The mass and spin structure of the proton

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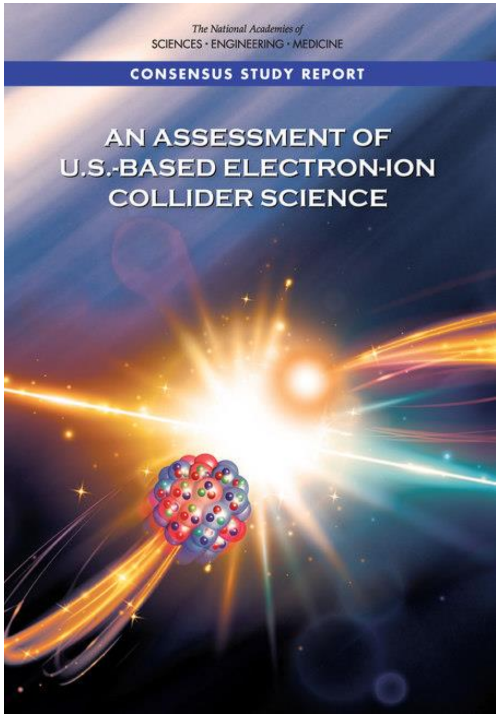
NAS report on EIC (2018/07)

An Assessment of
U.S.-Based Electron-Ion Collider Science

A Consensus Study Report of

The National Academies of
SCIENCES · ENGINEERING · MEDICINE

“The committee finds that the science that can be addressed by an EIC is compelling, fundamental and timely.”



Finding 1: An EIC can uniquely address three profound questions about nucleons-protons—and how they are assembled to form the nuclei of atoms:

- How does the mass of the nucleon arise?
- How does the spin of the nucleon arise?
- What are the emergent properties of dense systems of gluons? → Lectures by M. Sievert, A. Stasto

Outline

- Lecture 1: Proton spin decomposition
- Lecture 2: Orbital angular momentum in QCD
- Lecture 3: Proton mass and trace anomaly

Notations

Metric $\eta^{\mu\nu},\ g^{\mu\nu} = (+1,-1,-1,-1)$ $\mu,\nu = 0,1,2,3$ $i,j = 1,2$ (transverse)

Light-cone coordinates $P^{\pm} = \frac{1}{\sqrt{2}}(P^0 \pm P^3)$ $g^{+-} = 1$ $P^+ = P_-$

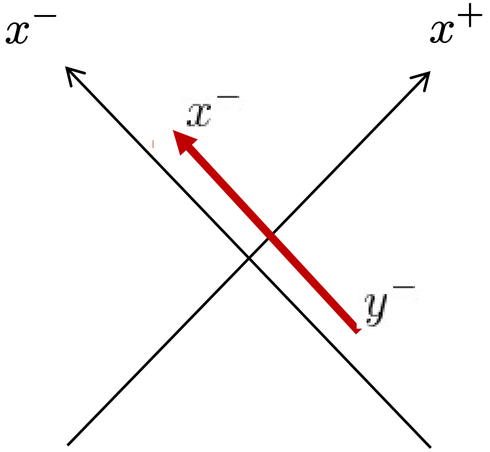
$$P \cdot x = P^+ x^- + P^- x^+ - P^i_\perp x^i_\perp$$

γ_5 , antisymmetric tensor $\gamma_5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$ $\epsilon^{0123} = +1$ $\epsilon^{12} = \epsilon_{12} = +1$

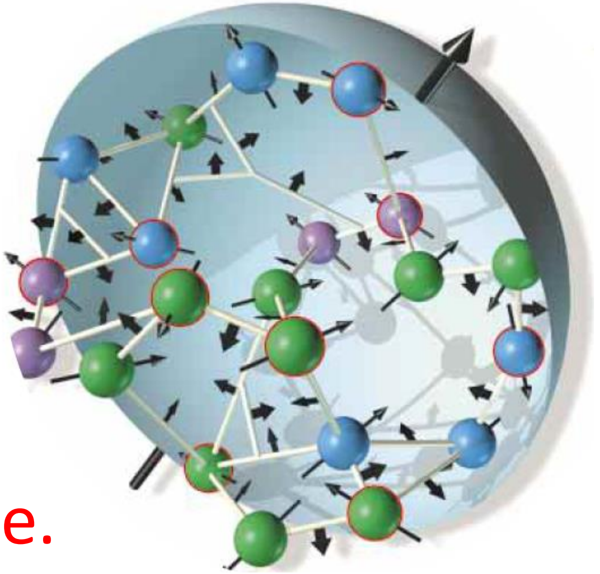
(opposite sign from Peskin) (same sign as Peskin)

Coupling constant $D^\mu = \partial^\mu + igA^\mu$

Wilson line $W[x^-, y^-, x_\perp] = P\exp\left(-ig\int_{y^-}^{x^-} dz^- A^+(z^-, x_\perp)\right)$



The proton spin problem



The proton has spin ½.

The proton is not an elementary particle.

$$\frac{1}{2} = \frac{1}{2}\Delta\Sigma + \Delta G + L^q + L^g$$

Quarks' helicity

Gluons' helicity

Orbital angular Momentum (OAM)

Note: each term is not RG invariant. The decomposition is scale and scheme dependent.

Quark helicity: definition

Proton single-particle state,

$$\boxed{2\Delta\Sigma S^\mu = \sum_f \langle PS|\bar{\psi}_f \gamma_5 \gamma^\mu \psi_f |PS\rangle}$$

spin 4-vector $S^\mu = \frac{1}{2}\bar{u}(PS)\gamma_5\gamma^\mu u(PS)$ $S^\mu \approx \pm P^\mu$ as $P^+ \to \infty$

Exercise : show that

$$P^\mu S_\mu = 0$$

$$S^2 = -M^2$$

$$u(P,S)\bar{u}(P,S) = (\not{P} + M)\frac{1 - \frac{\gamma_5 \not{S}}{M}}{2}$$

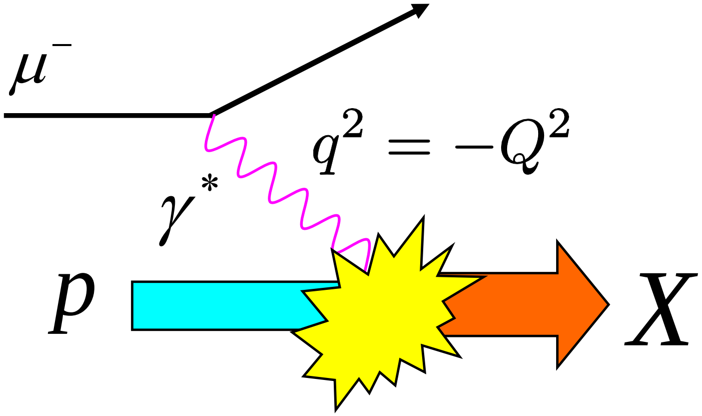
In the quark model,

$$|P,+\rangle = \frac{1}{3\sqrt{2}}\left\{ |uud\rangle(2|++-\rangle - |+-+\rangle - |-++\rangle) + perm \right\}$$

$$\frac{1}{2} = \frac{1}{2} + \frac{1}{2} - \frac{1}{2} \quad \Longrightarrow \quad \Delta\Sigma = 1$$

With relativistic effects, $\Delta\Sigma \approx 0.7$

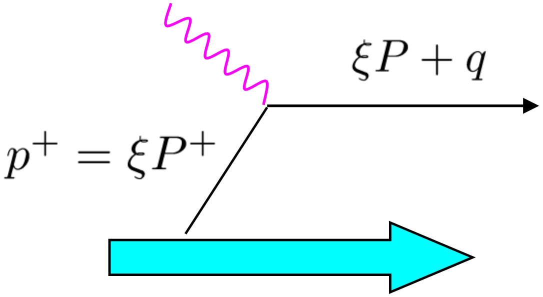
Deep inelastic scattering



Bjorken variable

$$\begin{aligned} x &= \frac{Q^2}{2P\cdot q} = \frac{Q^2}{(P+q)^2+Q^2-m_p^2} \\ &= \frac{Q^2}{Q^2+m_X^2-m_p^2} \\ &\sim \frac{Q^2}{s} \qquad (x \ll 1) \end{aligned}$$

Physical meaning of x : momentum fraction carried by the struck parton



$$(\xi P + q)^2 = \xi^2 m_p^2 + 2\xi P\cdot q - Q^2 = 0$$

$$\xi \approx \frac{Q^2}{2Pq} = x$$

The g_1 structure function

Unpolarized

$$\mathrm{Im}\frac{i}{2\pi}\int d^4y e^{iqy}\langle PS|T\{J^\mu(y)J^\nu(0)\}|PS\rangle\Big|_{sym}$$

$$\left(-\eta^{\mu\nu}+\frac{q^\mu q^\nu}{q^2}\right)F_1(x,Q^2)+\left(P^\mu-\frac{P\cdot q}{q^2}q^\mu\right)\left(P^\nu-\frac{P\cdot q}{q^2}q^\nu\right)\frac{F_2(x,Q^2)}{P\cdot q}$$

Polarized

$$\mathrm{Im}\frac{1}{2\pi}\int d^4y e^{iqy}\langle PS|T\{J^\mu(y)J^\nu(0)\}|PS\rangle\Big|_{asym}$$

$$=\epsilon^{\mu\nu\alpha\beta}q_\alpha\left(\frac{S_\beta}{P\cdot q}(\underline{g_1(x,Q^2)}+g_2(x,Q^2))-\frac{q\cdot SP_\beta}{(P\cdot q)^2}g_2(x,Q^2)\right)$$

Exercise

Forward Compton amplitude $q^0 > 0$

$$T^{\mu\nu} = \frac{i}{2\pi}\int d^4x e^{iq\cdot x}\langle PS|\mathrm{T}\{J^\mu(x)J^\nu(0)\}|PS\rangle = T_S^{\mu\nu} + iT_A^{\mu\nu}$$

symmetric antisymmetric

Hadronic tensor

$$W^{\mu\nu} = \frac{1}{2\pi}\int d^4x e^{iq\cdot x}\langle PS|[J^\mu(x), J^\nu(0)]|PS\rangle = W_S^{\mu\nu} + iW_A^{\mu\nu}$$

Show that

$$2\mathrm{Im}T_S^{\mu\nu} = W_S^{\mu\nu}$$

$$2\mathrm{Im}T_A^{\mu\nu} = W_A^{\mu\nu}$$

Relation between $g_1(x)$ and polarized quark distribution function

Operator product expansion c.f., Peskin (18.125)

$$\int d^4y e^{iqy}\underbracket{\bar\psi\gamma^\mu\overbracket{\psi(y)\bar\psi}\gamma^\nu\psi(0)} = \bar\psi i(i\partial_\alpha + q_\alpha)\gamma^\mu\gamma^\alpha\gamma^\nu\frac{-1}{Q^2}\sum_n\left(\frac{2iq\cdot\partial}{Q^2}\right)^n\psi(0)$$

$$+(\mu\to\nu, q\to -q) + \cdots$$

Pick up the antisymmetric part

$$\gamma^\mu\gamma^\alpha\gamma^\nu = g^{\mu\alpha}\gamma^\nu - g^{\mu\nu}\gamma^\alpha + g^{\alpha\nu}\gamma^\mu + i\epsilon^{\mu\alpha\nu\rho}\gamma_5\gamma_\rho$$

$$\int d^4y e^{iqy}\bar\psi\gamma^\mu\psi(y)\bar\psi\gamma^\nu\psi(0)$$

$$= 2\epsilon^{\mu\nu\lambda\alpha}q_\alpha\sum_n^{even}\frac{2q_{\mu_1}\cdots 2q_{\mu_n}}{Q^{2(n+1)}}\bar\psi\gamma_5\gamma_\lambda i\partial^{\mu_1}\cdots i\partial^{\mu_n}\psi(0) + \cdots$$

When $Q^2\to\infty$, naively, the most important operators are those with smallest dimensions (smallest n)

Twist expansion

However, in the proton matrix element, $i\partial^\mu \to P^\mu$, and $\dfrac{2P\cdot q}{Q^2} = \dfrac{1}{x}$ is not small in the Bjorken limit $Q^2 \to \infty$, $x = const$.

The most important operators are those with lowest twist

(twist) = (dimension) – (spin)

Twist-2 polarized quark operators
(symmetrized in all Lorentz indices and trace subtracted)

$$\bar{\psi}\gamma_5\gamma^{(\lambda}iD^{\mu_1}iD^{\mu_2}\ldots iD^{\mu_n)}\psi \; -(\mathrm{traces})$$

Convergent only when
$|x| > 1$

$$g_1(x) = \frac{1}{2\pi S^+}\mathrm{Im}\sum_{n=0}^{even}\frac{1}{(P^+)^n x^{n+1}}\langle PS|\bar{\psi}\gamma_5\gamma^+(iD^+)^n\psi|PS\rangle + \cdots$$

$$= \frac{1}{2\pi S^+}\mathrm{Im}\sum_{n=0}^{even}\frac{1}{x^{n+1}}\int\frac{dk^+}{2\pi}\left(\frac{k^+}{P^+}\right)^n\int dx^- e^{ik^+x^-}\langle PS|\bar{\psi}(0)\gamma_5\gamma^+W[0,x^-]\psi(x^-)|PS\rangle$$

Wilson line

$$= \frac{P^+}{4\pi S^+}\mathrm{Im}\int\frac{dk^+}{2\pi}\left(\frac{1}{xP^+ + k^+} + \frac{1}{xP^+ - k^+}\right)\int dx^- e^{ik^+x^-}\langle PS|\bar{\psi}(0)\gamma_5\gamma^+W[0,x^-]\psi(x^-)|PS\rangle$$

Analytic continuation from
$|x| > 1$ to $1 > x > 0$

$x \longrightarrow x - i\epsilon$

$$= \frac{P^+}{8\pi S^+}\int dx^- e^{ixP^+x^-}\langle PS|\bar{\psi}(0)\gamma_5\gamma^+W[0,x^-]\psi(x^-)|PS\rangle + (x \longrightarrow -x) \quad + \cdots$$

$$= \frac{1}{2}(\Delta q(x) + \Delta\bar{q}(x)) \quad + \cdots$$

Polarized quark and antiquark distributions

Digression: $g_2(x)$ structure function

Return to OPE (page 10)

$$\int d^4x e^{iqx} \bar{\psi}\gamma^\mu \psi(x) \bar{\psi}\gamma^\nu \psi(0) = \frac{2}{Q^2}\epsilon^{\mu\nu\lambda\alpha} q_\alpha \sum_n^{even} \bar{\psi}\gamma_5\gamma_\lambda \frac{i\partial_{\mu_1}\cdots i\partial_{\mu_n} 2q^{\mu_1}\cdots 2q^{\mu_n}}{Q^{2n}} \psi(0) + \cdots$$

Anti-symmetrize in λ and $\mu_1, \mu_2, \cdots$ → One twist higher (twist-3)

$g_2(x)$ is a mixture of twist-2 and twist-3 contributions

$$g_2(x) = \underline{-g_1(x) + \int_x^1 \frac{dz}{x} g_1(z)} + \underline{\bar{g}_2(x)}$$

Wandzura-Wilczek part

`Genuine twist-3’ part
(quark-gluon correlation)

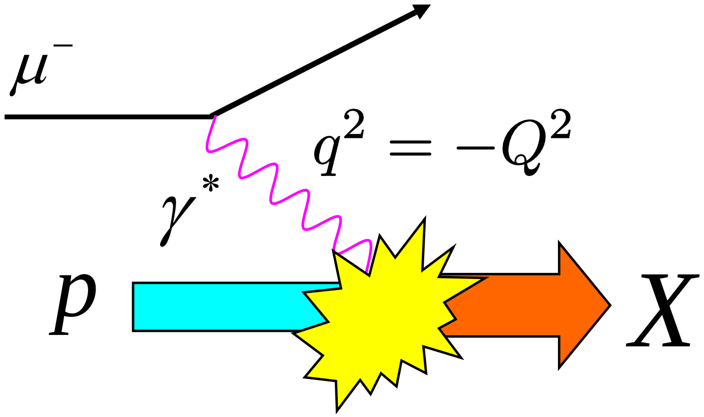
$$\int_0^1 dx x^2 \bar{g}_2(x) = \frac{d_2}{6} \qquad \langle PS|\bar{\psi}\gamma^+ gF^{+i}\psi|PS\rangle = 2d_2(P^+)^2\epsilon^{ij}S_j$$

Shuryak, Vainshtein (1982)

$\Delta\Sigma$ from polarized DIS

Longitudinal double spin asymmetry in polarized DIS

$$\begin{aligned} A_{LL} &= \frac{\mu^{\uparrow}p^{\downarrow} - \mu^{\uparrow}p^{\uparrow}}{\mu^{\uparrow}p^{\uparrow} + \mu^{\uparrow}p^{\downarrow}} \\ &\sim \left(1 + \frac{\sigma_L}{\sigma_T}\right) \frac{2x g_1}{F_2} \end{aligned}$$



$$\int_0^1 dx g_1(x,Q^2) = \frac{1}{2}\sum_f e_f^2 \int_0^1 dx(\Delta q_f(x,Q^2) + \Delta\bar{q}_f(x,Q^2)) + \cdots$$

Flavor SU(3) decomposition

$$\sum_f e_f^2 = \begin{pmatrix} \frac{4}{9} & & \\ & \frac{1}{9} & \\ & & \frac{1}{9} \end{pmatrix} = \frac{2}{9} + \frac{1}{6}\begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} + \frac{1}{18}\begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}$$

$$\begin{aligned}\int_0^1 dx g_1(x) \quad = \quad & \frac{1}{9}\overbrace{(\Delta u + \Delta d + \Delta s)}^{\Delta\Sigma} \\ & +\frac{1}{12}(\Delta u - \Delta d) \\ & +\frac{1}{36}(\Delta u + \Delta d - 2\Delta s) + \mathcal{O}(\alpha_s)\end{aligned}$$

isovector axial charge

$$_N\langle P|\bar{\psi}\gamma_5\gamma^\mu t^3\psi|P\rangle_N = g_A\bar{u}_N(P)\gamma_5\gamma^\mu t^3 u_N(P)$$

nucleon doublet

$$\Delta u - \Delta d = g_A \approx 1.2$$

octet axial charge $g_A^{(8)}$
Related by flavor SU(3) symmetry to
the semileptonic decay amplitude

$$\Xi^- \to \Lambda e^- \nu$$

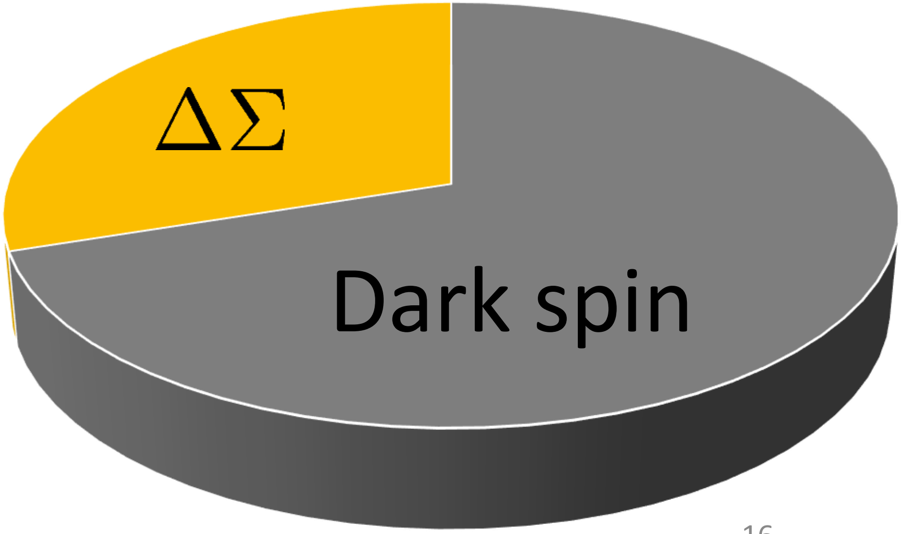
`Spin crisis’

In 1987, EMC (European Muon Collaboration) announced a very small value of the quark helicity contribution

$$\Delta\Sigma = 0.12 \pm 0.09 \pm 0.14 \ \ !?$$

Recent value from NLO QCD global analysis

$$\Delta\Sigma = 0.25 \sim 0.3$$



Gluon polarization ΔG

$$\Delta G = \int_0^1 dx \Delta G(x)$$

Polarized gluon distribution

$$\Delta G(x) = \frac{i}{xS^+} \int \frac{dy^-}{2\pi} e^{ixP^+y^-} \langle PS|F^{+\alpha}(0)\tilde{F}^+_{\ \alpha}(y^-)|PS\rangle$$

$$\epsilon^\mu_{R/L} = \frac{-1}{\sqrt{2}}(0, \pm 1, i, 0)$$

$$i F^{+i}\tilde{F}^+_{\ i} = (F^{+R})^\dagger F^{+R} - (F^{+L})^\dagger F^{+L}$$

$$\int_0^1 dx \Delta G(x) \ = \ -\frac{1}{2S^+} \int dy^- \theta(y^-) \langle PS|F^{+\alpha}(0)\tilde{F}^+_{\ \alpha}(y^-)|PS\rangle$$

Need to specify the prescription of the pole $1/x$

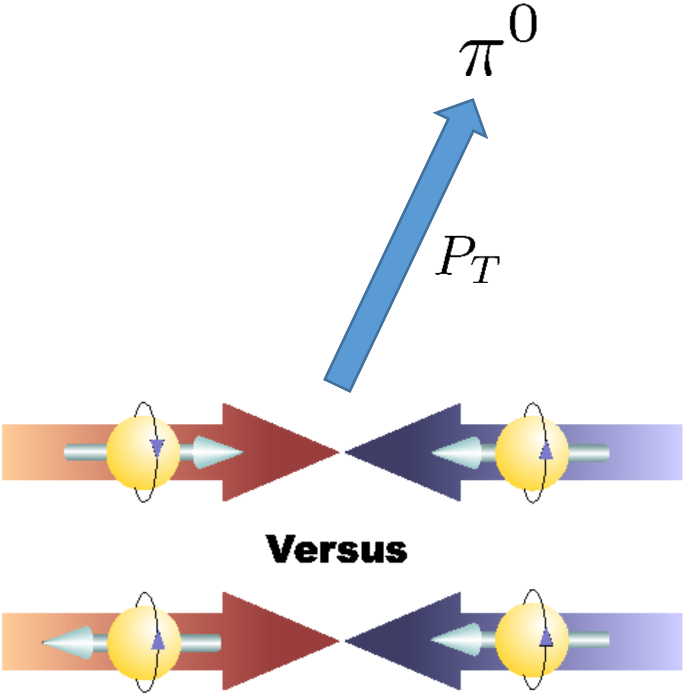
Non-local, even after taking a moment.

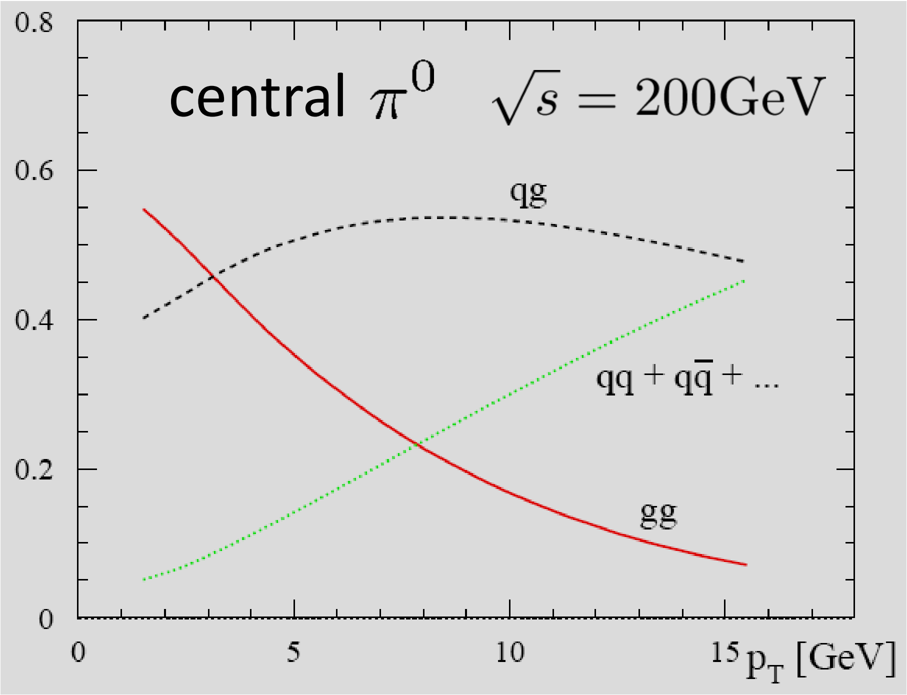
Determination of ΔG

Longitudinal double spin asymmetry in pp

$$A_{LL} = \frac{d\sigma^{++} - d\sigma^{+-}}{d\sigma^{++} + d\sigma^{+-}}$$

$$\propto \sum_{a,b} \Delta f_a \otimes \Delta f_b(x) \otimes \Delta\sigma_{ab}$$





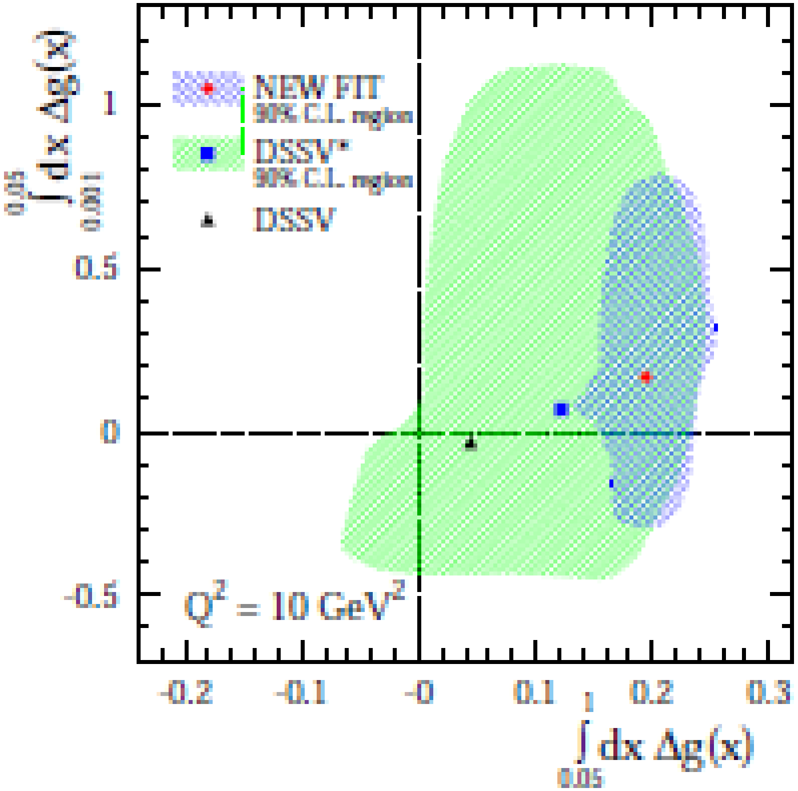
Evidence of nonzero ΔG

DeFlorian, Sassot, Stratmann, Vogelsang (2014)

Result from the NLO global analysis
after the RHIC 200 GeV pp data

$$\int_{0.05}^{1} dx \Delta G(x, Q^2) \approx 0.2 \pm^{0.06}_{0.07}$$

$$(Q^2 = 10\mathrm{GeV}^2)$$



HUGE uncertainty from the small-x region

→ RHIC 510GeV,
Electron-Ion Collider

QCD angular momentum tensor

QCD Lagrangian → Lorentz invariant $x^\mu \longrightarrow x^\mu + \omega^{\mu\nu} x_\nu$

$$\delta\psi = -\omega^{\mu\nu}\left(\frac{1}{2}(x_\nu\partial_\mu - x_\mu\partial_\nu)\psi - \frac{1}{8}[\gamma_\mu, \gamma_\nu]\psi\right)$$

→ Noether current $\partial_\mu M^{\mu\nu\lambda}_{can} = 0$

QCD angular momentum tensor (Exercise: derive this)

$$M^{\mu\nu\lambda}_{can} = x^\nu T^{\mu\lambda}_{can} - x^\lambda T^{\mu\nu}_{can} - \frac{1}{2}\epsilon^{\mu\nu\lambda\rho}\bar{\psi}\gamma_5\gamma_\rho\psi + F^{\mu\lambda}A^\nu - F^{\mu\nu}A^\lambda$$

quark helicity gluon helicity

canonical energy momentum tensor

$$T^{\mu\nu}_{can} = \bar{\psi} i\gamma^\mu \overleftrightarrow{\partial}^\nu \psi - F^{\mu\alpha}\partial^\nu A^\alpha - g^{\mu\nu}\mathcal{L}$$

→ Quark OAM → Gluon OAM

Jaffe-Manohar decomposition

$$\frac{1}{2} = \frac{1}{2}\Delta\Sigma + \Delta G + L^q_{can} + L^g_{can}$$

Based on the canonical angular momentum tensor $M^{\mu\nu\lambda}_{can}$

take $\mu\nu\lambda = +12$

Operators NOT gauge invariant.

$$\Delta G \sim \epsilon^{ij} F^{+i} A^j$$

$$L^q_{can} \sim \bar{\psi} x \times i\partial\psi$$

$$L^g_{can} \sim F x \times \partial A$$

Default gauge: light-cone gauge $A^+ = 0$

Improved (Belinfante) energy momentum tensor

Write $M^{\mu\nu\lambda}_{can} = x^\nu T^{\mu\lambda}_{can} - x^\lambda T^{\mu\nu}_{can} + H^{\mu\nu\lambda}$

Define $\tilde{T}^{\mu\nu} = T^{\mu\nu}_{can} + \partial_\rho G^{\rho\mu\nu}$ ← One can add a total derivative.

where $G^{\rho\mu\nu} = \frac{1}{2}(H^{\rho\mu\nu} - H^{\mu\rho\nu} - H^{\nu\rho\mu})$

Exercise: Show that $\tilde{T}^{\mu\nu}$ is symmetric and conserved.

Exercise: Show that in QCD,

$$\begin{aligned}\tilde{T}^{\mu\nu} &= \bar{\psi} i \gamma^{(\mu} \overleftrightarrow{D}^{\nu)} \psi - F^{\mu\rho} F^{\nu}{}_{\rho} - g^{\mu\nu} \mathcal{L} \\ &= \tilde{T}^{\mu\nu}_q + \tilde{T}^{\mu\nu}_g\end{aligned}$$

$$\tilde{M}^{\mu\nu\lambda} = x^\nu \tilde{T}^{\mu\lambda} - x^\lambda \tilde{T}^{\mu\nu}$$

Hint: A useful identity

$$\overleftrightarrow{D}^{\mu} = \frac{D^{\mu} - \overleftarrow{D}^{\mu}}{2}$$

$$\overleftarrow{D}^{\mu} = \overleftarrow{\partial}^{\mu} - igA^{\mu}$$

From the Dirac equation $(\not{D} + iM)\psi = \bar{\psi}(\overleftarrow{\not{D}} - iM) = 0$,

$$\begin{aligned} 0 &= \bar{\psi}\gamma^{\mu}\gamma^{\nu}(\not{D} + iM)\psi - \bar{\psi}(\overleftarrow{\not{D}} - iM)\gamma^{\nu}\gamma^{\mu}\psi \\ &= \bar{\psi}(g^{\mu\nu}\gamma^{\rho} + g^{\nu\rho}\gamma^{\mu} - g^{\mu\rho}\gamma^{\nu} + i\epsilon^{\mu\nu\rho\sigma}\gamma_5\gamma_{\sigma})D_{\rho}\psi \\ &\quad -\bar{\psi}\overleftarrow{D}_{\rho}(g^{\rho\nu}\gamma^{\mu} + g^{\nu\mu}\gamma^{\rho} - g^{\rho\mu}\gamma^{\nu} + i\epsilon^{\rho\nu\mu\sigma}\gamma_5\gamma_{\sigma})\psi + 2iMg^{\mu\nu}\bar{\psi}\psi \\ &= 2\bar{\psi}(\gamma^{\mu}\overleftrightarrow{D}^{\nu} - \gamma^{\nu}\overleftrightarrow{D}^{\mu})\psi + i\epsilon^{\rho\mu\nu\sigma}\partial_{\rho}(\bar{\psi}\gamma_5\gamma_{\sigma}\psi) \end{aligned}$$

Ji decomposition (1997)

$$\begin{aligned}\langle P|J^z_{q,g}|P\rangle &= \frac{1}{V}\langle P|\epsilon^{ij}\int d^3x x^i T^{0j}_{q,g}(x)|P\rangle \\ &= \frac{1}{V}\lim_{P'\to P}\langle P'|\epsilon^{ij}\int d^3x x^i T^{0j}_{q,g}(x)|P\rangle \\ &= -i\lim_{\Delta\to 0}\epsilon^{ij}\frac{\partial}{\partial \Delta^i}\langle P'|T^{0j}_{q,g}(0)|P\rangle\end{aligned}$$

$$\hat{O}(x) = e^{i\hat{P}x}\hat{O}(0)e^{-i\hat{P}x}$$

$$\Delta = P' - P$$

Gravitational form factors

$$\langle P'S|T^{\mu\nu}_{q,g}|PS\rangle = \bar{u}(P'S)\left[A_{q,g}\gamma^{(\mu}\bar{P}^{\nu)} + B_{q,g}\frac{\bar{P}^{(\mu}i\sigma^{\nu)\alpha}\Delta_\alpha}{2M} + \frac{D_{q,g}}{4}\frac{\Delta^\mu\Delta^\nu - g^{\mu\nu}\Delta^2}{M} + \bar{C}_{q,g}Mg^{\mu\nu}\right]u(PS)$$

$$A_q = \int_0^1 dx x(q(x) + \bar{q}(x))$$

$$A_g = \int_0^1 dx x g(x)$$

Momentum fraction of quarks and gluons $\Delta = 0$

`anomalous gravitomagnetic moment'

The D-term `pressure' inside the proton

Related to the trace anomaly (→ Lecture 3)

➡

$$\frac{1}{2} = \sum_q J_q + J_g \qquad J_{q,g} = \frac{1}{2}(A_{q,g} + B_{q,g})$$

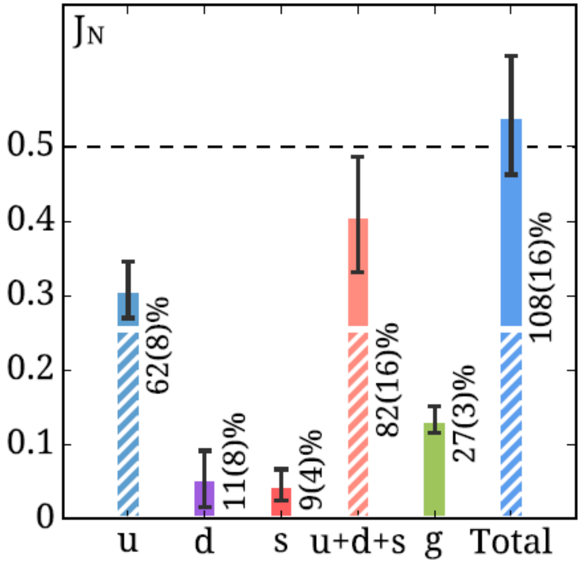
Further decomposition in the quark part possible (but not in the gluon part)

$$\bar{\psi} i\gamma^{(\mu} \overleftrightarrow{D}^{\nu)} \psi = \bar{\psi} i\gamma^{\mu} \overleftrightarrow{D}^{\nu} \psi - \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}\partial_\rho(\bar{\psi}\gamma_5\gamma_\sigma\psi)$$

(Exercise: show this)

$$J_q = \frac{1}{2}\Delta\Sigma + L^q_{\mathrm{Ji}}$$

All the operators involved are local, gauge invariant
→ calculable on a lattice



Alexandrou et al. (2017)

Relation to Generalized Parton Distribution (GPD)

GPD definition

non-forward matrix element

$$P^+ \int \frac{dy^-}{2\pi} e^{ixP^+y^-} \langle P'S'|\bar{\psi}(0)\gamma^\mu\psi(y^-)|PS\rangle$$

$$= H_q(x,\Delta)\bar{u}(P'S')\gamma^\mu u(PS) + E_q(x,\Delta)\bar{u}(P'S')\frac{i\sigma^{\mu\nu}\Delta_\nu}{2m}u(PS)$$

Multiply by x and integrate over x.

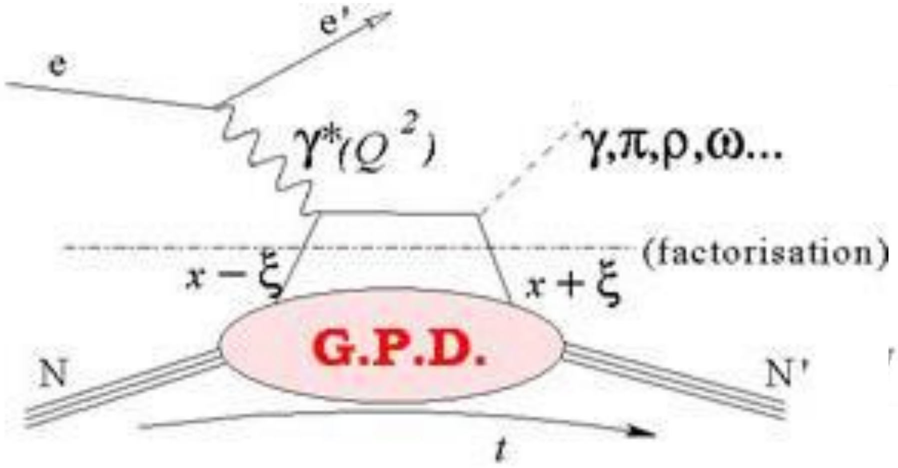
$$\int dx \int \frac{dy^-}{2\pi} x e^{ixP^+y^-} \langle\bar{\psi}(0)\gamma^+\psi(y^-)\rangle = \langle\bar{\psi}\gamma^+ iD^+\psi\rangle = \langle T_q^{++}\rangle$$

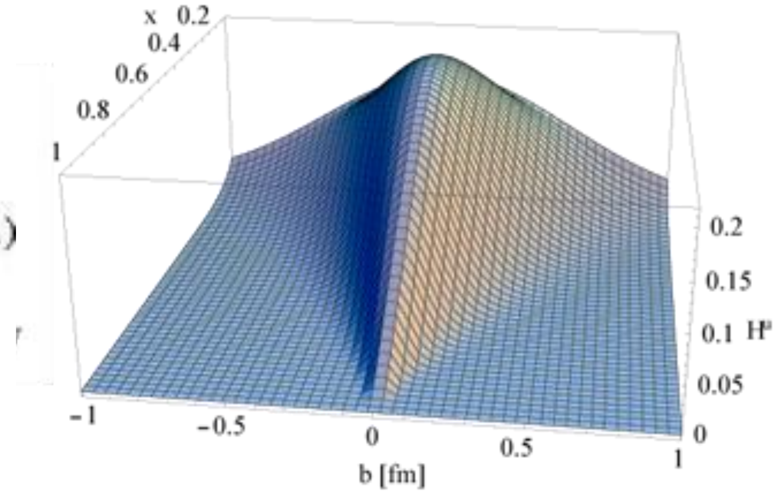
Ji sum rule

$$J^q = \frac{1}{2}\int dx x(H_q(x) + E_q(x)) \qquad J^g = \frac{1}{4}\int dx(H_g(x) + E_q(x))$$

H, E measurable in Deeply Virtual Compton Scattering (DVCS)

Deeply Virtual Compton Scattering (DVCS)





$$i\int d^4y e^{iqy}\langle P'|T\{J^\mu(y)J^\nu(0)\}|P\rangle$$

$$= -(g^{\mu+}g^{\nu-}+g^{\nu+}g^{\mu-}-g^{\mu\nu})\int\frac{dx}{2}\left(\frac{1}{x+\xi-i\epsilon}+\frac{1}{x-\xi+i\epsilon}\right)H_q(x,\eta,\Delta)\bar{u}(P')\gamma^+u(P)+\cdots$$

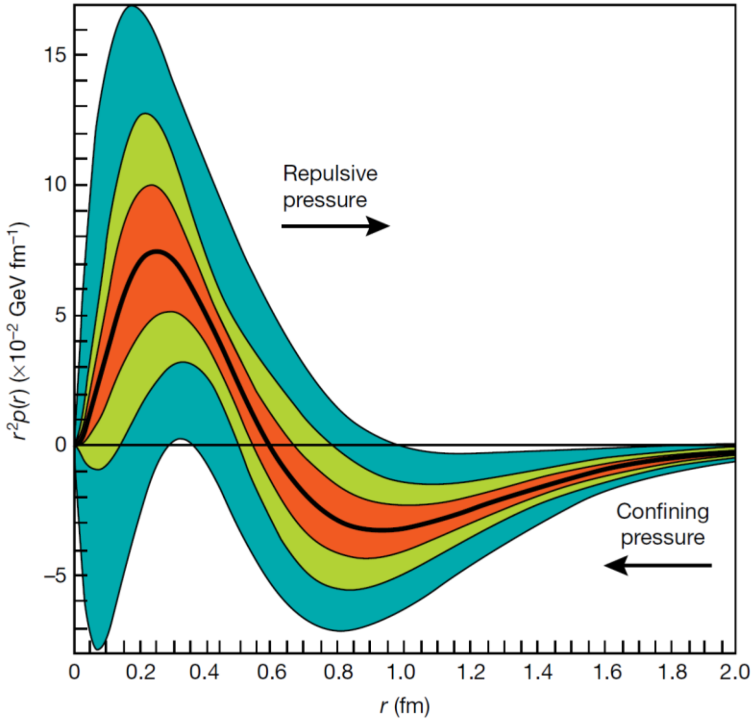
$$\xi=\frac{Q^2}{2P\cdot q}$$

Fourier transform $\Delta_\perp \leftrightarrow b_\perp$ Distribution of partons in impact parameter space

Digression: D-term—the last global unknown

$$\langle P'|T^{ij}|P\rangle \sim (\Delta^i\Delta^k - \delta^{ik}\Delta^2)D(t)$$

Burkert, Elouadrhiri, Girod (Nature, 2018)



$D(t=0)$ is a conserved charge of the nucleon, just like mass and spin!

Fourier transform $\vec{\Delta} \to \vec{r}$
can be interpreted as `radial force'
inside a nucleon   Polyakov, Schweitzer,…

$$T^{ij}(r) = \left(\frac{r^i r^j}{r^2} - \frac{1}{3}\delta^{ij}\right) s(r) + \delta^{ij} p(r)$$

First extraction at Jlab, large model dependence.
Need significant lever-arm in Q^2 to disentangle various moments of GPDs

➡ EIC

Two spin communities divided

measured by PHENIX, STAR, COMPASS, HERMES

Nonexistent in Ji's scheme…

Jaffe-Manohar

$$\frac{1}{2} = \frac{1}{2}\Delta\Sigma + \Delta G + L^q_{can} + L^g_{can}$$

common and well-known

not measured yet
not even well-defined?

Ji

$$\frac{1}{2} = \frac{1}{2}\Delta\Sigma + L^q_{\mathrm{Ji}} + J^g$$

Define rigorously.
Must be related to GPD!

accessible from GPD measured at JLab, COMPASS, HERMES, J-PARC
also calculable in lattice QCD

Gauge invariant completion of JM decomposition

For the gluon helicity, we know how to make it gauge invariant.

Compare $$\frac{1}{2S^+}\langle PS|\epsilon^{ij}F^{i+}A^j|PS\rangle = \Delta G = \int_0^1 dx \Delta G(x)$$ page 21

with $$\Delta G(x) = \frac{i}{xS^+}\int \frac{dy^-}{2\pi} e^{ixP^+y^-}\langle PS|F^{+\alpha}(0)\tilde{F}^+_{\alpha}(y^-)|PS\rangle$$ page 17

Just replace $A^\mu \to A^\mu_{phys}$ where

$$A^\mu_{phys}(x) = \frac{1}{D^+}F^{+\mu} = \int_{x^-}^{\infty} dz^- W[x^-, z^-]F^{+\mu}(z^-, x_\perp)$$

Gauge invariant completion of JM decomposition

Chen, Lu, Sun, Wang, Goldman (2008)
YH (2011)

$$\langle PS|\epsilon^{ij}F^{i+}A^{j}_{phys}|PS\rangle = 2S^{+}\Delta G$$

$$\lim_{\Delta\to 0}\langle P'S|\bar{\psi}\gamma^{+}i\overleftrightarrow{D}^{i}_{pure}\psi|PS\rangle = iS^{+}\epsilon^{ij}\Delta_{\perp j}L^{q}_{can}$$

$$\lim_{\Delta\to 0}\langle P'S|F^{+\alpha}\overleftrightarrow{D}^{i}_{pure}A^{phys}_{\alpha}|PS\rangle = -i\epsilon^{ij}\Delta_{\perp j}S^{+}L^{g}_{can}$$

where
(my choice)

$$A^{\mu}_{phys}(x) = \frac{1}{D^{+}}F^{+\mu} = \int_{x^{-}}^{\infty} dz^{-}W[x^{-},z^{-}]F^{+\mu}(z^{-},x_{\perp})$$

$$D^{\mu}_{pure} = D^{\mu} - iA^{\mu}_{phys}$$

Lecture 2: OAM and Wigner distribution

Wigner distribution in QM

Uncertainty principle: The position q and momentum p are not simultaneously measured.

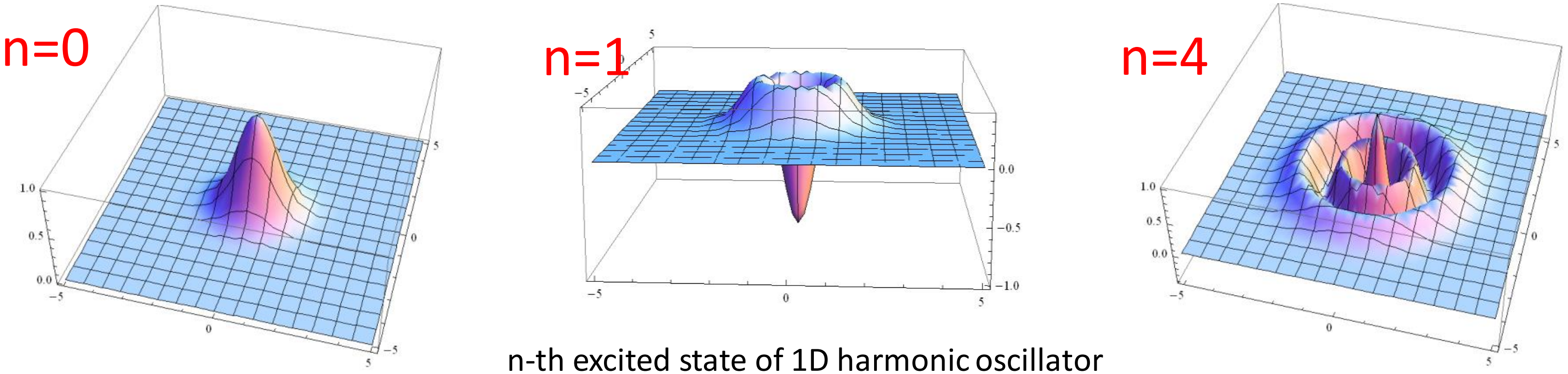
Still one can define a `phase space distribution’ in quantum mechanics

$$f_W(q,p) = \int dx e^{-ipx/\hbar} \langle \psi | q - x/2 \rangle \langle q + x/2 | \psi \rangle$$

Reduces to q and p distributions upon integration

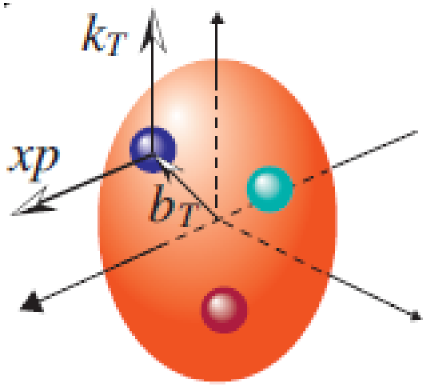
$$\int \frac{dq}{2\pi\hbar} f_W(q,p) = |\langle \psi | p \rangle|^2 \,, \qquad \int \frac{dp}{2\pi\hbar} f_W(q,p) = |\langle \psi | q \rangle|^2 \,.$$

Not positive definite, no probabilistic interpretation



n-th excited state of 1D harmonic oscillator

The QCD Wigner distribution



Phase space distribution of partons in QCD—the `mother distribution’

Belitsky, Ji, Yuan (2004)

$$W(x,\vec{k}_\perp,\vec{b}_\perp)$$
$$=\int\frac{d^2\Delta_\perp}{(2\pi)^2}\frac{dz^-d^2z_\perp}{16\pi^3}e^{ixP^+z^--i\vec{k}_\perp\cdot\vec{z}_\perp}\langle P-\frac{\Delta}{2}|\bar{q}(b-z/2)\gamma^+q(b+z/2)|P+\frac{\Delta}{2}\rangle$$

$$\begin{tikzcd}
 & W(x,\vec{k}_\perp,\vec{b}_\perp) \arrow[dl, "\int d\vec{b}_\perp"'] \arrow[dr, "\int d\vec{k}_\perp"] & & \\
\text{TMD}\quad f(x,\vec{k}_\perp) \arrow[dr, "\int d\vec{k}_\perp"'] & & \text{GPD}\quad f(x,\vec{b}_\perp) \arrow[dl, "\int d\vec{b}_\perp"] \arrow[dr, "\int dx"] & \\
 & \underset{\text{PDF}}{f(x)} \arrow[dr, "\int dx"'] & & F(\vec{b}_\perp)\quad \text{Form factor} \arrow[dl, "\int d\vec{b}_\perp"] \\
 & & \underset{\text{charge}}{Q} &
\end{tikzcd}$$

OAM from the Wigner distribution

Lorce, Pasquini (2011);
YH (2011);
Lorce, Pasquini, Xiong, Yuan (2011)

Define

$$L^q = \int dx \int d^2b_\perp d^2k_\perp (\vec{b}_\perp \times \vec{k}_\perp)_z W^q(x, \vec{b}_\perp, \vec{k}_\perp)$$

Go to the momentum space $b_\perp \to \Delta_\perp$ and look for the component

$$W^{q,g} = i\frac{S^+}{P^+}\epsilon^{ij}k_\perp^i\Delta_\perp^j f^{q,g}(x,k_\perp) + \cdots$$

Then

$$L^{q,g} = \int dx \int d^2k_\perp k_\perp^2 f^{q,g}(x,k_\perp)$$

Nice, but which OAM is this??

Canonical OAM from the light-cone staple Wilson line

YH (2011)

$$\int d^2k_\perp (b_\perp \times k_\perp) W_{LC}(b_\perp, k_\perp) = \langle \bar{\psi} b_\perp \times i D_\perp^{pure} \psi \rangle$$

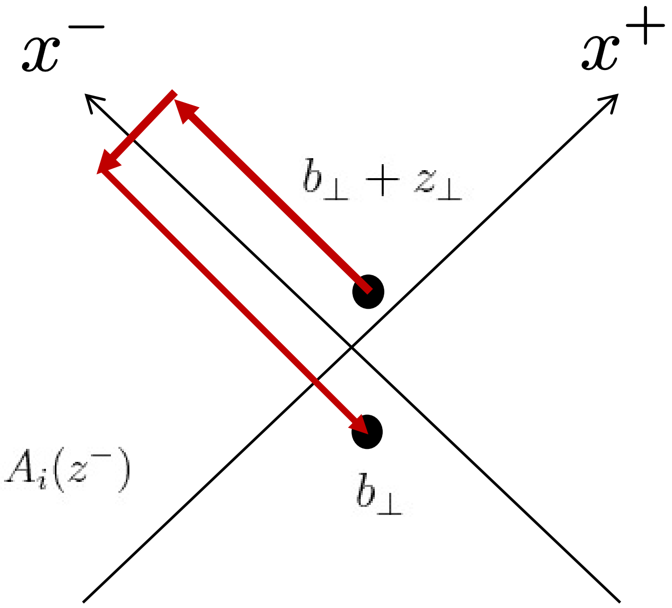
$$D_{pure}^\perp = D^\perp - \frac{i}{D^+} F^{+\perp}$$

$$W_{LC}(b_\perp, k_\perp) = \int d^2\Delta_\perp d^2 z_\perp e^{-ik_\perp z_\perp} \langle P' | \bar{\psi}(b_\perp) \gamma^+ W_{staple} \psi(b_\perp + z_\perp) | P \rangle$$

Quick proof: $k_\perp^i \to -i \frac{\partial}{\partial z_\perp^i}$

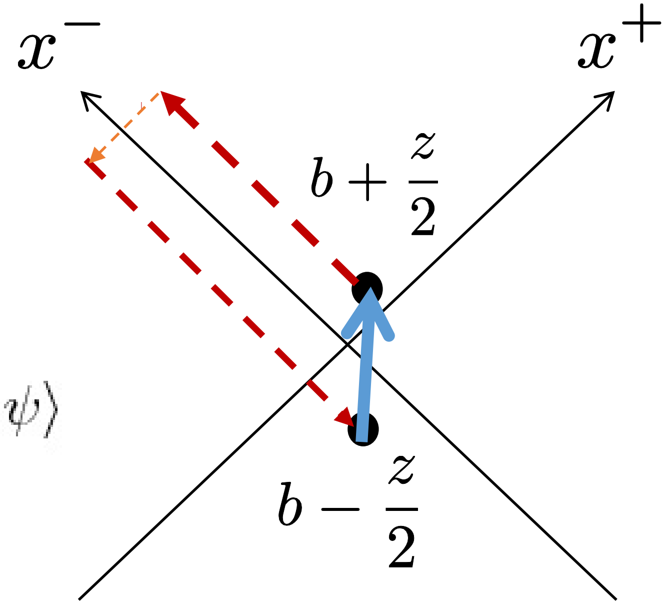
$$\frac{\partial}{\partial z_\perp^i} W[\infty, z^-, z_\perp] = -ig \int dx^- W[\infty, x^-] \underbrace{\frac{\partial}{\partial z_\perp^i} A^+(x^-)}_{F^{+i} - D^+ A^i} W[x^-, z^-]$$

$$= -ig \int dx^- W[\infty, x^-] F^{+i}(x^-) W[x^-, z^-] + igW[\infty, z^-] A_i(z^-)$$



Kinetic (Ji's) OAM from the straight Wilson line

Ji, Xiong, Yuan (2012)



$$\int d^2k_\perp (b_\perp \times k_\perp) W_{straight}(b_\perp, k_\perp) = \langle \bar{\psi} b_\perp \times iD_\perp \psi \rangle$$

The difference: Potential OAM

$$L_{pot} \equiv L^q_{\rm Ji} - L^q_{can} = \int dx^- \langle \epsilon^{ij} b^i F^{+j} \rangle$$

torque acting on a quark

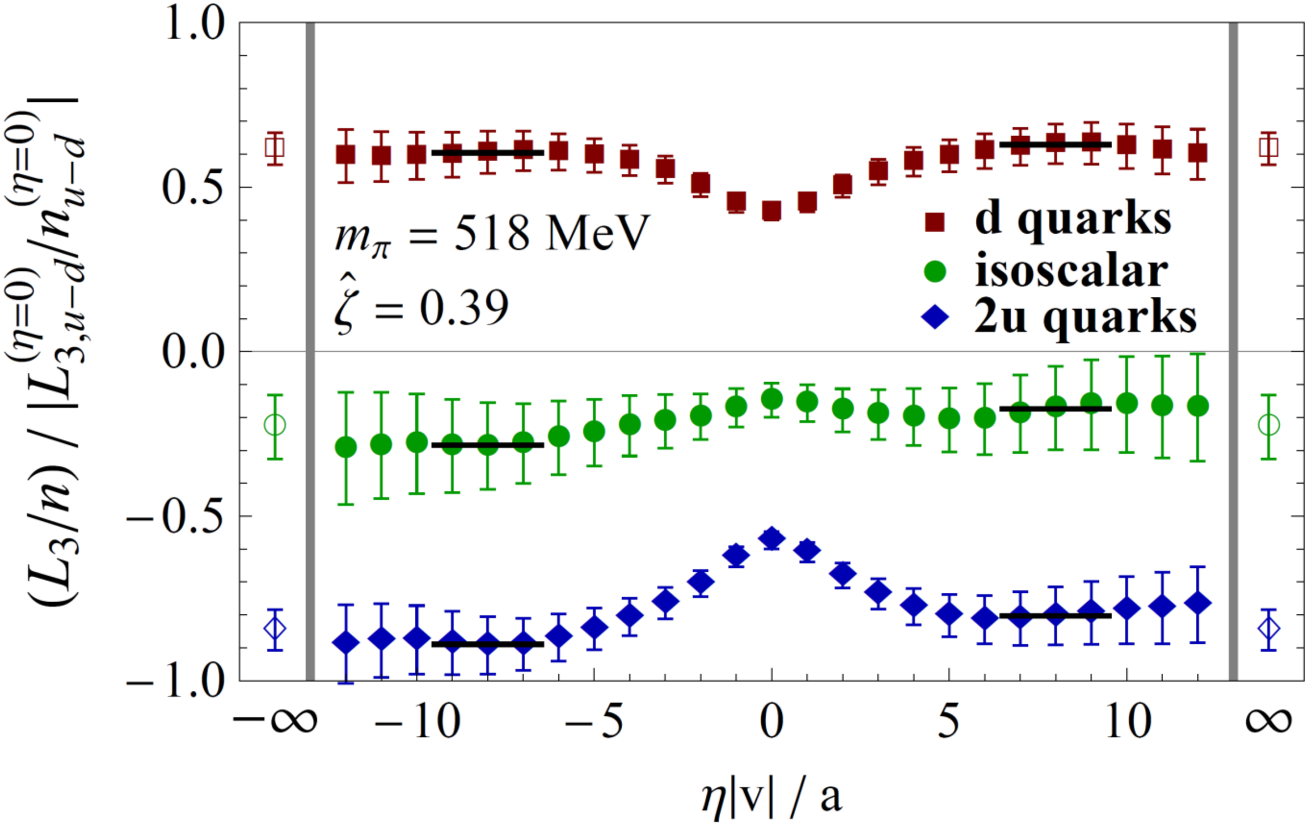
Burkardt (2012)

$$\sqrt{2}F^{+y} = -E^y + B^x = -(\vec{E} + \vec{v} \times \vec{B})^y \qquad \vec{v} = (0,0,-1)$$

Color Lorentz force

Jaffe-Manohar vs. Ji: First lattice result

Engelhardt (2017)



Staple length

Parton distribution for OAM

Define the x-distribution $L_{can} = \int dx L_{can}(x)$.

Hagler, Schafer (1998)
Harindranath, Kundu (1999)
YH, Yoshida (2012)

$$L^q_{can} = \int dx \int d^2b_\perp d^2k_\perp (\vec{b}_\perp \times \vec{k}_\perp)_z W^q(x, \vec{b}_\perp, \vec{k}_\perp)$$

➡ $$L^q_{can}(x) = \int d^2b_\perp d^2k_\perp (\vec{b}_\perp \times \vec{k}_\perp)_z W^q(x, \vec{b}_\perp, \vec{k}_\perp)$$??

It’s not an ordinary (twist-two) parton distribution function.

Must contain the twist-three part because the relevant operator is $\sim \bar{\psi}\gamma^+\vec{\partial}_\perp\psi$

Similar to $g_2(x)$ (page 13)

Deconstructing OAM

Ji’s OAM canonical OAM `potential OAM’

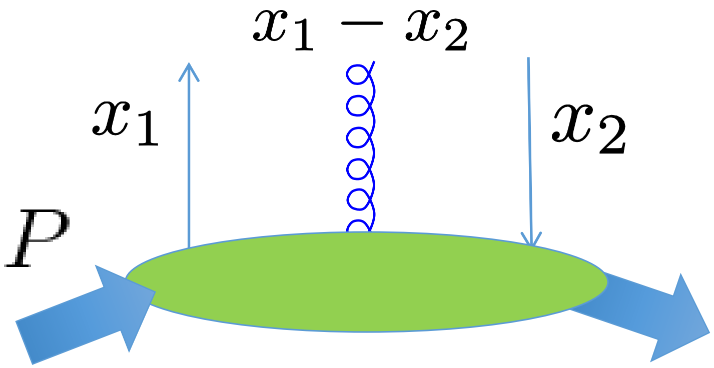
$$\langle \bar{\psi}\vec{b}\times\vec{D}\psi\rangle = \langle \bar{\psi}\vec{b}\times\vec{D}_{pure}\psi\rangle + \langle \bar{\psi}\vec{b}\times ig\vec{A}_{phys}\psi\rangle$$

$$A^{\mu}_{phys} = \frac{1}{D^{+}}F^{+\mu}$$

For a 3-body operator, it is natural to define the double density.

$$\int dz^- dy^- e^{i\frac{1}{2}(x_1+x_2)P^+y^- + i(x_1-x_2)P^+z^-} \langle P'S'|\bar{\psi}(-y^-/2) W D^i(z^-) W \psi(y^-/2)|PS\rangle$$

$$= \epsilon^{ij}\Delta_j S^+ \Phi_D(x_1,x_2)$$



Ji's OAM canonical OAM `potential OAM'

$$\langle \bar{\psi} \vec{b} \times \vec{D} \psi \rangle = \langle \bar{\psi} \vec{b} \times \vec{D}_{pure} \psi \rangle + \langle \bar{\psi} \vec{b} \times ig\vec{A}_{phys} \psi \rangle$$

doubly-unintegrate

$$\Phi_D(x_1, x_2) = \delta(x_1 - x_2) L^q_{can}(x_1) + \mathcal{P}\frac{1}{x_1 - x_2}\Phi_F(x_1, x_2)$$

Canonical OAM density

YH, Yoshida (2012)

It coincides with $L_{can}(x)$ defined via the Wigner distribution (page 39)

$$W[x^-, z^-]D^i(z^-)W[z^-, y^-]$$
$$= \underline{W[x^-, y^-]D^i(y^-)} + \frac{i}{P^+}\int_{y^-}^{z^-} dw^- W[x^-, w^-]gF^{+i}(w^-)W[w^-, y^-]$$

No z^- dependence,

→ $\delta(x_1 - x_2)$

Quark canonical OAM density

YH, Yoshida (2012)

Wandzura-Wilczek part

(part that is related to the twist-2 distribution)

$$L^q_{can}(x) = x\int_x^{\epsilon(x)} \frac{dx'}{x'}(H_q(x') + E_q(x')) - x\int_x^{\epsilon(x)} \frac{dx'}{x'^2}\Delta q(x')$$

$$-x\int_x^{\epsilon(x)} dx_1 \int_{-1}^{1} dx_2 \Phi_F(x_1, x_2) \mathcal{P}\frac{3x_1 - x_2}{x_1^2(x_1 - x_2)^2}$$

$$-x\int_x^{\epsilon(x)} dx_1 \int_{-1}^{1} dx_2 \tilde{\Phi}_F(x_1, x_2) \mathcal{P}\frac{1}{x_1^2(x_1 - x_2)}\,.$$

genuine twist-three part

$\langle \bar{\psi} F^{+i} \psi \rangle$

First moment: $J^q = \frac{1}{2}\Delta\Sigma + L^q_{can} + L_{pot}$

The bridge between JM and Ji

Gluon canonical OAM density

WW part

$$
\begin{aligned}
L^g_{can}(x) = {} & \frac{x}{2}\int_x^{\epsilon(x)}\frac{dx'}{x'^2}(H_g(x')+E_g(x')) - x\int_x^{\epsilon(x)}\frac{dx'}{x'^2}\Delta G(x') \\
& +2x\int_x^{\epsilon(x)}\frac{dx'}{x'^3}\int dX\,\Phi_F(X,x') + 2x\int_x^{\epsilon(x)}dx_1\int_{-1}^{1}dx_2\tilde{M}_F(x_1,x_2)\mathcal{P}\frac{1}{x_1^3(x_1-x_2)} \\
& +2x\int_x^{\epsilon(x)}dx_1\int_{-1}^{1}dx_2 M_F(x_1,x_2)\mathcal{P}\frac{2x_1-x_2}{x_1^3(x_1-x_2)^2}
\end{aligned}
$$

genuine twist-three

$$\langle F^{+i}F^{+j}F^{+}_{\ i}\rangle$$

first moment: $J^g + L_{pot} = \Delta G + L^g_{can}$

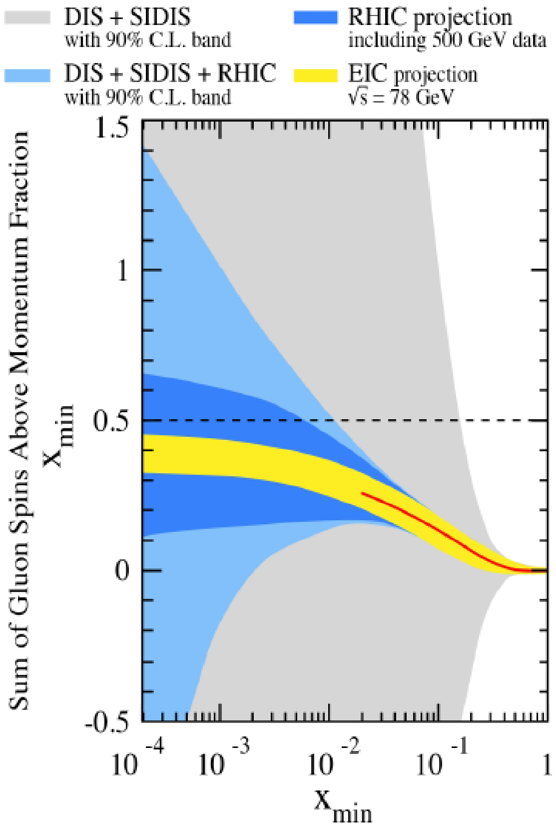
OAM challenges

Definition OK
A lot of theory progress in recent years.

BUT

Is it numerically important?
Huge uncertainty in ΔG in the small-x region,
can easily accommodate the missing spin.

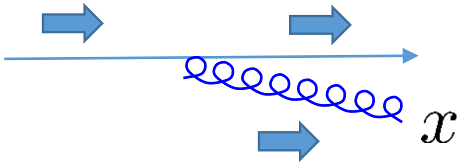
Is it measurable? Nothing known from experiments…
People tend to avoid talking about OAM.

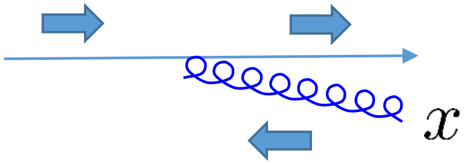


$$\int_{x_{min}}^{1} dx \Delta G(x)$$

Spin at small-x?

Consider q→ qg splitting





unpolarized splitting function

$$P_{gq}(x) = C_F \left(\frac{1}{x} + \frac{(1-x)^2}{x} \right)$$

$1/x$ enhancement (soft divergence)

polarized splitting function

$$\Delta P_{gq}(x) = C_F \left(\frac{1}{x} - \frac{(1-x)^2}{x} \right) = C_F(2-x)$$

No $1/x$ enhancement,
spin effects are always suppressed by

$$x \sim (\text{energy})^{-1}$$

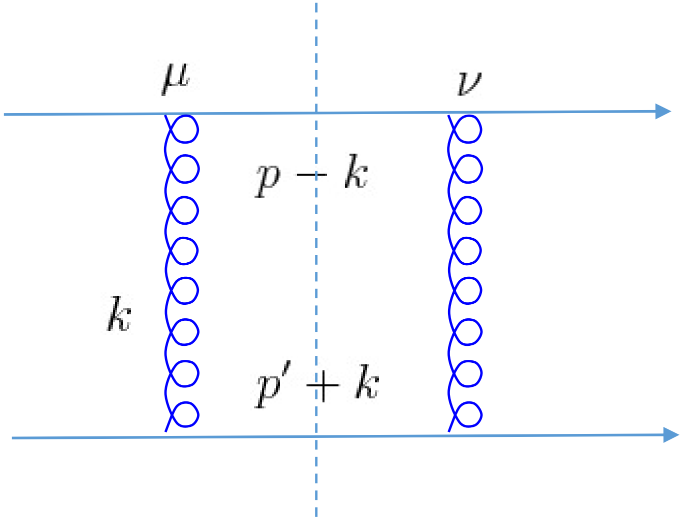
HOWEVER, they can be enhanced by double logarithms

$$(\alpha_s \ln^2 1/x)^n$$

Kirshner, Lipatov (1983)
Bartels, Ermolaev, Ryskin (1996),
Kovchegov, Pitonyak, Sievert (2015~)

Resummation very tough, but can be done!

Unpolarized

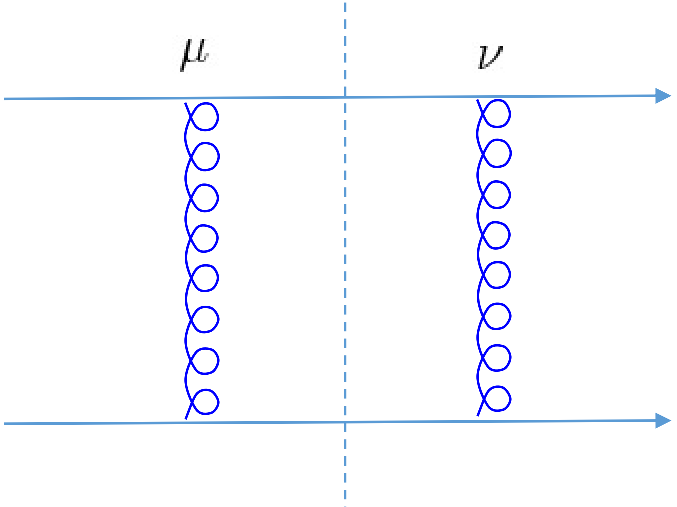


$$\mathrm{Tr}\,\slashed{p}\gamma^\mu(\slashed{p}-\slashed{k})\gamma^\nu \approx 8p^\mu p^\nu$$

$$g^4\frac{(p\cdot p')^2}{(k^2)^2} \sim \alpha_s^2\frac{s^2}{k_\perp^4}$$

Neglect k in the numerator
→ Eikonal approximation

Polarized



$$u(PS)\bar{u}(PS) = \frac{1}{2}(\slashed{p} - \gamma_5\slashed{S})$$

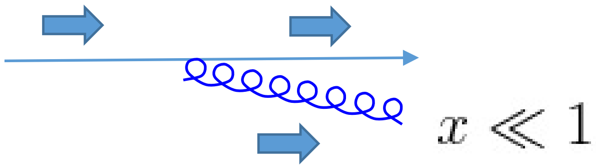
$$\mathrm{Tr}\gamma_5\slashed{S}\gamma^\mu(\slashed{p}-\slashed{k})\gamma^\nu \approx -4i\epsilon^{-\mu i\nu}S^+k_i$$

$$g^4\frac{p\cdot p' k_\perp^2}{(k^2)^2} \sim \alpha_s^2\frac{s}{k_\perp^2}$$

Either μ or ν is transverse (sub-eikonal)
$d^2k_\perp$ integral logarithmic

OAM at small-x

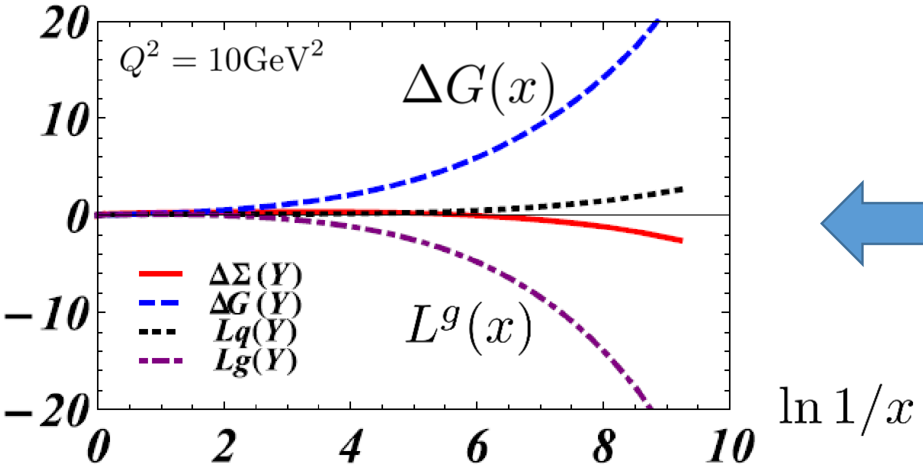
Suppose a quark emits a very soft gluon.
Quark helicity unchanged.



From angular momentum conservation, gluon spin and OAM have to cancel.

$$\frac{\partial}{\partial \ln Q^2}\Delta G(x) \sim \frac{\alpha_s}{2\pi}\int_x^1 \frac{dz}{z}(+2C_F + \cdots)\Delta\Sigma(x/z) + \cdots$$

$$\frac{\partial}{\partial \ln Q^2}L_g(x) \sim \frac{\alpha_s}{2\pi}\int_x^1 \frac{dz}{z}(-2C_F + \cdots)\Delta\Sigma(x/z) + \cdots$$

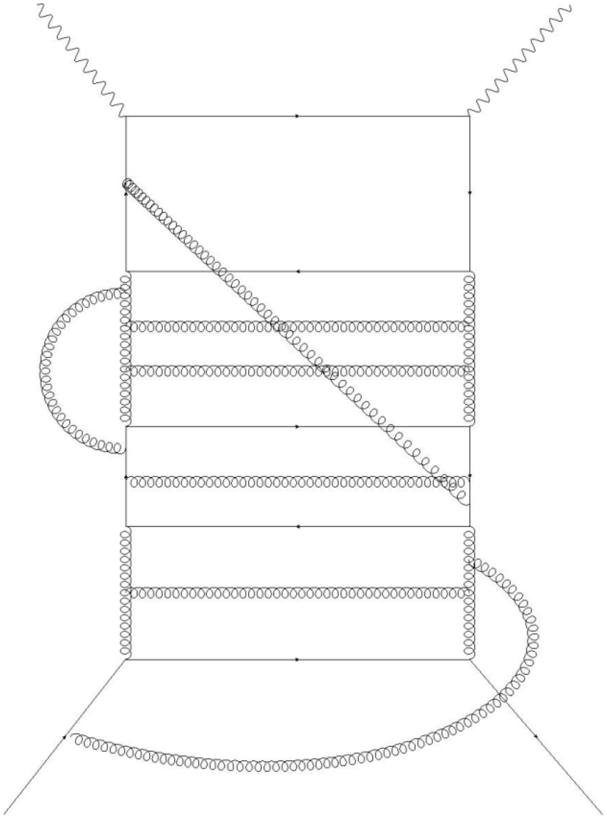


Significant cancellation at small-x from one-loop DGLAP
YH, Yang (2018)

Double logarithmic resummation

All-loop resummation of small-x double logarithms $(\alpha_s \ln^2 1/x)^n$ via InfraRed Evolution Equation

Kirshner, Lipatov (1983)
Bartels, Ermolaev, Ryskin (1996),



$$L_g(x) \approx -\frac{2}{1+\alpha}\Delta G(x) \sim \frac{1}{x^\alpha}$$

Boussarie, YH, Yuan (2019)

Helicity at small-x is more than canceled by OAM.

Resolution of the spin puzzle → OAM at medium to large x

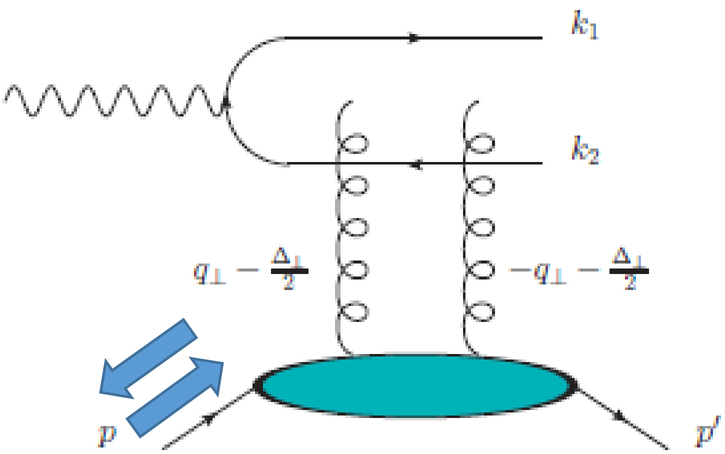
Measuring OAM at EIC

Ji, Yuan, Zhao (2016)
YH, Nakagawa, Xiao, Yuan, Zhao (2016)
Bhattacharya, Metz, Zhou (2017)

Exploit the connection between OAM and the Wigner distribution

$$L^{q,g} = \int dx \int d^2b_\perp d^2k_\perp (\vec{b}_\perp \times \vec{k}_\perp)_z W^{q,g}(x, \vec{b}_\perp, \vec{k}_\perp)$$

Longitudinal single spin asymmetry in diffractive dijet production



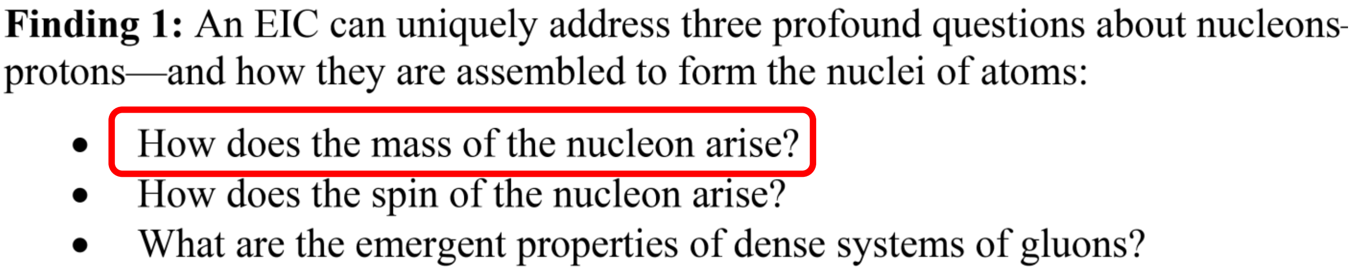
proton recoil momentum

$$\sigma^{\rightarrow} - \sigma^{\leftarrow} \propto \sin(\phi_P - \phi_\Delta)$$

dijet relative momentum

Need more work, more new ideas!

Lecture 3: Proton mass and trace anomaly

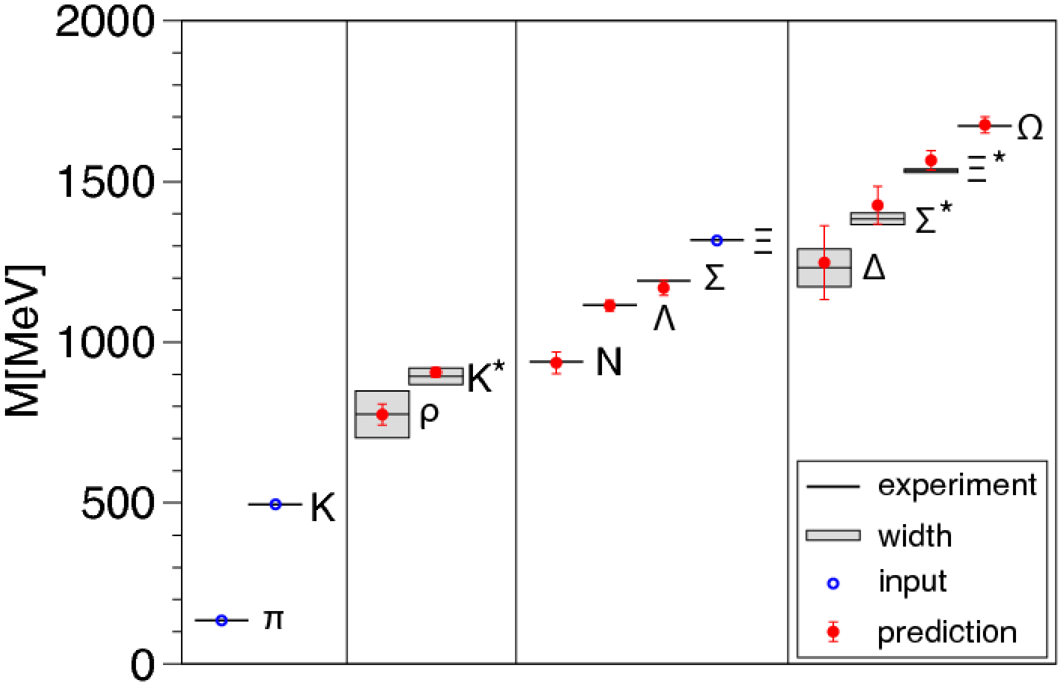


Finding 1: An EIC can uniquely address three profound questions about nucleons-protons—and how they are assembled to form the nuclei of atoms:

- How does the mass of the nucleon arise?
- How does the spin of the nucleon arise?
- What are the emergent properties of dense systems of gluons?

Nucleon mass: What's the issue?

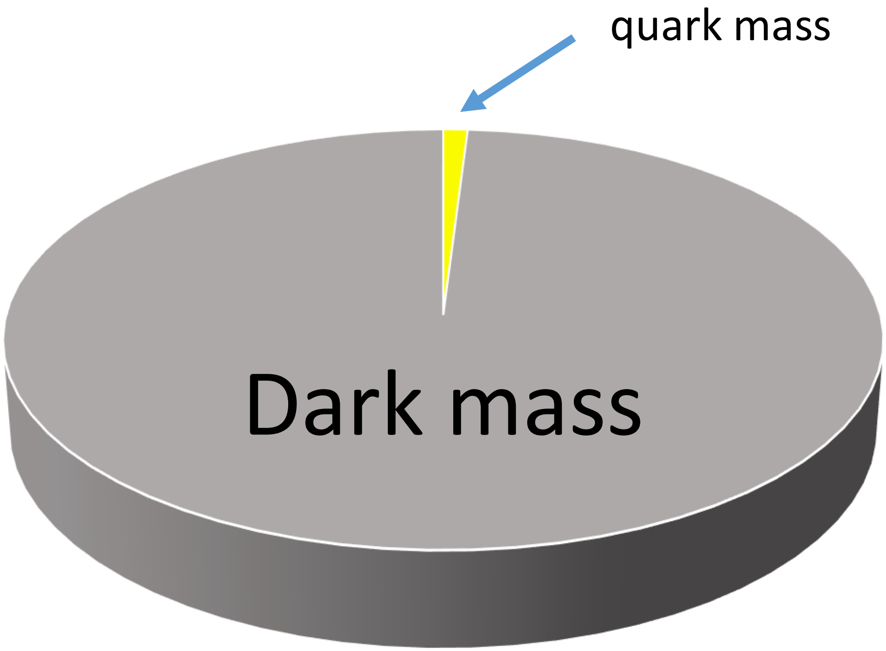
Durr, et al. (2009)



Lattice QCD can reproduce hadron masses very precisely.

Proton mass crisis

u,d quark masses add up to ~10MeV, only 1 % of the proton mass!



Higgs mechanism explains quark masses, but not hadron masses!

The trace anomaly

Approximate conformal symmetry of the Lagrangian explicitly broken by the quantum effects.

$$T^{\mu}_{\mu} = \frac{\beta(g)}{2g} F^2 + m(1 + \gamma_m(g))\bar{q}q$$

Collins, Duncan, Joglekar (1977)
N.K. Nielsen (1977)

$$\beta(g) = \frac{\partial g(\mu)}{\partial \ln \mu}$$

Beta-function

$$\gamma_m(g) = -\frac{1}{m}\frac{\partial m(\mu)}{\partial \ln \mu}$$

Mass anomalous dimension $\gamma_m > 0$

Fundamentally important in QCD. Trace anomaly is the origin of hadron masses

$$\langle P|T^{\mu\nu}|P\rangle = 2P^{\mu}P^{\nu}$$

$$\langle P|T^{\mu}_{\mu}|P\rangle = 2M^2$$

Proton mass decomposition

Ji (1995)

Traceless and trace parts of EMT

$$T^{\mu\nu} = \left(T^{\mu\nu} - \frac{\eta^{\mu\nu}}{d} T^{\alpha}_{\alpha}\right) + \frac{\eta^{\mu\nu}}{d} T^{\alpha}_{\alpha}$$

kinetic energy trace anomaly

Work in the rest frame. Mass is the eigenvalue of the Hamiltonian $H = \int d^3x T^{00}$

$$M = M_q^{kin} + M_g^{kin} + M_a + M_m$$

quark/gluon kinetic energy $M_q^{kin} = \frac{\langle P|\bar{\psi} i \vec{\gamma} \cdot \vec{D} \psi|P\rangle}{2M}$

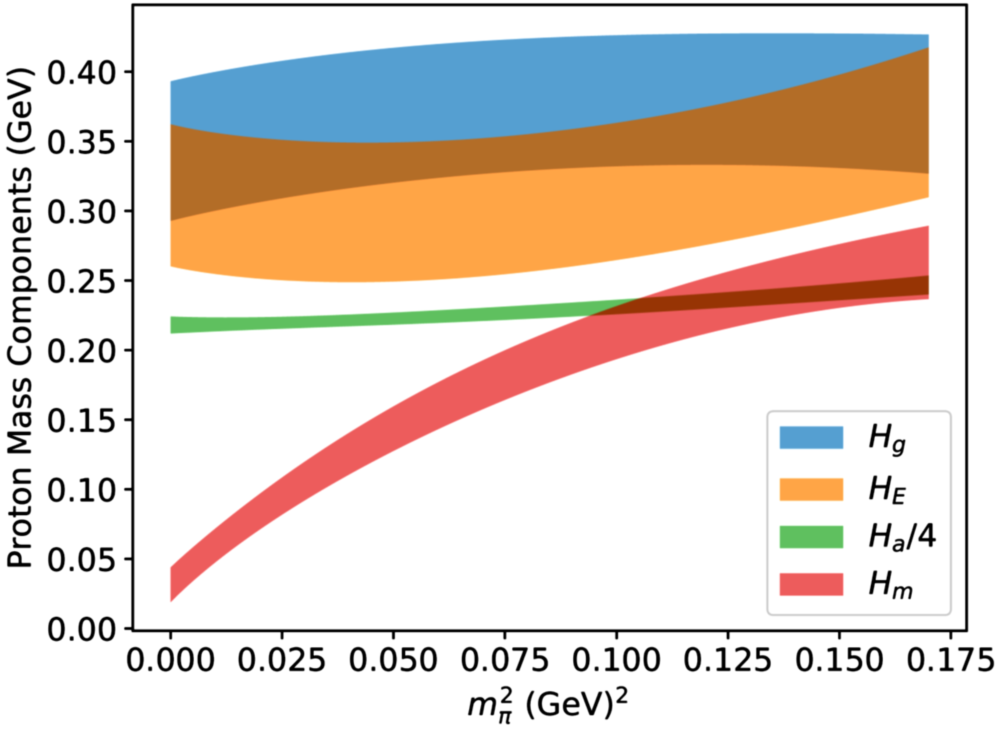
gluon condensate $M_a = \frac{\langle P|\frac{\beta}{8g} F^2|P\rangle}{2M}$

quark mass $M_m = \frac{\langle P|m(1+\frac{\gamma_m}{4})|P\rangle}{2M}$

$M_{q,g}^{kin}$ measurable in DIS

$$M_{q,g}^{kin} = \frac{3}{4} M A_{q,g} \qquad A_{q,g}(\mu) = \langle x \rangle_{q,g} = \int_0^1 dx x f_{q,g}(x,\mu)$$

$M_{q,g}^{kin}$, M_m calculable on a lattice, but M_a is very hard.



Yang, et al. (2018)
(χQCD collaboration)

The EMT consists of quark and gluon parts.

$$T^{\mu\nu} = \underbrace{-F^{\mu\lambda}F^{\nu}{}_{\lambda} + \frac{\eta^{\mu\nu}}{4}F^2}_{T_g^{\mu\nu}} + \underbrace{i\bar{q}\gamma^{(\mu}D^{\nu)}q}_{T_q^{\mu\nu}}$$

$$T^{\mu}_{\mu} = (T_q)^{\mu}_{\mu} + (T_g)^{\mu}_{\mu} = \frac{\beta}{2g}F^2 + m(1+\gamma_m)\bar{\psi}\psi$$

Exercise: Compute $(T_q)^{\mu}_{\mu}$ and $(T_g)^{\mu}_{\mu}$ separately.

Note: This is equivalent to computing the $\bar{C}_{q,g}$ gravitational form factor (cf. page 24)

$$\langle P|(T_{q,g})^{\mu}_{\mu}|P\rangle = 2M^2(A_{q,g} + 4\bar{C}_{q,g})$$

Trace anomaly in perturbation theory

First choose a regularization scheme.
e.g., dimensional regularization (DR), Pauli-Villars, etc.

Trace anomaly shows up by exploiting the pathologies of the chosen scheme.

→ The decomposition $T^\mu_\mu = (T_g)^\mu_\mu + (T_q)^\mu_\mu$ is scheme dependent.

In the following, I consider only DR in the $\overline{\mathrm{MS}}$ (modified minimal subtraction) scheme.

In $4-2\epsilon$ dimensions,

$$T^\mu_\mu = \underbrace{-2\epsilon\frac{F^2}{4}}_{\text{from } (T_g)^\mu_\mu} + \underbrace{m\bar{q}q}_{\text{from } (T_q)^\mu_\mu}$$

Operator renormalization and mixing

Under renormalization, bare fields are renormalized
$$A_R^\mu = \sqrt{Z_A} A^\mu$$
$$\psi_R = \sqrt{Z_\psi}\psi$$
etc.

Local, composite operators like $F^{\mu\nu}F_{\mu\nu}(x)$ gets additional renormalization

$$F_R^2 = Z_{F^2} F^2 + \cdots \qquad Z_{F^2} \neq (\sqrt{Z_A})^2 = Z_A$$

anomalous dimension $\gamma = -\dfrac{\partial Z_{F^2}}{\partial \ln \mu^2}$

In general, operators with the same quantum numbers and the same (or lower) dimension mix.

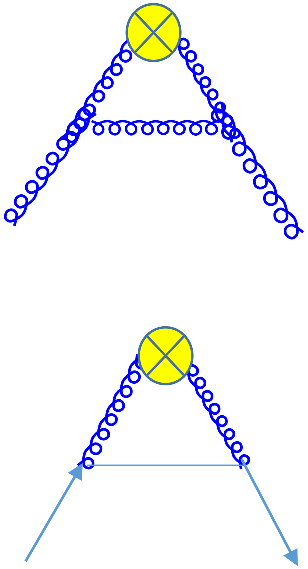
$$O_1^R = Z_{11} O_1 + Z_{12} O_2 + \ldots$$

The anomalous dimension becomes a matrix.

Renormalization of F^2

Tarrach, Nucl. Phys. B196 (1982), 45

The bare operator F^2 is divergent



$$F^2 = \left(1 + \beta_0 \frac{\alpha_s}{4\pi\epsilon}\right)(F^2)_R - \frac{2\gamma_m^0}{\epsilon}(m\bar{q}q)_R + \cdots$$

$$T^\mu_\mu = -2\epsilon \frac{F^2}{4} + m\bar{q}q$$

$$= \underbrace{\frac{\beta(g)}{2g}(F^2)_R + \gamma_m (m\bar{q}q)_R}_{\text{from } (T_g)^\mu_\mu} + \underbrace{(m\bar{q}q)_R}_{\text{from } (T_q)^\mu_\mu}$$

For the bare EMT, in DR, the anomaly entirely comes from the gluon part $T_g^{\mu\nu}$

Renormalization of the trace

RG-invariant in DR

For the bare operators,

$$(T_q)^\mu_\mu = (m\bar{q}q)_R = m\bar{q}q$$

$$(T_g)^\mu_\mu = \frac{\beta}{2g}(F^2)_R + \gamma_m (m\bar{q}q)_m = \frac{\beta}{2g}F^2 + \gamma_m m\bar{q}q$$

What about the renormalized trace operators?

$$(T_q^R(\mu))^\alpha_\alpha$$

$$(T_g^R(\mu))^\alpha_\alpha$$

$$\langle P|(T^R_{q,g}(\mu))^\alpha_\alpha|P\rangle = 2M^2(A^R_{q,g}(\mu) + 4\bar{C}^R_{q,g}(\mu))$$

Renormalized trace: naïve look

Now consider the energy momentum tensors renormalized in DR

$$T^{\mu\nu}_{gR} = -(F^{\mu\lambda}F^{\nu}{}_{\lambda})_R + \frac{\eta^{\mu\nu}}{4}(F^2)_R$$

$$T^{\mu\nu}_{qR} = i(\bar{\psi}\gamma^{\mu}D^{\nu}\psi)_R$$

$(F^2)_R$ is now a finite operator. $d \to 4$ limit can be safely taken

$$(T^R_g)^{\mu}_{\mu} = 0$$

$$(T^R_q)^{\mu}_{\mu} = (m\bar{\psi}\psi)_R$$

??

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$$(T^R_g)^{\mu}_{\mu} = 0$$

$$(T^R_q)^{\mu}_{\mu} = (m\bar{\psi}\psi)_R$$

??

This is wrong because, in DR, trace operation and renormalization do not commute

$$\eta_{\mu\nu}(F^{\mu\lambda}F^{\nu}{}_{\lambda})_R \neq (F^{\mu\lambda}F_{\mu\lambda})_R \qquad \eta_{\mu\nu}(\bar{\psi}\gamma^{\mu}D^{\nu}\psi)_R \neq (\bar{\psi}\not{D}\psi)_R$$

Renormalization → trace

$$
\begin{aligned}
\int \frac{d^dp}{(2\pi)^d}\frac{p^\mu p^\nu}{(p^2+\Delta)^2} &= \frac{g^{\mu\nu}}{d}\int \frac{d^dp}{(2\pi)^d}\frac{p^2}{(p^2+\Delta)^2} \\
&= \frac{g^{\mu\nu}}{2(4\pi)^2}\left(\frac{1}{\epsilon}-\gamma_E+\ln 4\pi-\ln\Delta\right)\Delta \\
&\to \frac{g^{\mu\nu}}{2(4\pi)^2}(-\ln\Delta)\Delta \qquad \overline{\mathrm{MS}} \\
&\to \frac{4}{2(4\pi)^2}(-\ln\Delta)\Delta \qquad \text{trace}
\end{aligned}
$$

Trace → Renormalization

$$
\begin{aligned}
\int \frac{d^dp}{(2\pi)^d}\frac{p^2}{(p^2+\Delta)^2} &= \frac{d}{2(4\pi)^2}\left(\frac{1}{\epsilon}-\gamma_E+\ln 4\pi-\ln\Delta\right)\Delta \qquad \text{trace} \\
&= \frac{4}{2(4\pi)^2}\left(\frac{1}{\epsilon}-\frac{1}{2}-\gamma_E+\ln 4\pi-\ln\Delta\right)\Delta \\
&\to \frac{4}{2(4\pi)^2}\left(-\frac{1}{2}-\ln\Delta\right)\Delta \qquad \overline{\mathrm{MS}}
\end{aligned}
$$

Renormalized trace $(T^R_{q,g})^\alpha_\alpha$: calculation

Choose the basis of operators

$$
\begin{aligned}
O_1 &= -F^{\mu\lambda}F^{\nu}{}_{\lambda},\\
O_2 &= \eta^{\mu\nu}F^2,\\
O_3 &= i\bar{\psi}\gamma^{(\mu}\overleftrightarrow{D}^{\nu)}\psi,\\
O_4 &= \eta^{\mu\nu}m\bar{\psi}\psi.
\end{aligned}
$$

$$T^{\mu\nu} = O_1 + \frac{O_2}{4} + O_3.$$

Mixing under renormalization

$$
\begin{aligned}
O_1^R &= Z_T O_1 + Z_M O_2 + Z_L O_3 + Z_S O_4,\\
O_2^R &= Z_F O_2 + Z_C O_4,\\
O_3^R &= Z_\psi O_3 + Z_K O_4 + Z_Q O_1 + Z_B O_2,\\
O_4^R &= O_4.
\end{aligned}
$$

Impose two conditions. First condition is simply $T^{\mu\nu} = T^{\mu\nu}_R$

Second condition:

Make the operators twist-2 by subtracting the trace

$$A_q^R = O_3^R - (\mathrm{trace})$$
$$A_g^R = O_1^R - (\mathrm{trace})$$

They satisfy the usual RG (DGLAP) equation for the twist-2, spin-2 operators.

$$\frac{\partial}{\partial \ln \mu}\begin{pmatrix} A_q^R \\ A_g^R \end{pmatrix} = \frac{\alpha_s}{4\pi}\begin{pmatrix} -\frac{16}{3}C_F & \frac{4n_f}{3} \\ \frac{16}{3}C_F & -\frac{4n_f}{3} \end{pmatrix}\begin{pmatrix} A_q^R \\ A_g^R \end{pmatrix},$$

cf. Peskin's Eq.(18.186)

Be careful when subtracting the trace.
Renormalization and trace operation do not commute.

$$\eta_{\mu\nu}(F^{\mu\lambda}F^{\nu}{}_{\lambda})_R = x(F^2)_R + y(m\bar{\psi}\psi)_R$$

Introduce more unknown constants $x = 1 + \mathcal{O}(\alpha_s)$ $y = \mathcal{O}(\alpha_s)$

Result in $\overline{\mathrm{MS}}$ at one-loop

YH, Rajan, Tanaka, JHEP 1812 (2018) 008

$$\eta_{\mu\nu} T^{\mu\nu}_{gR} = \frac{\alpha_s}{4\pi}\left(-\frac{11C_A}{6}(F^2)_R + \frac{14C_F}{3}(m\bar{\psi}\psi)_R\right),$$

$$\eta_{\mu\nu} T^{\mu\nu}_{qR} = (m\bar{\psi}\psi)_R + \frac{\alpha_s}{4\pi}\left(\frac{n_f}{3}(F^2)_R + \frac{4C_F}{3}(m\bar{\psi}\psi)_R\right)$$

n_f term in the 1-loop beta function

$$\lim_{\mu\to\infty}(T^R_q(\mu))^\alpha_\alpha \neq (T_q)^\alpha_\alpha$$

$$\lim_{\mu\to\infty}(T^R_g(\mu))^\alpha_\alpha \neq (T_g)^\alpha_\alpha$$

Finite renormalization

Result in $\overline{\mathrm{MS}}$ at two-loops

YH, Rajan, Tanaka, JHEP 1812 (2018) 008

$$\eta_{\mu\nu}\left(T_g^{\mu\nu}\right)_R = \frac{\alpha_s}{4\pi}\left(\frac{14}{3}C_F\left(m\bar{\psi}\psi\right)_R - \frac{11}{6}C_A\left(F^2\right)_R\right) + \left(\frac{\alpha_s}{4\pi}\right)^2 \times\left[\left(C_F\left(\frac{812C_A}{27}-\frac{22n_f}{27}\right)+\frac{85C_F^2}{27}\right)\left(m\bar{\psi}\psi\right)_R + \left(\frac{28C_An_f}{27}-\frac{17C_A^2}{3}+\frac{5C_Fn_f}{54}\right)\left(F^2\right)_R\right]$$

n_f terms contributes to the gluon part

$$\eta_{\mu\nu}\left(T_q^{\mu\nu}\right)_R = \left(m\bar{\psi}\psi\right)_R + \frac{\alpha_s}{4\pi}\left(\frac{4}{3}C_F\left(m\bar{\psi}\psi\right)_R + \frac{1}{3}n_f\left(F^2\right)_R\right) + \left(\frac{\alpha_s}{4\pi}\right)^2 \times\left[\left(C_F\left(\frac{61C_A}{27}-\frac{68n_f}{27}\right)-\frac{4C_F^2}{27}\right)\left(m\bar{\psi}\psi\right)_R + \left(\frac{17C_An_f}{27}+\frac{49C_Fn_f}{54}\right)\left(F^2\right)_R\right]$$

Result in $\overline{\text{MS}}$ at three-loops

Tanaka, JHEP 1901 (2019) 120

$$
\begin{aligned}
\eta_{\mu\nu}\left(T_g^{\mu\nu}\right)_R = {} & \frac{\alpha_s}{4\pi}\left(\frac{14}{3}C_F\left(m\bar{\psi}\psi\right)_R - \frac{11}{6}C_A\left(F^2\right)_R\right) + \left(\frac{\alpha_s}{4\pi}\right)^2 \\
& \times\left[\left(C_F\left(\frac{812C_A}{27} - \frac{22n_f}{27}\right) + \frac{85C_F^2}{27}\right)\left(m\bar{\psi}\psi\right)_R + \left(\frac{28C_An_f}{27} - \frac{17C_A^2}{3} + \frac{5C_Fn_f}{54}\right)\left(F^2\right)_R\right] \\
& + \left(\frac{\alpha_s}{4\pi}\right)^3\left[\left\{n_f\left(\left(\frac{368\zeta(3)}{9} - \frac{25229}{729}\right)C_F^2 - \frac{2}{243}(4968\zeta(3) + 1423)C_AC_F\right)\right.\right. \\
& + \left(\frac{32\zeta(3)}{3} - \frac{91753}{1458}\right)C_AC_F^2 + \left(\frac{294929}{1458} - \frac{32\zeta(3)}{9}\right)C_A^2C_F - \frac{554}{243}C_Fn_f^2 \\
& \left. + \left(\frac{95041}{729} - \frac{64\zeta(3)}{9}\right)C_F^3\right\}\left(m\bar{\psi}\psi\right)_R \\
& \left. + \left\{n_f\left(\left(\frac{1123}{162} - \frac{52\zeta(3)}{9}\right)C_AC_F + \left(4\zeta(3) + \frac{293}{36}\right)C_A^2 + \frac{16}{729}(81\zeta(3) - 98)C_F^2\right) + n_f^2\left(\frac{655C_A}{2916} - \frac{361C_F}{729}\right) - \frac{2857C_A^3}{108}\right\}\left(F^2\right)_R\right]
\end{aligned}
$$

$$
\begin{aligned}
\eta_{\mu\nu}\left(T_q^{\mu\nu}\right)_R = {} & \left(m\bar{\psi}\psi\right)_R + \frac{\alpha_s}{4\pi}\left(\frac{4}{3}C_F\left(m\bar{\psi}\psi\right)_R + \frac{1}{3}n_f\left(F^2\right)_R\right) + \left(\frac{\alpha_s}{4\pi}\right)^2 \\
& \times\left[\left(C_F\left(\frac{61C_A}{27} - \frac{68n_f}{27}\right) - \frac{4C_F^2}{27}\right)\left(m\bar{\psi}\psi\right)_R + \left(\frac{17C_An_f}{27} + \frac{49C_Fn_f}{54}\right)\left(F^2\right)_R\right] \\
& + \left(\frac{\alpha_s}{4\pi}\right)^3\left[\left\{n_f\left(\left(\frac{64\zeta(3)}{9} - \frac{8305}{729}\right)C_F^2 - \frac{2}{243}(864\zeta(3) + 1079)C_AC_F\right)\right.\right. \\
& - \frac{8}{729}(972\zeta(3) + 143)C_AC_F^2 + \left(\frac{32\zeta(3)}{9} + \frac{6611}{729}\right)C_A^2C_F - \frac{76}{243}C_Fn_f^2 \\
& \left. + \frac{8}{729}(648\zeta(3) - 125)C_F^3\right\}\left(m\bar{\psi}\psi\right)_R \\
& + \left\{n_f\left(\left(\frac{52\zeta(3)}{9} - \frac{401}{324}\right)C_AC_F + \left(\frac{134}{27} - 4\zeta(3)\right)C_A^2 + \left(\frac{2407}{1458} - \frac{16\zeta(3)}{9}\right)C_F^2\right)\right. \\
& \left.\left. + n_f^2\left(-\frac{697C_A}{729} - \frac{169C_F}{1458}\right)\right\}\left(F^2\right)_R\right],
\end{aligned}
$$

Scale dependence of $\bar{C}_{q,g}(\Delta = 0)$

$$
\begin{aligned}
\bar{C}_q^R(\mu) = {} & -\frac{1}{4}\left(\frac{n_f}{4C_F+n_f}+\frac{2n_f}{3\beta_0}\right)+\frac{1}{4}\left(\frac{2n_f}{3\beta_0}+1\right)\frac{\langle P|\left(m\bar{\psi}\psi\right)_R|P\rangle}{2M^2} \\
& -\frac{4C_F A_q^R(\mu_0)+n_f\left(A_q^R(\mu_0)-1\right)}{4(4C_F+n_f)}\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\frac{8C_F+2n_f}{3\beta_0}} \\
& +\frac{\alpha_s(\mu)}{4\pi}\Bigg[\frac{n_f\left(-\frac{34C_A}{27}-\frac{49C_F}{27}\right)}{4\beta_0}+\frac{\beta_1 n_f}{6\beta_0^2} \\
& \qquad +\frac{1}{4}\left(\frac{n_f\left(\frac{34C_A}{27}+\frac{157C_F}{27}\right)}{\beta_0}+\frac{4C_F}{3}-\frac{2\beta_1 n_f}{3\beta_0^2}\right)\frac{\langle P|\left(m\bar{\psi}\psi\right)_R|P\rangle}{2M^2}\Bigg]+\cdots, \\
\simeq {} & \boxed{-0.146}-0.25\left(A_q^R(\mu_0)-0.36\right)\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\frac{50}{81}}-0.01\alpha_s(\mu) \\
& +\left(0.306+0.08\alpha_s(\mu)\right)\frac{\langle P|\left(m\bar{\psi}\psi\right)_R|P\rangle}{2M^2},
\end{aligned}
$$

Asymptotic value
in the chiral limit
$(n_f = 3)$

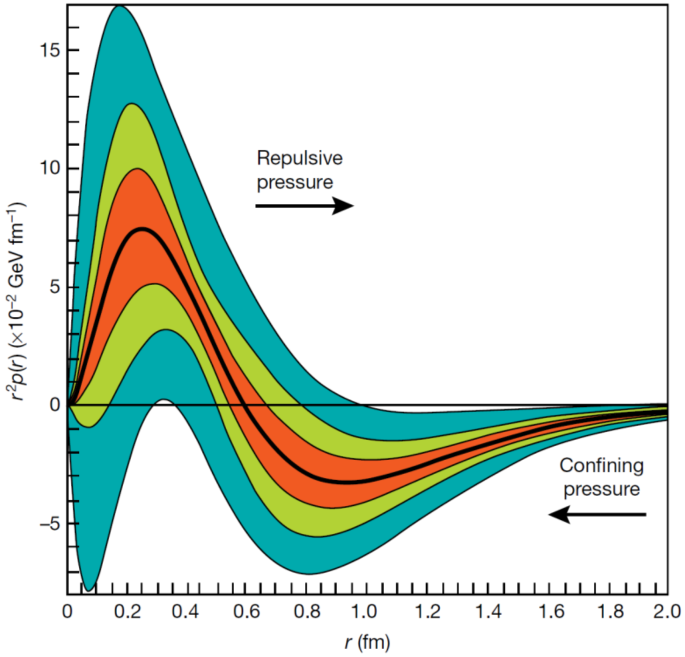
Force inside the proton from quark and gluon subsystems

Fourier transform of the D-term → radial force inside a nucleon (page 28)

One can decompose it into quark and gluon contributions   Polyakov, Schweitzer (2018)

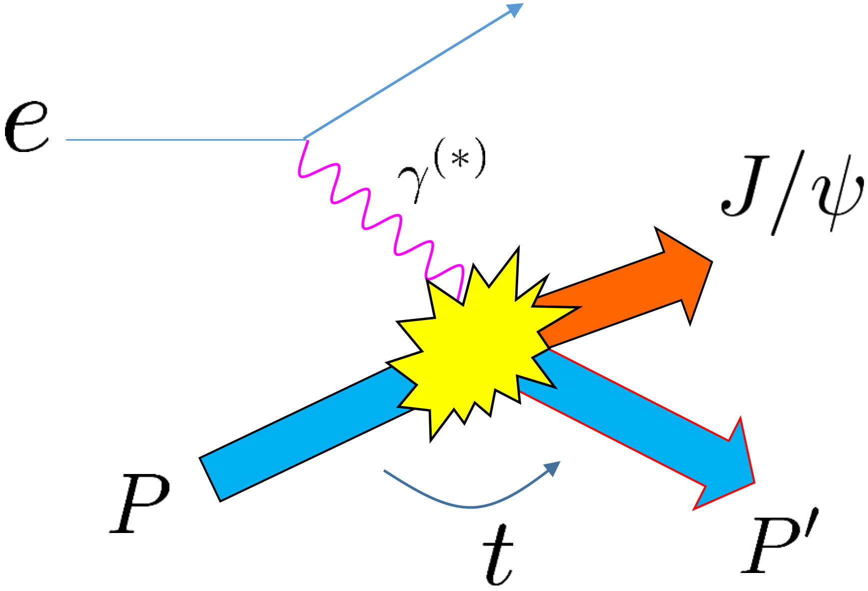
$$p_{q,g}(r) = \frac{1}{6Mr^2}\frac{d}{dr}r^2\frac{d}{dr}D_{q,g}(r) - M\bar{C}_{q,g}(r)$$

$$\bar{C}_q(\Delta = 0) = -\bar{C}_g(\Delta = 0) < 0$$



Toward measuring the gluon condensate $\langle P|F^2|P\rangle$ in experiment

Photo-production of J/ψ near threshold

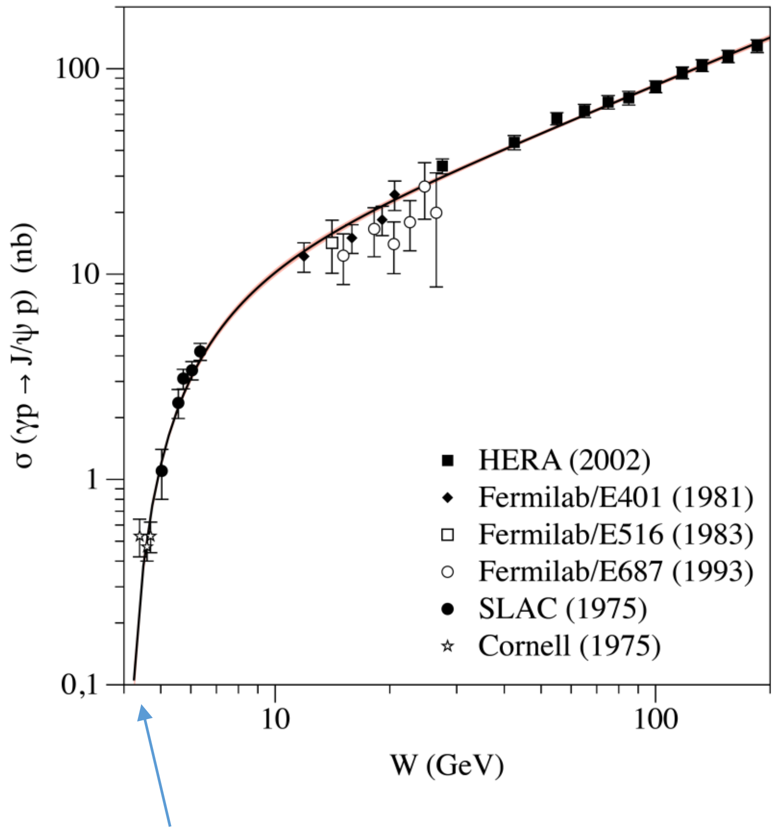


Sensitive to the non-forward matrix element $\langle P'|F^{\mu\nu}F_{\mu\nu}|P\rangle$

Straightforward to measure.
Ongoing experiments at Jlab.

Difficult to compute from first principles
(need nonperturbative approaches)

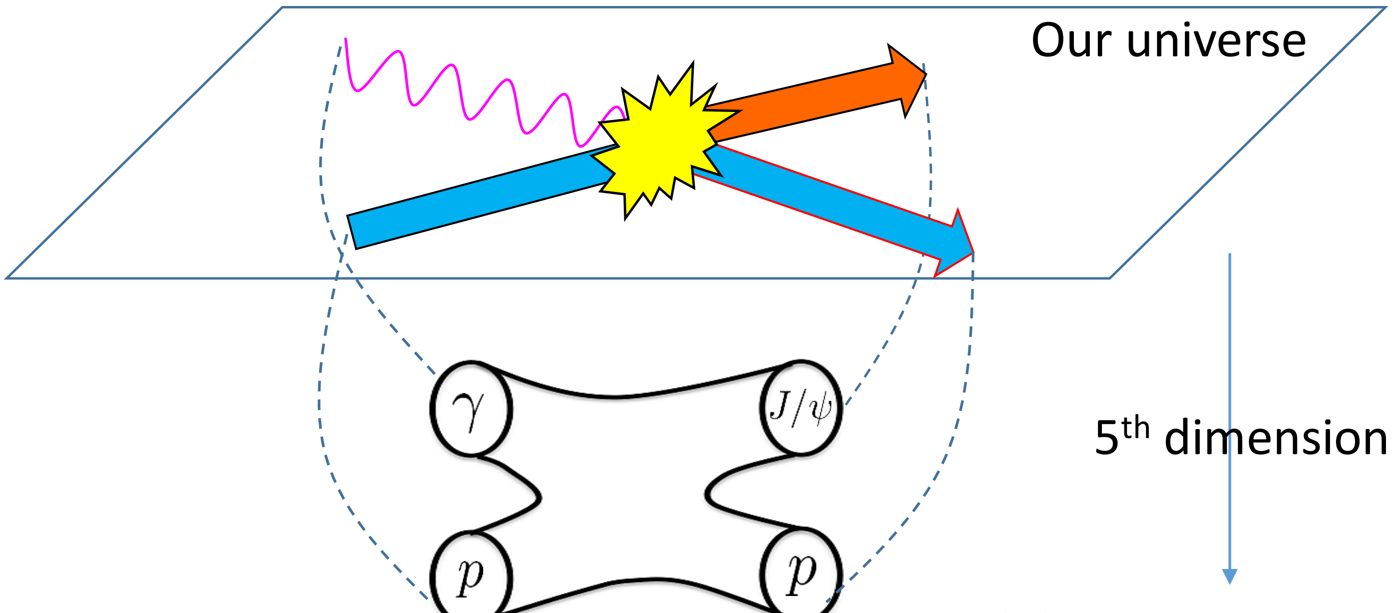
Kharzeev, Satz, Syamtomov, Zinovjev (1998)
Brodsky, Chudakov, Hoyer, Laget (2000)



$W_{th} \approx 4.04\text{GeV}$

Holographic approach

YH, Yang (2018)

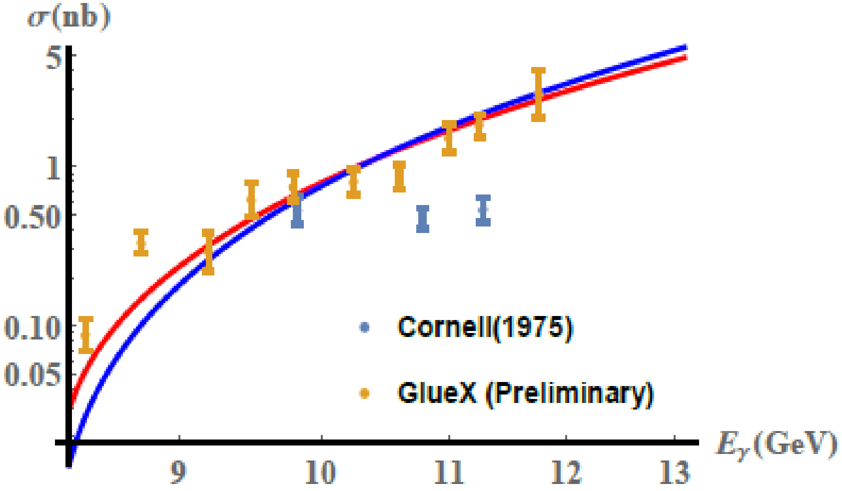


Our universe

5th dimension

The operator $F^{\mu\nu}F_{\mu\nu}$ is dual to a massless string called dilaton in AdS

Suppressed compared to graviton exchange at high energy, but not at very low energy.



Data from GlueX collaboration 1905.10811

Red: with gluon condensate. Blue: without