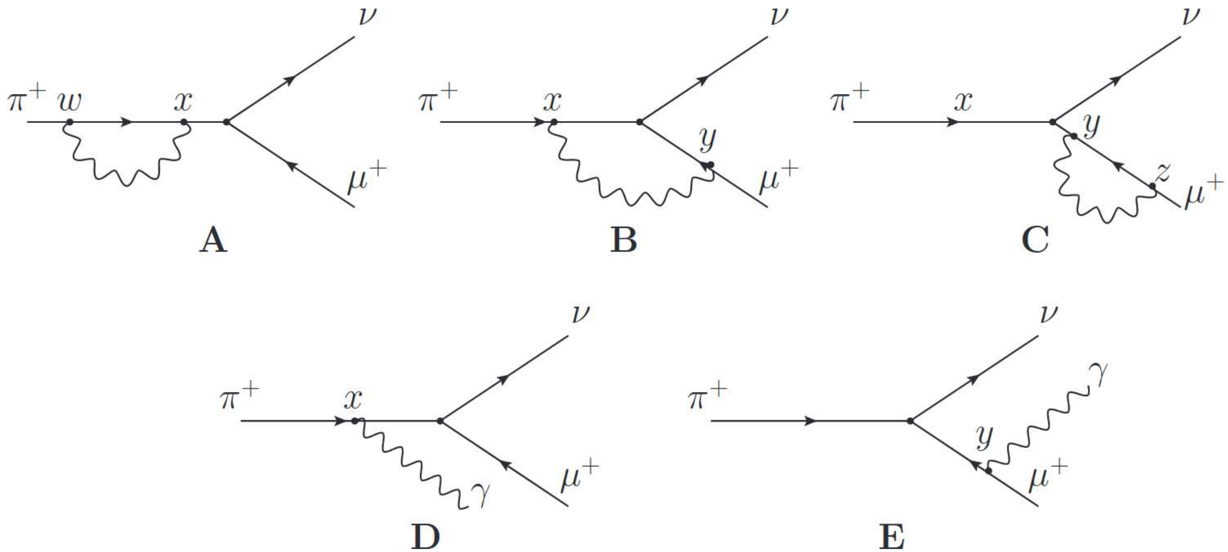
- The long distance part:

$$\mathcal{I}^{(l,L)} \;=\; \int_{-L/2}^{L/2} d^3x\, \mathcal{H}^L_{\mu,\nu}(t_s,\vec{x})\, L_{\mu,\nu}(t_s,\vec{x})$$

- For Feynman gauge:

$$S^\gamma_{\mu,\nu}(x) \;=\; \frac{\delta_{\mu,\nu}}{4\pi^2 x^2}$$

$$L_{\mu,\nu}(t_s,\vec{x}) \;=\; \frac{\delta_{\mu,\nu}}{2\pi^2}\int_0^\infty dp\, \frac{\sin(p|\vec{x}|)}{2(p+E_p-M)|\vec{x}|}\, e^{-pt_s}$$



- Diagram D:

  → $-t_s \leqslant x_t \leqslant t_s$: directly evaluate on the lattice.

  → $x_t \leqslant -t_s$: can be resconstructed with the lattice data at $x_t = -t_s$.

  The final result should be expressed as a multiple sum over pairs of gauge configurations together with the positions of the two electromagnetic currents.

- The beauty of the present approach is that IR divergence comes from those parts of the calculation which are performed analytically.

- Therefore, these IR divergence shall cancel analytically as it always does.

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The general idea & comparison

The guideline of QED_L, Massive photon, and C^* boundary condition, etc are:

- Formulate a consistant, finite volume, QED-like, field theory in the discrete lattice.
- Find connections between the result calculated in the discrete finite volume box and the needed continuum infinite volume result.
- Usually the difference is suppressed by some certain power of the size of the box.

  $\rightarrow$ Power-law suppressed finite volume errors.

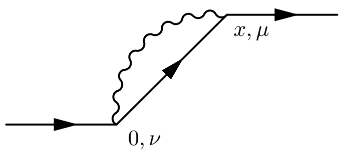
If we are not interested in the non-perturbative aspect of QED, we don't have to apply a non-perturbative treatment of the QED.

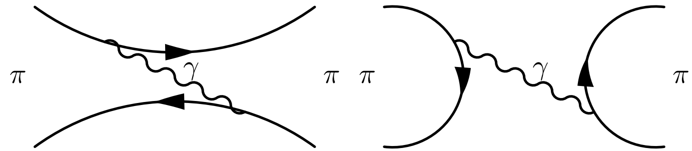
Instead, we are interested in calculating the hadronic correlation function, that if combined with QED part, can give the most accurate final result.

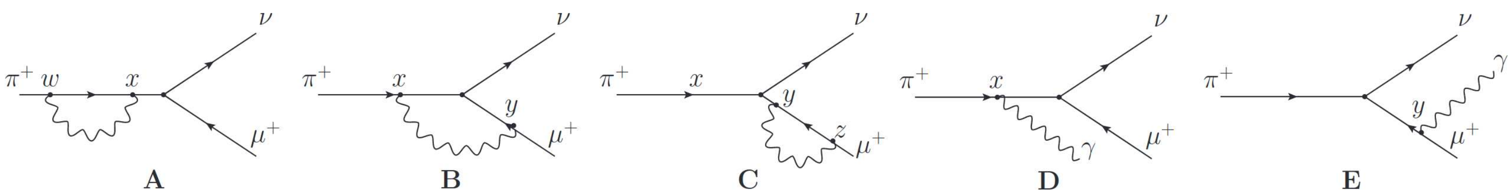
- Write down the problem of interest in the infinite volume in coordinate space.
- Evaluate the expression in the (continuum) infinite volume (semi-)analytically as much as possible.
- Hopefully, the remaining hadronic correlation function can be calculated directly,

  or can be reconstructed from some different correlation function that can be calculated in some modest size lattice with only exponentially suppressed errors.

- **Infinite volume reconstruction method: Feng and Jin (2018)** apply to mass corrections and charged meson decays.







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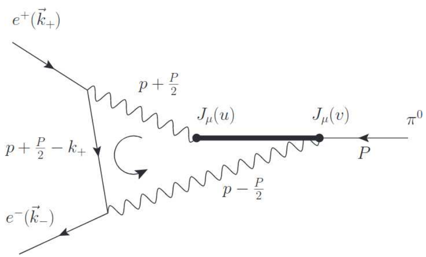

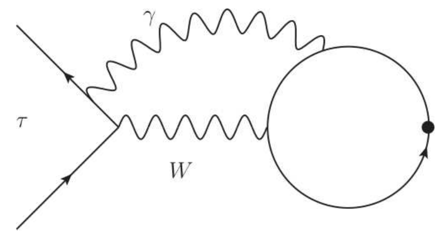

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Thank You!

- Reconstruct the hadronic part, $\mathcal{H}_{\mu,\nu}(x)$ – also in continuum infinite volume – to some correlation functions that can be calculated in finite volume.

Consider the pion correlation function:

$$\langle \bar{u}\gamma_5 d(x)\, \bar{d}\gamma_5 u(y)\rangle$$

- At large $|x-y|$, we shall have:

$$\langle \bar{u}\gamma_5 d(x)\, \bar{d}\gamma_5 u(y)\rangle \;=\; Z_\pi \frac{m_\pi\, \mathrm{BesselK}(1, m_\pi|x-y|)}{4\,\pi^2|x-y|} \tag{7}$$

- Reconstruct step: The correlation function at large $|x-y|$ can be reconstructed by two constants: Z_π and m_π, upto exponentially suppressed errors.
- Calculation step: These two constant can be calculated in a modest volume with only exponentially suppressed finite volume errors.
- Both the reconstruction step and the calculation step are absence of power-law suppressed finite volume effects!

(a) Please comment on the ensemble sizes used (30 configs), and whether this is anticipated to be sufficient, particularly at the smallest volume taking account of any volume averaging.

Our measurement strategy, in more detail explained in the 2018 proposal, is to sample a fixed number of vertices with an importance-sampling technique similar to our g-2 HLbL work. Since we keep this number fixed per configuration (and small compared to the total volume), we sample approximately the same effective physical volume on all ensembles and we expect (also from $\mathrm{QED_L}$ experience) a similar statistical noise on all ensembles. From $\mathrm{QED_L}$ experience, 30 configurations are sufficient for a below 10% precision of QED corrections. Having fewer than 30 configurations makes a statistical analysis with reasonably controlled error estimates difficult.

(b) The calculation of the real part of pi -> e+ e- sounds intriguing. Please describe this calculation in more detail, and outline the expected uncertainties in the calculation.

The proposed calculation of the $\pi^0 \to e^+e^-$ involves an interesting extension of the lattice QCD methods that have been used to compute the hadronic light-by-light (HLbL) contribution to $g_\mu - 2$. The QED portion of the proposed calculation involves only one loop instead of the two-loops needed in the HLbL calculation. However, since this is a decay process it involves time-like momenta and Minkowski-/Euclidean-space issues that can be avoided in the HLbL calculation.

In fact, because of the light mass of the pion it is possible to write an exact expression for the complex $\pi^0 \to e^+e^-$ decay amplitude as a product of a known complex function coming from the photon-photon-electron loop and a hadronic amplitude that can be computed using lattice QCD. The resulting expression gives the exact, $O(\alpha_{\rm EM}^2)$, $\pi^0 \to e^+e^-$ decay amplitude with finite volume corrections that arise because the lattice QCD amplitude is computed in finite volume. However, these finite-volume corrections decrease exponentially in the size of the QCD box. This approach is planned as the topic of two talks at Lattice 2019 by N. Christ and Y. Zhao.

Cost budget

12 sloppy 48I solves on 32 KNL nodes	380 seconds
12 exact 48I solves on 32 KNL nodes	1630 seconds
12 sloppy 48D solves on 32 KNL nodes	250 seconds
12 exact 48D solves on 32 KNL nodes	1100 seconds
12 sloppy 32D solves on 32 KNL nodes	75 seconds
12 exact 32D solves on 32 KNL nodes	330 seconds
12 sloppy 24D solves on 32 KNL nodes	31 seconds
12 exact 24D solves on 32 KNL nodes	140 seconds
Number of configurations per ensemble	30
Number of sloppy solves per configuration	900×12
Number of exact solves per configuration	15×12
Total computational cost on 48I in Mio KNL core hours	6.3
Total computational cost on 48D in Mio KNL core hours	4.1
Total computational cost on 32D in Mio KNL core hours	1.2
Total computational cost on 24D in Mio KNL core hours	0.6
Total request	12.2 Mio KNL core hours

Table 1: Cost estimates for the proposed computation. We intend to use an AMA [16] setup with parameters described in this table.