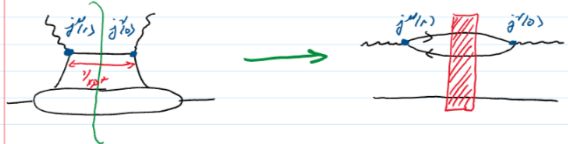
L3.1 – Degrees of Freedom at Small x

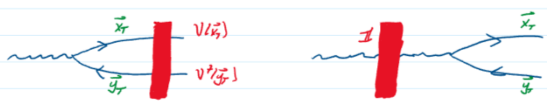
<u>Previously:</u> Dipole picture of DIS at small x
Light-front (time-ordered) perturbation theory
Wilson lines
Exercises: Eikonal gluon radiation

At small x $(x \lesssim 0.01)$ the dipole picture of DIS looks like



Because the gluon structure of the proton (target) is localized to a δ-fn along the light cone, there are only 2 distinct time orderings to consider:

Splitting Before | **Splitting After**



$\psi^{\gamma^* \to q\bar{q}}_{(\vec{x}_T, \vec{y}_T)} \left[V(\vec{x}_T) V^\dagger(\vec{y}_T) \right]$ $\qquad$ $- \psi^{\gamma^* \to q\bar{q}}_{(\vec{x}_T, \vec{y}_T)} \left[\mathbb{1} \right]$

Then, when we square the amplitude, there are 4 total terms:

$$\sigma^{\gamma^* p} \sim \psi(\vec{x}_T, \vec{y}_T) \otimes \underbrace{\mathrm{tr}\left[\left(V(\vec{x}_T) V^\dagger(\vec{y}_T) - \mathbb{1} \right) \left(V(\vec{x}_T) V^\dagger(\vec{y}_T) - \mathbb{1} \right)^\dagger \right]}_{\mathrm{tr}\left[\cancel{V(\vec{x}_T)} \cancel{V^\dagger(\vec{y}_T)} \cancel{V(\vec{y}_T)} \cancel{V^\dagger(\vec{x}_T)} - V(\vec{x}_T) V^\dagger(\vec{y}_T) - V(\vec{y}_T) V^\dagger(\vec{x}_T) + \mathbb{1} \right]} \otimes \psi^*(\vec{x}_T, \vec{y}_T)$$

<u>note</u>: approximately Hermitian (C-even)

$$\sigma^{\gamma^* p} \sim (2N_c)\, \psi(\vec{x}_T, \vec{y}_T) \otimes \Big(1 - \underbrace{\tfrac{1}{N_c} \mathrm{tr}\left[V(\vec{x}_T) V^\dagger(\vec{y}_T) \right]}_{q\bar{q} \text{ scattering amplitude}} \Big) \otimes \psi^*(\vec{x}_T, \vec{y}_T)$$

Including everything:

$$\boxed{\sigma^{\gamma^* q} = 2N_c \int_0^1 \frac{dz}{4\pi z(1-z)} \int d^2x\, d^2y \; \left| \psi(\vec{x}_T - \vec{y}_T, z) \right|^2 \left[1 - \frac{1}{N_c} \mathrm{tr}\left[V(\vec{x}_T) V(\vec{y}_T) \right] \right]}$$

The elementary ingredients are:

<u>Time-ordered wave functions</u>: $\quad \psi \sim \dfrac{\langle q\bar{q} | \mathcal{H}_{LFPT} | \gamma^* \rangle}{E^{LF}_{initial} - E^{LF}_{final}}$

* Note: Finite life time $\longleftrightarrow$ Energy fluctuations

<u>Wilson line scattering</u>: $\quad V(\vec{x}_T) = P_{exp}\left[ig \int dz_- A^{+a} t^a \right]$

At lowest order, the Wilson lines are just built out of the qq scattering cross section we calculated previously:

which was energy-independent.

We thus conclude that DIS in the valence region $(x \sim 1/3)$ proceeds by a quark knockout mechanism which scales as $(x)^1$ at small x.

At kinematically small x $(x \lesssim 0.01)$ DIS transitions to a dipole scattering mechanism, which at leading order is x-independent: $(x)^0$.

This looks like it should be the whole story...

L3.2 — Small-x Evolution

Higher order corrections in pQCD will modify the process, but usually these are small:

e.g.) 

N.L.O. $\sim \mathcal{O}(\alpha_s)$ smaller.

but we saw in the Exercises that there is a special subset of higher order corrections which are especially important at small x / high energy: eikonal radiation of soft gluons.

$$\left.\frac{d\sigma}{d^2l}\right|_{LO}^{qq} \sim \alpha_s$$

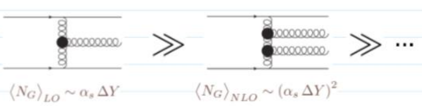
$$\left.\frac{d\sigma}{d^2l}\right|_{NLO}^{qq+G} \sim \int d^2k \int_{-Y/2}^{+Y/2} dy \, \frac{d\sigma^{qqG}}{d^2l\, d^2k\, dy}$$

$$\sim \alpha_s \cdot Y \cdot \left.\frac{d\sigma^{qq}}{d^2l}\right|_{LO}$$

These corrections are $\mathcal{O}(\alpha_s)$ small, but are systematically enhanced by the longitudinal phase space $Y \sim \ln s/Q^2 \sim \ln 1/x$ for soft gluon emission at high energy.

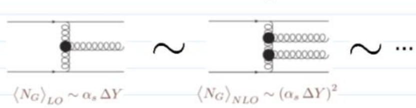
Perturbation theory relies on a well-separated hierarchy of contributions

$$\sigma = (\alpha_s)^1 \sigma_{LO} + (\alpha_s)^2 \sigma_{NLO} + (\alpha_s)^3 \sigma_{NNLO} + \cdots$$



$\langle N_g \rangle \ll 1$

But if some of the higher order corrections are systematically enhanced, it can break this hierarchy:



$\langle N_g \rangle \gtrsim 1$

$$\begin{aligned}
\sigma = \; & (\alpha_s)^1 \sigma_{LO} && + (\alpha_s^2)\, \sigma_{NLO}^{(0)} && \cdots \\
& + \alpha_s (\alpha_s \ln 1/x)\, \sigma_{NLO}^{(1)} && + \alpha_s^2 (\alpha_s \ln 1/x)\, \sigma_{NNLO}^{(1)} && \cdots \\
& + \alpha_s (\alpha_s \ln 1/x)^2\, \sigma_{NNLO}^{(2)} && + \alpha_s^2 (\alpha_s \ln 1/x)^2\, \sigma_{N3LO}^{(2)} && \cdots \\
& \quad \vdots && \quad \vdots && \ddots
\end{aligned}$$

When the large logarithm becomes big enough to compensate the extra factor of α_s, these enhanced contributions are comparable to the lower order ones, and the perturbative hierarchy breaks:

$$\boxed{\alpha_s \ln 1/x \gtrsim \mathcal{O}(1) \longrightarrow x \lesssim e^{-1/\alpha_s} \quad \underline{\text{exponentially small } x}}$$

Ballpark guess: for $\alpha_s \simeq 0.2 \rightarrow x \lesssim 5\times 10^{-3}$

When all of the systematically enhanced corrections are comparable we must re-sum them all, instead of keeping them order by order. This restructures the perturbative hierarchy:

$$\sigma = \underbrace{\begin{matrix} (\alpha_s)^1 \sigma_{LO} \\ + \alpha_s (\alpha_s \ln 1/x) \sigma^{(1)}_{NLO} \\ + \alpha_s (\alpha_s \ln 1/x)^2 \sigma^{(2)}_{NNLO} \\ \vdots \end{matrix}}_{\substack{\alpha_s \Sigma_n (\alpha_s \ln 1/x)^n \sigma^{(n)}_{N(n)LO} \\ \equiv \alpha_s \sigma_{LLx}}} + \underbrace{\begin{matrix} (\alpha_s^2) \sigma^{(0)}_{NLO} \\ + \alpha_s^2 (\alpha_s \ln 1/x) \sigma_{NNLO} \\ + \alpha_s^2 (\alpha_s \ln 1/x)^2 \sigma^{(2)}_{N3LO} \\ \vdots \end{matrix}}_{\substack{\alpha_s^2 \Sigma_n (\alpha_s \ln 1/x)^n \sigma^{(n)}_{N(n+1)LO} \\ \equiv \alpha_s^2 \sigma_{NLLx}}} \begin{matrix} \cdots \\ \cdots \\ \cdots \\ \ddots \end{matrix}$$

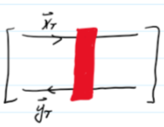
$$\boxed{\begin{matrix} \sigma = \alpha_s \sigma_{LLx} + \alpha_s^2 \sigma_{NLLx} + \cdots \\ \text{New, re-ordered perturbative hierarchy} \end{matrix}}$$

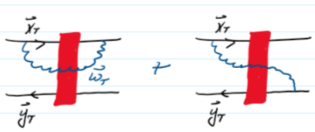
"Quantum Evolution"

The strategy to perform the resummation is to write the systematically enhanced correction as a recursive (differential) equation, then try to solve it.

For DIS at exponentially small x, we need to find the corrections to the dipole scattering operator $\frac{1}{N_c} tr[V(\vec{x}_T) V^\dagger(\vec{y}_T)]$ which are systematically enhanced by $Y = \ln 1/x$.

We've already seen what those corrections are: eikonal soft gluon radiation.

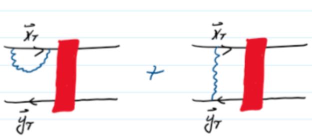
$\frac{d}{dY}$ [] =



+ mirror images

"Real" Corrections
(Gluon scatters)

$U^{ba}(\vec{w}_T)\ tr[V(\vec{x}_T) t^a V^\dagger(\vec{y}_T) t^b]$



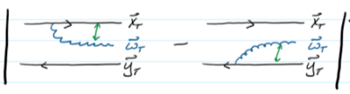
+ mirror images

"Virtual" Corrections
(Only $q\bar{q}$ scatter)

$tr[V(\vec{x}_T) V^\dagger(\vec{y}_T)]$

The only difference between the terms is what scattering operator you get.

All terms have the same spatial dependence: the eikonal ($q \to qG$) WF squared.



$$\left| \ldots \right|^2 \sim \left(\frac{\vec{\omega}_T - \vec{x}_T}{(\omega-x)_T^2} - \frac{\vec{\omega}_T - \vec{y}_T}{(\omega-y)_T^2} \right)^2$$

$$= \frac{1}{(\omega-x)_T^2} - \frac{2(\overrightarrow{\omega-x})_T \cdot (\overrightarrow{\omega-y})_T}{(\omega-x)_T^2 (\omega-y)_T^2} + \frac{1}{(\omega-y)_T^2}$$

$$= \frac{1}{(\omega-x)_T^2 (\omega-y)_T^2} \Big[\underbrace{(\omega-y)_T^2 - 2(\overrightarrow{\omega-x})_T \cdot (\overrightarrow{\omega-y})_T + (\omega-x)_T^2}_{\left[(\cancel{\vec{\omega}}-y)_T - (\cancel{\vec{\omega}}-x)_T \right]^2 = (x-y)_T^2} \Big]$$

$$= \boxed{\frac{(x-y)_T^2}{(\omega-x)_T^2 (\omega-y)_T^2}} \quad \text{Universal Kernel}$$

Schematically, the small-x evolution equation is

$$\frac{\partial}{\partial Y} \left\langle tr(V(\vec{x}_T) V^\dagger(\vec{y}_T)) \right\rangle \sim \alpha_s \int d^2\omega \, \frac{(x-y)_T^2}{(\omega-x)_T^2 (\omega-y)_T^2} \left\langle \mathcal{U}^{ba}(\vec{\omega}_T)\, tr[V(\vec{x}_T) t^a V^\dagger(\vec{y}_T) t^b] - tr[V(\vec{x}_T) V^\dagger(\vec{y}_T)] \right\rangle$$

Enhanced correction: $\alpha_s Y = \alpha_s \ln 1/x$

$|\psi|^2$ gluon splitting kernel

Real corrections: $q\bar{q}G$ scatters

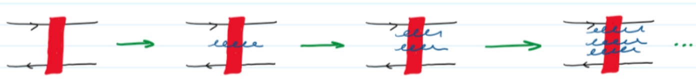
Virtual corrections: only original $q\bar{q}$ scatters

Including all the factors + color algebra:

$$\boxed{\frac{\partial}{\partial Y} \left\langle \frac{1}{N_c} tr\left[V(\vec{x}_T) V^\dagger(\vec{y}_T)\right] \right\rangle_{(s)} = \frac{\alpha_s N_c}{2\pi^2} \int_{\Lambda^2/s}^{1} \frac{dz}{z} \int d^2\omega \, \frac{(x-y)_T^2}{(x-\omega)_T^2 (y-\omega)_T^2} \left[\left\langle \frac{1}{N_c^2} tr\left[V(\vec{\omega}_T) V^\dagger(\vec{y}_T)\right] tr\left[V(\vec{x}_T) V^\dagger(\vec{\omega}_T)\right] \right\rangle_{(zs)} - \left\langle \frac{1}{N_c} tr\left[V(\vec{x}_T) V^\dagger(\vec{y}_T)\right] \right\rangle_{(zs)} \right]}$$

- Nonlinear — gluon also scatters

  In principle, an infinite series of coupled equations....



$q\bar{q}$ $\quad$ $q\bar{q}G$ $\quad$ $q\bar{q}GG$ $\quad$ $q\bar{q}GGG$....

– "Balitsky Operator Hierarchy"

– "JIMWLK equation":

(Jalilian-Marian / Iancu / McLerran / Weigert / Leonidov / Kovner)

Equivalent formulation in terms of functional integrals

– "BK Equation": (Balitsky / Kovchegov)

Mean field approximation (satisfied for $N_c \to \infty$)

$$\left\langle \frac{1}{N_c^2} tr\left[V(\vec{\omega}_T) V^\dagger(\vec{y}_T)\right] tr\left[V(\vec{x}_T) V^\dagger(\vec{\omega}_T)\right] \right\rangle = \left\langle \frac{1}{N_c} tr\left[V(\vec{\omega}_T) V^\dagger(\vec{y}_T)\right] \right\rangle \left\langle \frac{1}{N_c} tr\left[V(\vec{x}_T) V^\dagger(\vec{\omega}_T)\right] \right\rangle$$

Becomes a closed nonlinear equation $\longrightarrow$ can solve.

– "BFKL": (Balitsky / Fadin / Kuraev / Lipatov)

Linearized approximation: $1 - \left\langle \frac{1}{N_c} tr[V(\vec{x}_T) V^\dagger(\vec{y}_T)] \right\rangle \ll 1$

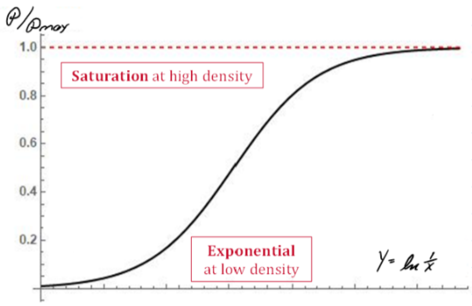
Neglects nonlinear term $\longrightarrow$ can solve

L3.3 — Gluon Saturation and High-Density QCD

The resummation of leading-$\ln\frac{1}{x}$ corrections strongly modifies the tree level behavior:

$$F_2(x) \sim xG,\, xq \sim \sigma_{tot}^{\gamma^* p} \sim (x)^0 = \text{const.}$$

The essential physics of the eikonal emission of multiple soft gluons is captured by the GLR-MQ equation, which we solved in an approximate form in the Exercises.



The details of exactly how this equation approaches saturation in the nonlinear regime are specific to the equation and the approximations used. But the general physics of the linear / nonlinear regimes are universal.

They are reflected in the solutions to the true Balitsky hierarchy in the appropriate limits.

<u>Linearized (BFKL):</u> (Saddle point solution)

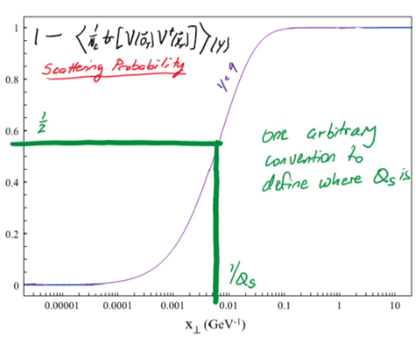
$$\left\langle \frac{1}{N_c} tr\left[V(\vec{x}_T) V^\dagger(\vec{y}_T)\right]\right\rangle \sim e^{\left(\frac{4\alpha_s N_c}{\pi}\ln 2\right)Y} \sim \left(\frac{1}{x}\right)^{\frac{4\alpha_s N_c}{\pi}\ln 2}$$

↳ exponential growth as a function of rapidity
↳ power law growth as a function of $s \sim \frac{1}{x}$
↳ calculable exponent ("Pomeron intercept")
⇒ $\left(\frac{1}{x}\right)^{0.79}$ for $\alpha_s \approx 0.3$

<u>Nonlinear + Mean Field (BK):</u>

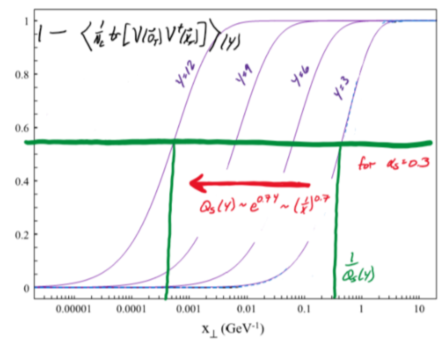
↳ A characteristic length / momentum scale emerges.
⇒ At distances long enough that the gluon number $\sim 1/\alpha_s$, the density saturates to a maximum.
⇒ The $q\bar{q}$ scattering probability is as large as it can get (Black Disk Limit)

↳ The dynamically-generated momentum scale is called the saturation scale Q_s.

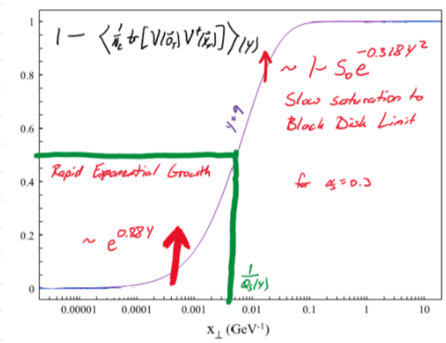


The characteristic momentum scale $Q_s(Y)$ describes the typical transverse momentum acquired by multiple scattering in the dense gluon fields.

$Q_s(Y)$ increases as a function of $Y \sim \ln 1/x$



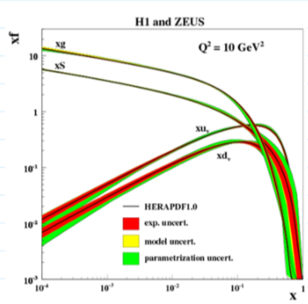
The unsaturated modes grow exponentially, while the nearly saturated ones slowly converge to the black disk limit.



L3.4 — Conclusion and Perspective

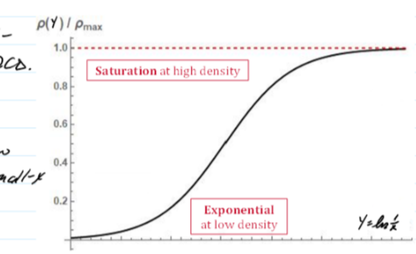
The physics of the "radiation regime" of proton structure is very different from the "valence regime"

- Many-body
- Gluon (+ sea quark) dominated
- High density

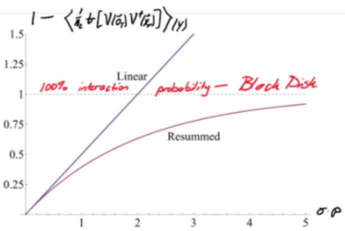


The emergence of the small-x, high-density regime is inevitable in QCD.

- Any QCD system, no matter how dilute, radiates a proliferation of small-x gluons leading to high density.



Just stopping at the linear regime (with endless exponential growth) is not an option.



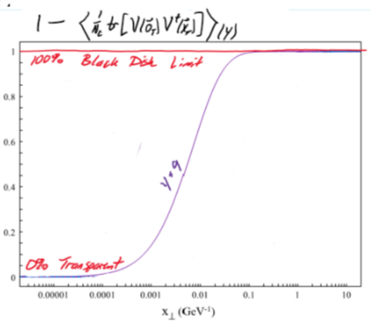
- An exponential growth of gluon density would violate the "black disk" unitary limit: scattering probabilities > 100%

- Equivalently, the Froissart bound translates the unitarity constraint into an upper limit on how fast a cross section can grow with energy:

$$\sigma_{tot} \le \frac{\pi \Delta^2}{2m^2} \ln^2\left(\frac{s}{m^2}\right)$$

vs

$$\sigma_{tot}^{\gamma^* p}\Big|_{\substack{\text{linear}\\ \text{evolution}}} \sim s^{\left(\frac{4\alpha_s N_c}{\pi}\ln 2\right)} \sim s^{0.79}$$



- Nonlinear interactions which saturate the growing density are necessary to restore unitarity

"Saturation is a non-negotiable consequence of QCD"
— N. Armesto

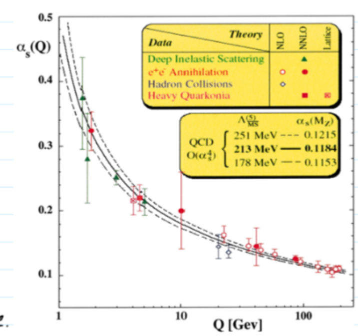
This issue is bigger than just QCD. Any non-Abelian theory will proliferate radiation at high energy.

 $\propto [t^a, t^b]$

- If you want asymptotic freedom, you must deal with the consequences for radiation at small x.

- Asymptotic freedom is critical to our understanding of the Standard Model

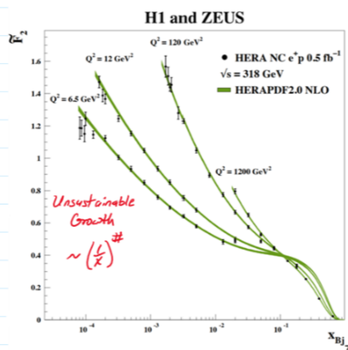
- QCD is the only archetype for a UV-complete QFT we have.



"Saturation is as fundamental a consistency test of the Standard Model as the discovery of the Higgs."
— M. Sievert

The current experimental status is unsustainable.

- We see direct evidence of the gluon cascade
- But no definitive evidence yet for nonlinear gluon saturation.
- This cannot be the whole story.



- Finding evidence of the onset of gluon saturation is one of the design missions of the EIC
- In part, by taking advantage of the density enhancement in a heavy ion

$\Rightarrow$ Anna Stasto: eA Lectures

- We don't see saturation yet. We may never see it. But finding it is an essential test of our theory. And it reveals a different and unique many-body sector of hadron structure.