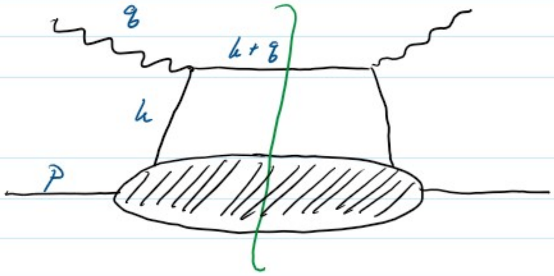
L2.1 — The Meaning of Small x

Recall that the kinematics of the handbag diagram gave another interpretation for Bjorken x:

$$k^+ = (q\cancel{+k})^+ - q^+ \approx x p^+$$

$$\boxed{x = \frac{k^+}{p^+} = \text{fraction of proton momentum carried by struck parton.}}$$

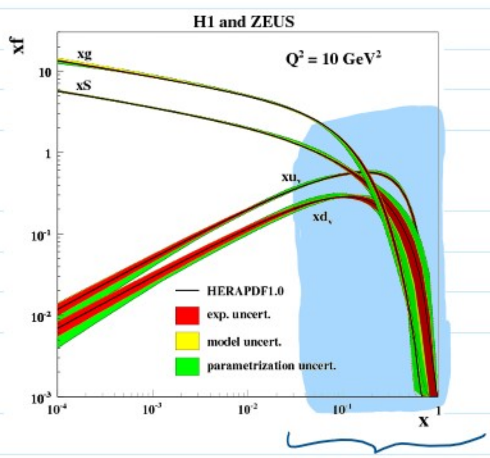


The value of the kinematic variable "Bjorken x" $x_B = \frac{Q^2}{2P\cdot q}$, which randomly fluctuates from event to event, uniquely identifies the 1D kinematics "Feynman x" $x_F = k^+/p^+$ of the struck parton.

On-shell condition of struck quark:

$$\begin{aligned}\delta\left((k+q)^2\right) &\approx \delta\left(2(k^+ + q^+)q^-\right)\\ &= \frac{1}{2q^-}\,\delta(k^+ + q^+)\\ &= \frac{1}{2q^-}\,\delta(k^+ - x_B p^+)\\ &= \frac{1}{2p^+q^-}\,\delta\left(\frac{k^+}{p^+} - x_B\right)\\ &= \frac{1}{2p^+q^-}\,\delta(x_F - x_B)\end{aligned}$$

Then different regimes in the kinematic variable x_B reveal different aspects of the physics of proton structure.

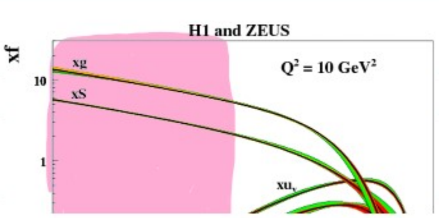


Valence Region

The partons that carry the largest fraction of the proton energy are the valence quarks

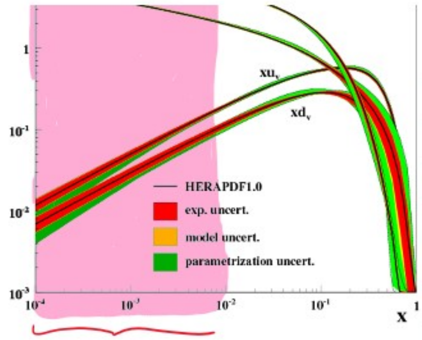
$$\boxed{p \approx (uud) \quad \text{around} \quad x \approx 1/3}$$

* Actually $x_{peak} < 1/3$, because in an interacting theory like QCD, some of this energy is bled off into radiation.



The region with $x \ll 1$ describes the radiation content of the proton

$$\boxed{p \approx (\{G\}, \{q\bar{q}\}) \quad \text{for } x \ll 1}$$



Radiation Region

$$\boxed{p \approx \left(\{G\}, \{q\bar{q}\} \right) \quad \text{for } x \ll 1}$$

- Gluon radiation (from valence quarks + other gluons)
- Sea quarks (pair produced from gluons)
- Huge amount of radiation, each carrying a very small fraction of the proton energy.

The small-x regime corresponds to the highest collision energies for a given Q^2:

$$x_B = \frac{Q^2}{s + Q^2} \quad \longleftrightarrow \quad \frac{Q^2}{s} = \frac{x}{1-x}$$

So if $x \ll 1$ then $x \approx Q^2/s \ll 1$

Compare:

Bjorken Kinematics $s \sim Q^2 \gg \Lambda_{QCD}^2$ ($Q^2 \to \infty$ with $x \sim \mathcal{O}(1)$)

Regge Kinematics $s \gg Q^2 \gg \Lambda_{QCD}^2$ ($Q^2 \to \infty$ as $x \to 0$)

At fixed Q^2, going to smaller x means higher energies (s)

↳ Boosting the proton enhances relativistic time dilation

↳ Makes more ephemeral quantum fluctuations ("wee partons") resolvable with the same DIS microscope

The kinematic reach of any accelerator into small x is always constrained by its finite collision energy:

If $Q^2_{,min} = (1\ \text{GeV})^2$ when we stop trusting pQCD

and $s_{\gamma p} \leq s_{max}$ the maximum energy that can be delivered to the $\gamma^* p$ collision while still detecting the recoil electron

Then $x \gtrsim \dfrac{(1\ \text{GeV})^2}{s_{max}}$ is the minimum accessible value of x.

Kinematic Coverage of the EIC

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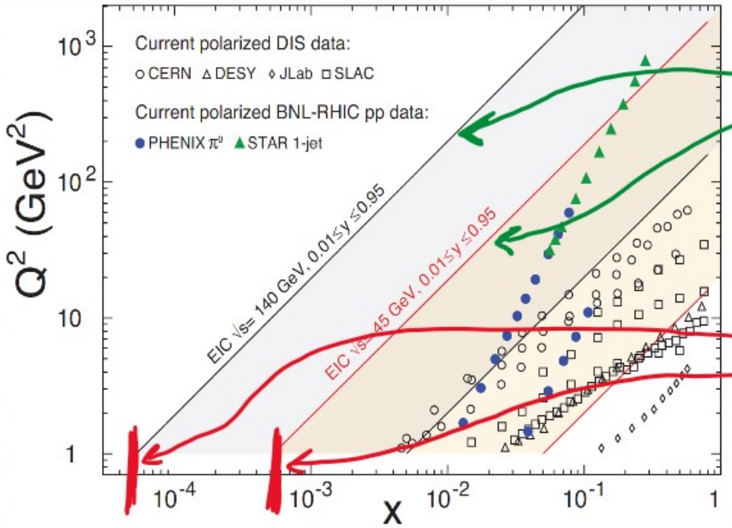


Line determined by collider energy:

$$Q^2 \approx x \cdot s \quad \text{at small } x$$

Minimum value of x accessible when pQCD is still applicable.

This fundamental limitation at small x is a challenge to our ability to understand proton structure.

The global properties of the proton arise from sum rules which integrate the contributions of parton distributions over the whole physical range $0 \le x \le 1$

e.g.) Proton Spin : Jaffe – Manohar Sum Rule

$$\frac{1}{2} = \frac{1}{2}\Delta\Sigma + \Delta G + L_q + L_G$$

$$\Delta\Sigma = \sum_f \int_0^1 dx \cdot \left(\Delta q_f(x) + \Delta\bar{q}_f(x)\right)$$

$$\Delta G = \int_0^1 dx \cdot \Delta g(x)$$

$\Rightarrow$ Extrapolation to $x \to 0$ will always be necessary to understand the origin of the proton structure.

L2.2 — Dipole Picture of DIS

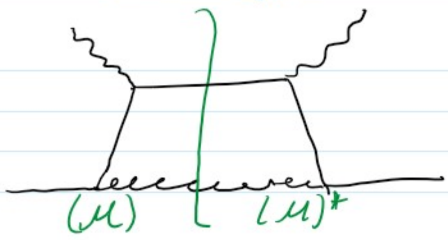
The "handbag diagram" we calculated previously in the quark target model describes how DIS occurs in the valence regime $x \sim \mathcal{O}(1)$

"Handbag DIS":

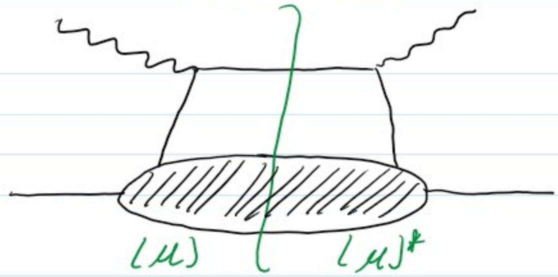
Quark Target

Real Proton

<u>Quark Target</u>



<u>Real Proton</u>



- The γ^* interacts electromagnetically with a quark inside the proton.
- It delivers a hard momentum kick which knocks the quark out of the proton
- The outgoing quark fragments and hadronizes into the "current jet"
- Spectator remnants of the proton fragment into the "beam jet"

In accordance with the physics of the valence region, we found the scaling of the "knockout process" of DIS to be:

$$\begin{cases} F_2(x) \sim (\alpha_s)^1 & \text{leading order in coupling} \\ F_2(x) \sim x^1 & \text{linear dependence on } x \end{cases}$$

The particular coefficients we found are specific to the quark target model. But these features are general.

> When we go to small x kinematics $x \ll 1$, this mechanism is suppressed.
>
> At a certain point (say, $x \lesssim 0.01$) the suppression from small x is more severe than a higher order correction in α_s.
>
> Then the physical picture of DIS changes.

If x specifies a longitudinal momentum (fraction) k^+/p^+, then by the uncertainty principle it is also related to a distance scale.

The general expression for the DIS cross section is in terms of two electromagnetic interactions (currents):

$$E' \frac{d\sigma}{d^3\ell'} = \frac{\alpha_{em}^2}{2\pi m} \frac{1}{EQ^4} \left[\ell_\mu \ell'_\nu + \ell'_\mu \ell_\nu - (\ell\cdot\ell') g_{\mu\nu}\right] \int d^4r \, e^{iq\cdot r} \langle p | j^\mu_{em}(r) \, j^\nu_{em}(0) | p \rangle$$

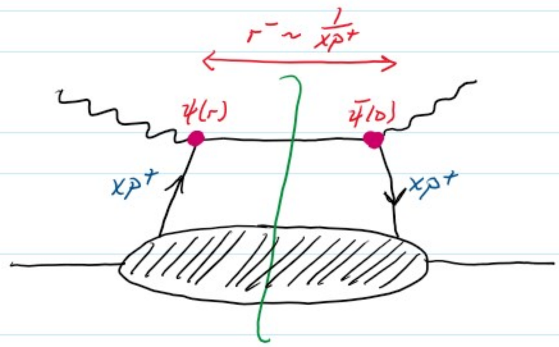
$$E' \frac{d\sigma}{d^3\ell'} = \frac{\alpha_{em}^2}{2\pi m} \frac{1}{E Q^4} \left[\ell_\mu \ell'_\nu + \ell'_\mu \ell_\nu - (\ell \cdot \ell') g_{\mu\nu} \right] \int d^4 r \, e^{iq\cdot r} \langle p | j^\mu_{em}(r) \, j^\nu_{em}(0) | p \rangle$$

And the parton distribution functions extracted from them are similar:

$$f(x, Q^2) = \int \frac{dr^-}{2\pi} e^{ixp^+ r^-} \langle p | \bar{\psi}(0) \, U[0, r^-] \frac{\gamma^+}{2} \psi(r^-) | p \rangle$$

The longitudinal separation between the operators is

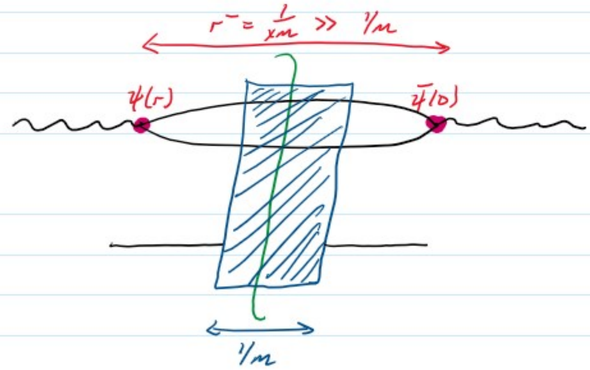
$$\boxed{\begin{aligned} r^- &\sim \frac{1}{xp^+} \qquad (\text{the "Ioffe time" or "Ioffe length"}) \\ &\sim \frac{1}{mx} \quad \text{in the proton rest frame} \end{aligned}}$$



If $x \sim O(1)$, then the electromagnetic interactions happen inside the proton, hitting a quark via the handbag diagram.

But if $x \ll 1$ (say, $x \lesssim 0.01$) then the electromagnetic interactions happen far outside the proton, where there are no quarks to interact with.

Instead, the photon interacts electromagnetically by pair producing a $q\bar{q}$ dipole far outside the proton which then collides hadronically.



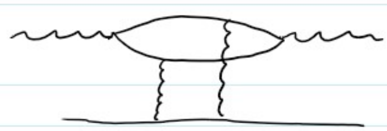
This process is higher order in α_s than the handbag diagram, but unsuppressed by x:

Quark Target:

Handbag

$$\langle |\mathcal{M}|^2 \rangle \sim \alpha_{em} \, \alpha_s \, (x)^1$$

Dipole



$$\langle |\mathcal{M}|^2 \rangle \sim \alpha_{em} \, \alpha_s^2 \, (x)^0$$

So at small x kinematics, DIS (ep scattering) becomes hadronic scattering ($q\bar{q}\,p$ scattering).

> To understand proton structure at small x ($x \to 0$), we need to understand hadronic scattering at high energy ($s \to \infty$).

L2.3 — Eikonal Hadronic Scattering

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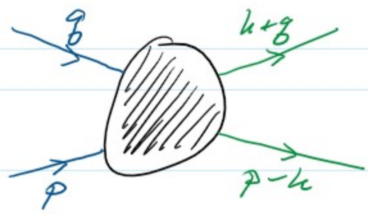
We showed previously that

$$\sigma_{tot}^{\gamma^* p} = \int \frac{d^2k}{(2\pi)^2} \left|\frac{\mathcal{M}}{2s}\right|^2$$



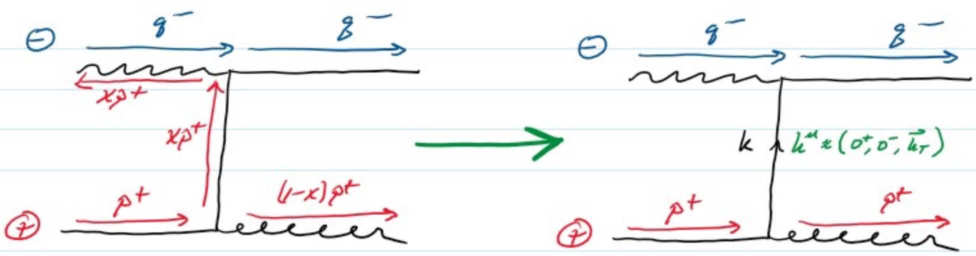
and hence

$$\frac{d\sigma}{d^2k} = \frac{1}{(2\pi)^2} \left|\frac{\mathcal{M}}{2s}\right|^2$$

which is valid for any $2 \to 2$ scattering process

Thus the energy dependence of a scattering cross section is controlled by $|\mathcal{M}/2s|^2$, and we are interested in the high-energy asymptotics $s \gg k_T^2$. since $x \approx Q^2/s$. as $x \to 0$.

We found that the $(\gamma^* q \to q G)$ knockout process was suppressed by $(x)^1$ at small x



$$|\mathcal{M}|^2 \sim \frac{1}{(k^2)^2} \, tr\left[(\not{k}+\not{q})\not{k}\not{p}\not{k}\right]$$

$$\sim \frac{1}{k_T^4} \underbrace{p^+ q^-}_{\sim s} \underbrace{tr\left[\gamma^+ \not{k} \gamma^- \not{k}\right]}_{\sim k_T^2}$$

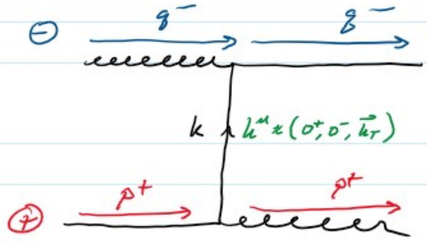
$$|\mathcal{M}|^2 \sim \frac{s}{k_T^2}$$

So that $\frac{d\sigma}{d^2k} \sim \left|\frac{\mathcal{M}}{2s}\right|^2 \sim \frac{1}{s} \cdot \frac{1}{k_T^2}$ is suppressed by one power of s as $s \to \infty$ (or one power of x as $x \to 0$).

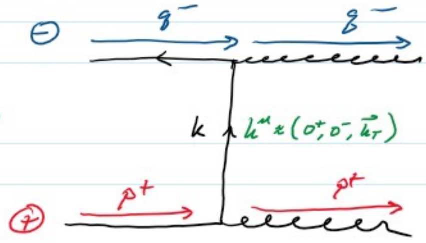
[Exactly the same thing is true in high-energy hadronic scattering as long as a quark is exchanged between the colliding hadrons
$\Longrightarrow$ Very little baryon stopping in high energy collisions.

$\underline{qG \to qG}$

$\underline{q\bar{q} \to GG}$



or



$$|\mathcal{M}|^2 \sim \left(\frac{1}{k^2}\right)^2 \mathrm{tr}[\not{q}\not{k}\not{p}\not{k}] \qquad\qquad |\mathcal{M}|^2 \sim \left(\frac{1}{k^2}\right)^2 \mathrm{tr}[\not{q}\not{k}\not{p}\not{k}]$$

$$\sim \frac{1}{k_T^4}\cdot(p^+q^-)\cdot(k_T^2) \qquad\qquad \sim \frac{1}{k_T^4}(p^+q^-)(k_T^2)$$

$$|\mathcal{M}|^2 \sim s/k_T^2 \qquad\qquad |\mathcal{M}|^2 \sim s/k_T^2$$

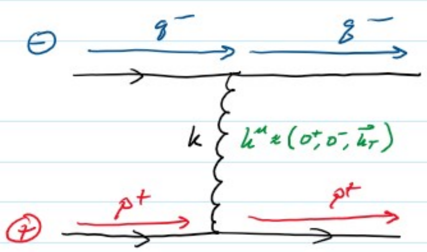
so $\boxed{\frac{d\sigma}{d^2k} \sim \frac{1}{s}\cdot\frac{1}{k_T^2}}$ so $\boxed{\frac{d\sigma}{d^2k} \sim \frac{1}{s}\cdot\frac{1}{k_T^2}}$

(and even worse suppression for s- or u-channel diagrams, where $\left(\frac{1}{k^2}\right) \sim \frac{1}{s}$ rather than $\frac{1}{k_T^2}$!)

It is very hard to stop / reflect a high-energy quark.

Instead, it is far more preferable for them to punch right through each other, being deflected by each others' gluon fields.

$qG \to qG$



$$|\mathcal{M}|^2 \sim \left(\frac{1}{k^2}\right)^2 \underbrace{\mathrm{tr}[\not{p}\gamma_\mu\not{p}\gamma_\nu]}_{\sim p_\mu p_\nu}\ \underbrace{\mathrm{tr}[\not{q}\gamma^\mu\not{q}\gamma^\nu]}_{\sim q^\mu q^\nu}$$

$$\sim \left(\frac{1}{k_T^4}\right)(p^+q^-)^2$$

$$\sim s^2/k_T^4$$

so $\boxed{\frac{d\sigma}{d^2k} \sim \frac{1}{k_T^4}}$ **Energy independent** (unsuppressed by x)

High energy scattering with "eikonal" $s \to \infty$ kinematics is dominated by gluon exchange, which delivers only a transverse momentum kick ("Glauber" or "Coulomb" gluon exchange).

This analysis can be generalized to any exchanged particle based only on its spin j:

$$|\mathcal{M}|^2 \sim s^{(2j)} \quad \text{so} \quad \frac{d\sigma}{d^2k} \sim \frac{1}{s^2}\left(\frac{s}{k_T^2}\right)^{2j} \sim s^{2j-2}\cdot\left(\frac{1}{k_T^2}\right)^{2j}$$

... energy on its spin j

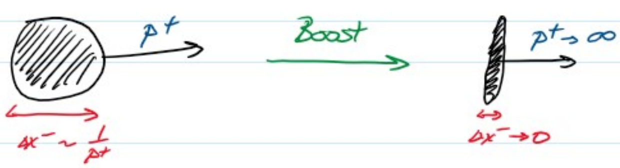
$$|\mathcal{M}|^2 \sim s^{(2j)} \qquad \text{so} \qquad \frac{d\sigma}{d^2k_T} \sim \frac{1}{s^2}\left(\frac{s}{k_T^2}\right)^{2j} \sim s^{2j-2}\cdot\left(\frac{1}{k_T^2}\right)^{2j}$$

$j=0$	scalar	Higgs exchange	$\frac{d\sigma}{d^2k_T} \sim \frac{1}{s^2}$
$j=1/2$	spinor	Quark exchange	$\frac{d\sigma}{d^2k_T} \sim \frac{1}{s}\cdot\frac{1}{k_T^2}$
$j=1$	vector	Gluon exchange	$\frac{d\sigma}{d^2k_T} \sim \frac{1}{k_T^4}$

Underlies the framework of pre-QCD "Regge theory" which classified effective particles being exchanged by their energy dependence (spin) and other quantum numbers.

"Pomeron": $j=1$ energy independent $\Rightarrow$ gluons

In addition to being dominated by Glauber gluon exchange, eikonal scattering is also highly Lorentz-boosted.

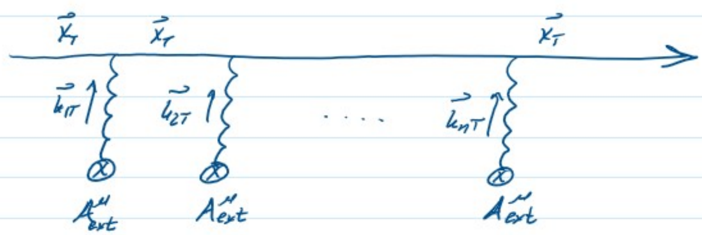


For $s \to \infty$, the collision region shrinks to a delta function

$$\begin{cases} p^+ \to \infty \\ \rho(x^-) \to \delta(x^-) \\ F^{\mu\nu}(x^-) \to f^{\mu\nu}\cdot\delta(x^-) \end{cases}$$

→ Instantaneous interaction takes zero time
→ Gluon "shockwave"
→ Finite momentum kick through instantaneous impulse
→ Transverse position has no time to drift during interactions

Multiple scattering all $\mathcal{O}(1)$ in terms of energy.
↳ Controlled only by α_s
↳ If momenta are soft, α_s large, then may all be $\mathcal{O}(1)$



Corrections to this $s \to \infty$ shockwave picture (sub-eikonal corrections) are power suppressed by one or more factors of s.

High-energy scattering in a background gauge field is particularly simple in transverse coordinate space, since the transverse position does not have time to change.

All that can occur is a pure phase rotation due to the external potential — which for QCD is a color rotation matrix

For an eikonal particle propagating along p^+:

QED: $\psi(\vec{x}_T) \rightarrow e^{i\phi(\vec{x}_T)}\psi(\vec{x}_T)$ $\qquad \phi(\vec{x}_T) = -e\int_{-\infty}^{\infty} dz^+ A^-(z^+,\vec{x}_T)$

QCD: $\psi(\vec{x}_T) \rightarrow V(\vec{x}_T)\,\psi(\vec{x}_T)$ $\qquad \underbrace{V(\vec{x}_T) = P\exp\left[ig\int_{-\infty}^{\infty} dz^+ A^{-a}(z^+,\vec{x}_T)\, t^a\right]}_{\text{"Wilson Line"}}$

A Wilson line resums any number of multiple eikonal gluon scatterings

It is a special case of the more general "gauge link" which can follow an arbitrary contour:

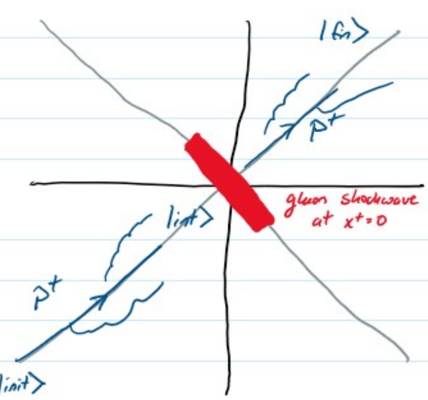
$$\mathcal{U}^{[C]}[x_f,x_i] \equiv P\exp\left[ig\int_{x_i,\,C}^{x_f} dz_\mu\, A^{\mu a}(z)\, t^a\right]$$

The expressions for an eikonal gluon are equivalent, but use the adjoint representation of the $SU(N_c)$ color matrices:

$$(T^a_{adj})_{bc} = -i f^{abc}$$

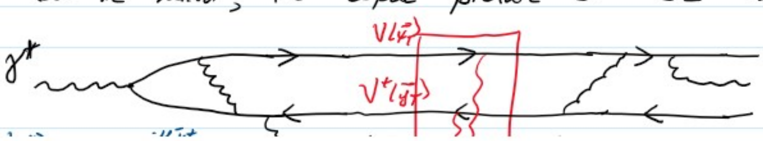
Eikonal kinematics actually simplify the dynamics of QCD considerably because they are naturally "time-ordered" along the light front x^+

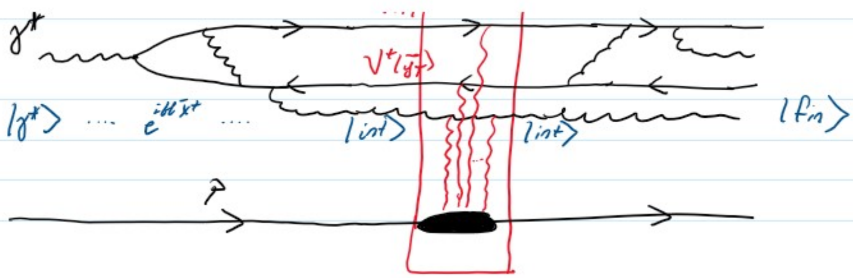
- A "projectile" moves with large p^+ along the $+z$ light cone
- The "target" is highly Lorentz contracted to a δ-fn shockwave at $x^+=0$.
- The evolution of the projectile before and after the collision is governed by "old-fashioned" time-dependent perturbation theory, with x^+ playing the role of time.



"Light-front perturbation theory"

With this in mind, the dipole picture of DIS looks like:





$$\underbrace{\langle fin|\, e^{iH^-x^+}\, |int\rangle}_{\psi^*(\vec{x}_T,\vec{y}_T)} \; V(\vec{x}_T)\, V^\dagger(\vec{y}_T)\cdots \; \underbrace{\langle int \,|\, e^{iH^-x^+}\, |\gamma^*\rangle}_{\psi_{\gamma^*}(\vec{x}_T,\vec{y}_T)}$$

Intuitive ingredients: Wave functions $\otimes$ Wilson Lines.