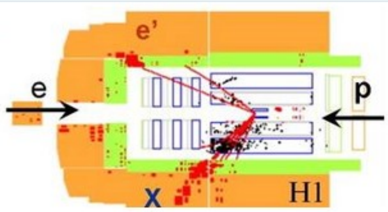
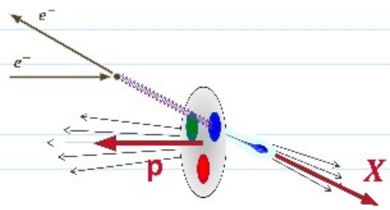
L1.1 – Deep Inelastic Scattering

The way to study the structure of a complex object (proton) is to hit it with a simple object (electron)

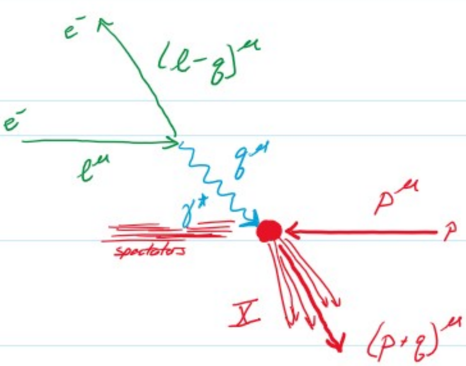
Deep → Shatters the proton
Inelastic → High recoil
Scattering

$$\boxed{\text{Inclusive DIS: } \quad e^- + p \to e^- + X}$$





The nice thing about inclusive DIS is that you only need to measure the recoil electron to fix the event kinematics



Known: $p^\mu = (E, \vec{p}) \approx (|\vec{p}|, \vec{p})$
$l^\mu = (E_e, \vec{l}) \approx (|\vec{l}|, \vec{l})$
$l'^\mu \equiv (l-q)^\mu = (E_e', \vec{l}') \approx (|\vec{l}'|, \vec{l}')$

$\hookrightarrow \quad q^\mu = (l-l')^\mu = (|\vec{l}| - |\vec{l}'|, \vec{l} - \vec{l}')$

$q^2 = (l-l')^2 = \cancel{l^2} - 2l\cdot l' + \cancel{l'^2} \approx -2l\cdot l'$

$\approx -2[E_e E_e' - \vec{l}\cdot\vec{l}']$

$q^2 \approx -2E_e E_e' \underbrace{[1-\cos\theta]}_{\geq 0} \rightarrow q^2 < 0$ (spacelike)

$$\boxed{Q^2 \equiv -q^2 = +2E_e E_e'(1-\cos\theta) = 4E_e E_e' \sin^2\theta/2}$$

$\hookrightarrow$ Q^2 tells you how hard the impact with the proton is

$\hookrightarrow$ Q^2 is a transverse variable (can always choose a frame where $q_T^2 = Q^2$)

$\hookrightarrow$ $r_T = 1/Q$ tells you the transverse resolution scale of the collision.

To complete the picture of the collision, need a longitudinal variable to quantify how much energy is delivered to the proton.

$\Rightarrow$ $p\cdot q$ is a measure of the $\gamma^* p$ impact

ie) in the proton rest frame, $p^\mu = (m, \vec{0})$? $p\cdot q = m(E_e - E_e')$

ic) in the proton rest frame,

$$\left.\begin{aligned} p^\mu &= (m, \vec{0}) \\ q^\mu &= (E_i - E_e', \vec{e} - \vec{e}\,') \end{aligned}\right\} \quad p\cdot q = m(E_i - E_e')$$

What to compare it to? Different choices / conventions

$$s_{\gamma p} = (p+q)^2 = \cancel{p^2}^{\,0} + 2p\cdot q + \underbrace{q^2}_{-Q^2}$$

= invariant mass (squared) of $\gamma^* p$ collision

= invariant mass (squared) of shattered proton

$$2p\cdot q = s_{\gamma p} + Q^2$$

Bjorken x: $$\boxed{x_B \equiv \frac{Q^2}{2p\cdot q} = \frac{Q^2}{s_{\gamma p} + Q^2} \qquad 0 \le x_B \le 1}$$

↳ x_B is a longitudinal variable describing the energy transfer

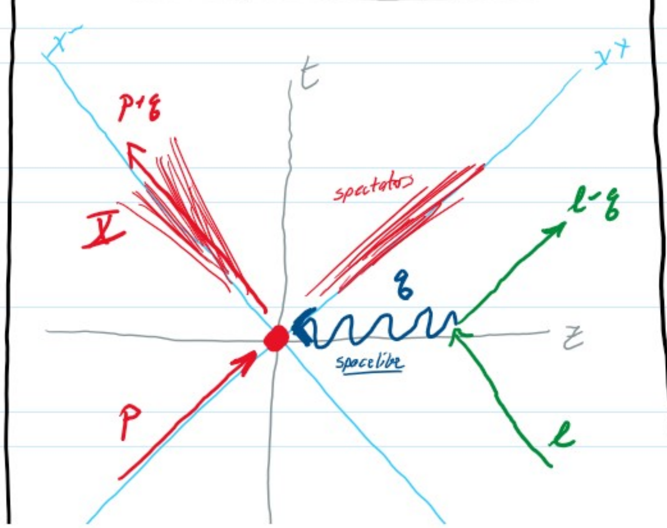
↳ The higher the $\gamma^* p$ collision energy, the smaller x_B:

as $s_{\gamma p} \to \infty$ (for fixed Q^2)
$x_B \to 0$

↳ The smaller the x value, the more time-dilated the proton structure is.

> Inclusive DIS takes a snapshot of proton substructure:
> - Q^2 controls the transverse resolution scale
> - x_B controls the exposure time.

In the $\gamma^* p$ CMS frame:

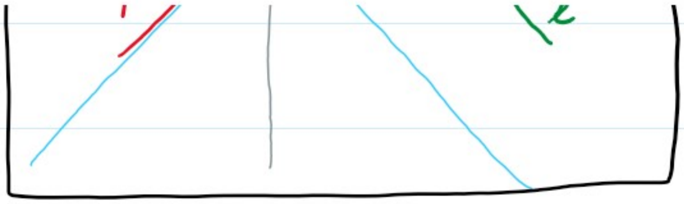


The electrons can basically be removed from the process — all they do is determine the kinematics of the virtual photon

All the real action is in the $\gamma^* p$ collision.

Pick a frame (such as the $\gamma^* p$ CMS frame) where the proton moves along the $+z$ axis at the speed of light and the incoming electron moves along the $-z$ axis at the speed of light

"Brick wall frame": the photon hits a constituent



"Brick wall frame": the photon hits a constituent of the proton and knocks the fragments backwards along the $-z$ axis

What do we learn about proton substructure?

- In general there can be 2 independent structure functions (corresponding to the electric and magnetic form factors)
- In general, they can depend on both x and Q^2

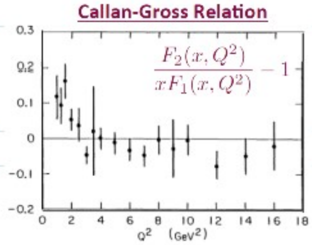
$$\frac{d\sigma}{d\Omega\, dE'} = \sigma_{\text{point}} \left[\frac{2mx}{Q^2} F_2(x, Q^2) + \frac{1}{m} F_1(x, Q^2) \tan^2 \frac{\theta}{2} \right]$$

But experimentally, F_1 and F_2 are not independent:

$$\boxed{F_2 \approx x F_1 \qquad (\text{Callan-Gross})}$$

Consistent with γ^* hitting a fermion

Callan-Gross Relation



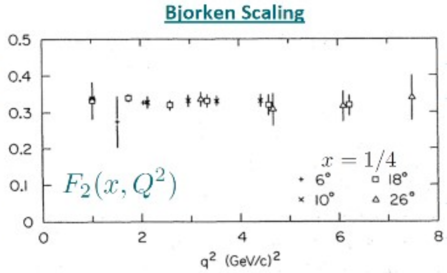
Note: $F_1^{\text{Yuri}} = 2F_1^{\text{Kendall}}$

And they depend only on x_B and are Q^2 independent

$$\boxed{F_2(x, Q^2) \approx F_2(x) \qquad (\text{Bjorken Scaling})}$$

↳ Fermions are pointlike at any resolution
↳ ... even when the uncertainty principle should be going crazy
↳ ... need asymptotic freedom
↳ discovery of QCD.

Bjorken Scaling



L1.2 — DIS on a Quark Target

High energy scattering is conveniently described in terms of light-front coordinates:

$$p^+ \equiv \frac{1}{\sqrt{2}}(p^0 + p^3) \qquad p^\mu = (p^+, p^-, \vec{p}_T)$$
$$p^- \equiv \frac{1}{\sqrt{2}}(p^0 - p^3)$$

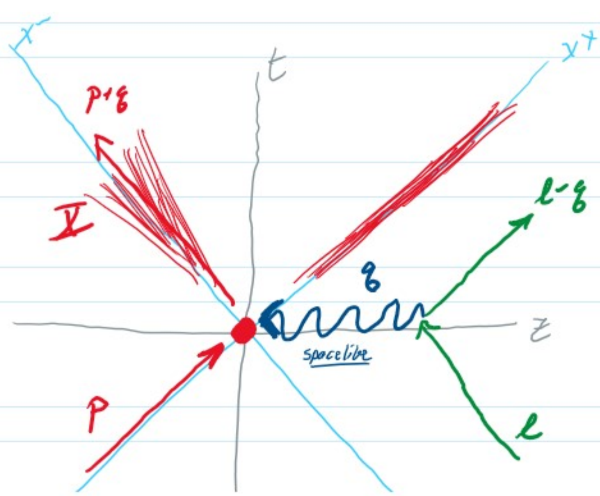
$$p \cdot q = \underbrace{p^+ q^- + p^- q^+}_{} - \vec{p}_T \cdot \vec{q}_T \qquad p^2 = 2p^+p^- - p_T^2$$

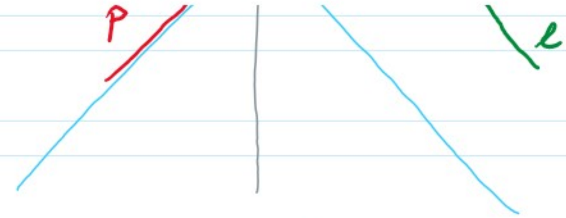
$$\frac{1}{2}\left[\begin{matrix} p^0q^0 + \cancel{p^3q^0} - \cancel{p^0q^3} - p^3q^3 \\ + p^0q^0 - \cancel{p^3q^0} + \cancel{p^0q^3} - p^3q^3 \end{matrix} \right]$$

$$= p^0q^0 - \vec{p}_T \cdot \vec{q}_T - p^3q^3 \ \checkmark$$

$$p^\pm \to e^{\pm \Delta y} p^\pm$$

$$d^4p = dp^+ dp^- d^2p_T \qquad \frac{d^3p}{E_p} = d^2p_T \frac{dp^+}{p^+} = d^2p_T \frac{dp^-}{p^-}$$





$$- p q - p_T \cdot q_T - p q$$

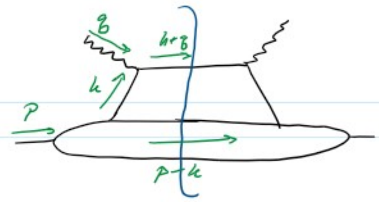
$$d^4p = dp^+ dp^- d^2p_T \qquad \frac{d^3p}{E_p} = d^2p_T \frac{dp^+}{p^+} = d^2p_T \frac{dp^-}{p^-}$$

$$\delta^4(p-q) = \delta(p^+ - q^+)\,\delta(p^- - q^-)\,\delta^2(\vec{p}_T - \vec{q}_T)$$

A particle moving near the speed of light in the $(+z)$ direction has large p^+

A particle moving near the speed of light in the $(-z)$ direction has large p^-

Take the proton to be moving along the $+z$ axis:



$$p^\mu = (p^+, p^-, \vec{0}_T) = \left(p^+, \frac{m^2}{2p^+}, \vec{0}_T\right) \approx (p^+, 0^-, \vec{0}_T)$$

$$p^2 = m^2 \rightarrow 2p^+p^- = m^2, \qquad p^- = \frac{m^2}{2p^+}$$

Take the photon to collide head on with the proton:

$$q^\mu = (q^+, q^-, \vec{0}_T) = (-Q^2/2q^-, q^-, \vec{0}_T)$$

$$q^2 = -Q^2 \rightarrow 2q^+q^- = -Q^2, \qquad q^+ = -Q^2/2q^-$$

$$s_{\gamma p} = (p+q)^2 = \cancel{p^2}^{\,0} + 2p\cdot q + \underbrace{q^2}_{-Q^2}$$

$$2p\cdot q = s_{\gamma p} + Q^2 = 2p^+q^- + \underbrace{\cancel{2p^-q^+}}_{\frac{m^2Q^2}{2p^+q^-}\to 0}$$

$$\boxed{2p^+q^- \approx s_{\gamma p} + Q^2}$$

Then $$q^+ = -\frac{Q^2}{2q^-} = -\left(\frac{Q^2}{2p^+q^-}\right)p^+ = -\left(\frac{Q^2}{s+Q^2}\right)p^+$$

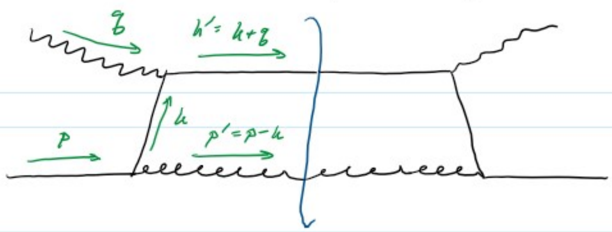
$$\boxed{q^+ = -x_B\, p^+}$$

$$q^\mu = (-x_B p^+, q^-, \vec{0}_T)$$

Evaluate the photon/proton cross-section

$$F_2 = \frac{Q^2}{4\pi^2 \alpha_{EM}}\, \sigma_{tot}^{\gamma^* p}$$

For illustration, consider the quark target model



For simplicity, neglect all masses

$$p^\mu = (p^+, 0^-, \vec{0}_T)$$

$$q^\mu = (-xp^+, q^-, \vec{0}_T)$$

$$d\sigma = \frac{1}{2s}\,\frac{d^3k'}{2(2\pi)^3 E_k'}\,\frac{d^3p'}{2(2\pi)^3 E_p'}\,|\mathcal{M}|^2\,(2\pi)^4\,\delta^4(p+q-p'-k')$$

with $(k')^2 = 0$ and $(p')^2 = 0$ from on-shell conditions

$$\sigma_{tot}^{\gamma^* p} = \frac{1}{2s}\,\frac{1}{4(2\pi)^2}\int d^2k_T'\, d^2p_T' \int \frac{dp'^+}{p'^+} \int \frac{dk'^-}{k'^-}\,|\mathcal{M}|^2\,\delta(p^+ + q^+ - p'^+ - k'^+)\,\delta(p^- + q^- - p'^- - k'^-)\,\delta^2(\vec{p}_T' + \vec{k}_T')$$

Use the δ-fn to set the 3 independent components of p'^μ:

$$\begin{cases} p'^+ = p^+ + q^+ - k'^+ \\ \vec{p}_T' = -\vec{k}_T' \end{cases}$$

with $$p'^- = \frac{p_T'^2}{2p'^+} = \frac{k_T'^2}{2(p^+ + q^+ - k'^+)}$$ from on-shell condition.

$$\sigma_{tot}^{\gamma^* p} = \frac{1}{2s}\,\frac{1}{4(2\pi)^2}\int d^2k_T' \int \frac{dk'^-}{k'^-}\,|\mathcal{M}|^2 \left(\frac{1}{p'^+}\right)\delta(p^- + q^- - p'^- - k'^-)$$

$$\sigma_{tot}^{\gamma^* p} = \frac{1}{2s}\,\frac{1}{4(2\pi)^2}\int d^2k_T' \int \frac{dk'^-}{k'^-}\, |\mathcal{M}|^2 \left(\frac{1}{p'^+}\right) \delta(p^- + q^- - p'^- - k'^-)$$

[change variables to $k = k' - q$ for clarity.
Then $(k')^\mu = (k+q)^\mu$ and $(p')^\mu = (p-k)^\mu$

$$\sigma_{tot}^{\gamma^* p} = \frac{1}{2s}\,\frac{1}{4(2\pi)^2}\int d^2k_T \int dk^- \frac{1}{(p^+-k^+)(k^-+q^-)}\, |\mathcal{M}|^2\, \delta(p^- - (p-k)^- - k^-)$$

[Last δ-fn fixes k^- together with on-shell conditions.

$$\begin{cases} k^+ = (q+k)^+ - q^+ = \cancel{\dfrac{k_T^2}{2(q^-+k^-)}} + xp^+ \approx xp^+ \\ k^- = \cancel{p^-} - (p-k)^- = -\dfrac{k_T^2}{2(p^+-k^+)} = -\dfrac{k_T^2}{2(1-x)p^+} \\ k^\mu = \left(xp^+, -\dfrac{k_T^2}{2(1-x)p^+}, \vec{k}_T\right) \end{cases}$$

Then $\dfrac{1}{(p^+-k^+)(\cancel{k^-}+q^-)} \approx \dfrac{1}{(1-x)p^+q^-}$

$$\sigma_{tot}^{\gamma^* p} = \frac{1}{2s}\,\frac{1}{4(1-x)p^+q^-}\int \frac{d^2k}{(2\pi)^2}|\mathcal{M}|^2$$

[And
$$\begin{aligned} 4(1-x)p^+q^- &= 2(1-x)(2p^+q^-) \\ &= 2\left(1-\frac{Q^2}{s+Q^2}\right)(s+Q^2) \\ &= 2s \end{aligned}$$

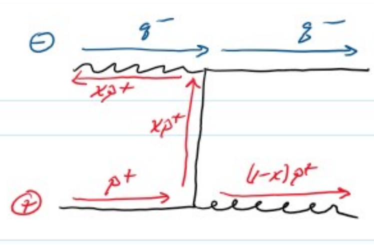
Giving

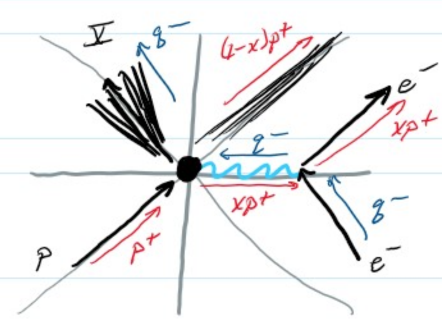
$$\boxed{\sigma_{tot}^{\gamma^* p} = \int \frac{d^2k}{(2\pi)^2}\left|\frac{\mathcal{M}}{2s}\right|^2 \quad \text{and} \quad F_2 = \frac{Q^2}{(2\pi)^2\alpha_{em}}\int\frac{d^2k}{(2\pi)^2}|\mathcal{M}|^2}$$

with overall kinematics

$$\begin{cases} p^\mu = (p^+, 0^-, \vec{0}_T) \\ q^\mu = (-xp^+, q^-, \vec{0}_T) \\ k^\mu = \left(xp^+, -\dfrac{k_T^2}{2(1-x)p^+}, \vec{k}_T\right) \end{cases}$$

$$\begin{cases} k^2 = -\dfrac{k_T^2}{1-x} \\ 2p^+q^- = \dfrac{s}{1-x} \end{cases}$$





Calculate the amplitude from the Feynman rules:

$$\frac{\cancel{i}\mathcal{M}}{2s} = \frac{\cancel{i}}{2s}\,\bar{u}(k+q)\left[\cancel{-}ie\,\not{\epsilon}(q)\right]\left[\frac{i\not{k}}{k^2}\right]\left[igt^a\,\not{\epsilon}^*(p-k)\right]u(p)$$

$$\frac{\mathcal{M}}{2s} = \frac{iegt^a}{2s}\left(\frac{1}{k^2}\right)\left[\bar{u}(k+q)\,\not{\epsilon}(q)\,\not{k}\,\not{\epsilon}^*(p-k)\,u(p)\right]$$

$$= -\frac{ieg\,t^a}{2s}\,\frac{(1-x)}{k_T^2}\left[\bar{u}(k+q)\,\not\epsilon(q)\,\not k\,\not\epsilon^*(p-k)\,u(p)\right]$$

Then square the amplitude and average over quantum numbers:

- Color: $\frac{1}{N_c}$ trace [color matrices]
- Quark Spins: $\frac{1}{2}\sum_{spins} \longrightarrow \frac{1}{2}$ trace [Dirac matrices]
- Photon / Gluon Spins: $\frac{1}{2}\sum_{polariz} \epsilon^\mu \epsilon^{\nu *} \longrightarrow -g^{\mu\nu}$

$$\left\langle \left|\frac{\mathcal{M}}{2s}\right|^2 \right\rangle = \frac{e^2 g^2}{4s^2}\,\frac{(1-x)^2}{k_T^4}\left(\frac{1}{N_c}\,tr\left[t^a t^a\right]\right)\left(\frac{1}{2}\,tr\left[(\not k+\not q)\gamma_\mu \not k \gamma_\nu \not p \gamma^\nu \not k \gamma^\mu\right]\right)$$

Color Algebra: $(t^a t^a) = C_F \cdot \mathbb{1}$ with $C_F = \frac{N_c^2-1}{2N_c} = \frac{4}{3}$ for $SU(3)$

so $\frac{1}{N_c} tr[t^a t^a] = C_F \cdot \frac{1}{N_c} tr[\mathbb{1}] = C_F \cdot \cancel{\frac{1}{N_c}} \cdot \cancel{N_c} = C_F$

$$\left\langle \left|\frac{\mathcal{M}}{2s}\right|^2 \right\rangle = \frac{e^2 g^2 C_F}{8s^2}\,\frac{(1-x)^2}{k_T^4}\; tr\left[(\not k+\not q)\gamma_\mu \not k \gamma_\nu \not p \gamma^\nu \not k \gamma^\mu\right]$$

Dirac Algebra: $\gamma_\mu \not A \gamma^\mu = -2\not A$

and $\gamma_\mu \not A \not B \not C \gamma^\mu = -2\not C \not B \not A$

so $tr\left[(\not k+\not q)\gamma_\mu \not k \gamma_\nu \not p \gamma^\nu \not k \gamma^\mu\right] = -2\, tr\left[(\not k+\not q)\gamma_\mu \not k \not p \not k \gamma^\mu\right]$

$= +4\, tr\left[(\not k+\not q)\not k \not p \not k\right]$

$$\left\langle \left|\frac{\mathcal{M}}{2s}\right|^2 \right\rangle = \frac{e^2 g^2 C_F}{2s^2}\,\frac{(1-x)^2}{k_T^4}\; tr\left[(\not k+\not q)\not k \not p \not k\right]$$

Note that p^μ has only a large p^+ component: $\not p \approx p^+\gamma^-$

and $(k+q)^\mu$ has only a large $(k+q)^-$ component: $(\not k+\not q) \approx (k+q)^-\gamma^+ \approx q^-\gamma^+$

Then $tr\left[(\not k+\not q)\not k \not p \not k\right] \approx p^+ q^-\, tr\left[\gamma^+ \not k \gamma^- \not k\right] = \frac{1}{2}\frac{s}{1-x}\, tr\left[\gamma^+ \not k \gamma^- \not k\right]$

$$\left\langle \left|\frac{\mathcal{M}}{2s}\right|^2 \right\rangle = \frac{e^2 g^2 C_F}{4s}\,\frac{(1-x)}{k_T^4}\; tr\left[\gamma^+ \not k \gamma^- \not k\right]$$

Last γ-matrix identity: $tr[\not A \not B \not C \not D] = 4\left[(A\cdot B)(C\cdot D) - (A\cdot C)(B\cdot D) + (A\cdot D)(B\cdot C)\right]$

so $tr\left[\gamma^+ \not k \gamma^- \not k\right] = 4\left[2k^+k^- - k^2\right]$

$= 4\left[2k^+k^- - (2k^+k^- - k_T^2)\right]$

$= 4k_T^2$

Giving
$$\boxed{\left\langle \left|\frac{\mathcal{M}}{2s}\right|^2 \right\rangle = \frac{e^2 g^2 C_F}{s}\,\frac{(1-x)}{k_T^2}}$$

so that $\sigma_{tot}^{\gamma^* p} = \frac{e^2 g^2 C_F}{s}(1-x)\int \frac{d^2k}{(2\pi)^2}\frac{1}{k_T^2}$

Using $e^2 = 4\pi\alpha_{em}$

so that σ_{tot} [illegible]

Using $e^2 = 4\pi\alpha_{em}$

$g^2 = 4\pi\alpha_s$

and $\int \frac{d^2k_T}{k_T^2} = 2\pi \int \frac{dk_T}{k_T} = 2\pi \ln k_T \Big|_{min}^{max}$

with limits set by $k_T^2 \leq s$ above

and $k_T^2 \gtrsim m^2$ below (artifact of massless approximation)

$$\sigma_{tot}^{\gamma^* p} = 8\pi\alpha_s\alpha_{em} C_F \left(\frac{1-x}{s}\right) \ln \sqrt{s}/m$$

And then $F_2 = \frac{Q^2}{(2\pi)^2\alpha_{em}} \sigma_{tot}^{\gamma^* p}$

giving $F_2 = \left(\frac{2\alpha_s C_F}{\pi}\right) (1-x)\frac{Q^2}{s} \ln \sqrt{s}/m$

and since $\frac{Q^2}{s} = \frac{x}{1-x}$, the final answer is:

$$\boxed{F_2(x) = \frac{2\alpha_s C_F}{\pi}\, x \ln \sqrt{s}/m}$$