THE 2019 CFNS SUMMER SCHOOL ON THE PHYSICS OF THE ELECTRON ION COLLIDER

LATTICE QCD
AND
NUCLEON(US) STRUCTURE

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LECTUTE I:
LATTICE QCD FORMALISM AND METHODOLOGY

LECTUTE II:
NUCLEON STRUCTURE FROM LATTICE QCD

LECTUTE III:
TOWARDS NUCLEAR STRUCTURE FROM LATTICE QCD

LECTUTE I:
LATTICE QCD FORMALISM AND METHODOLOGY

Quantum chromodynamics (QCD) in continuum:

QCD is a SU(3) Yang-Mills theory augmented with several flavors of massive quarks:

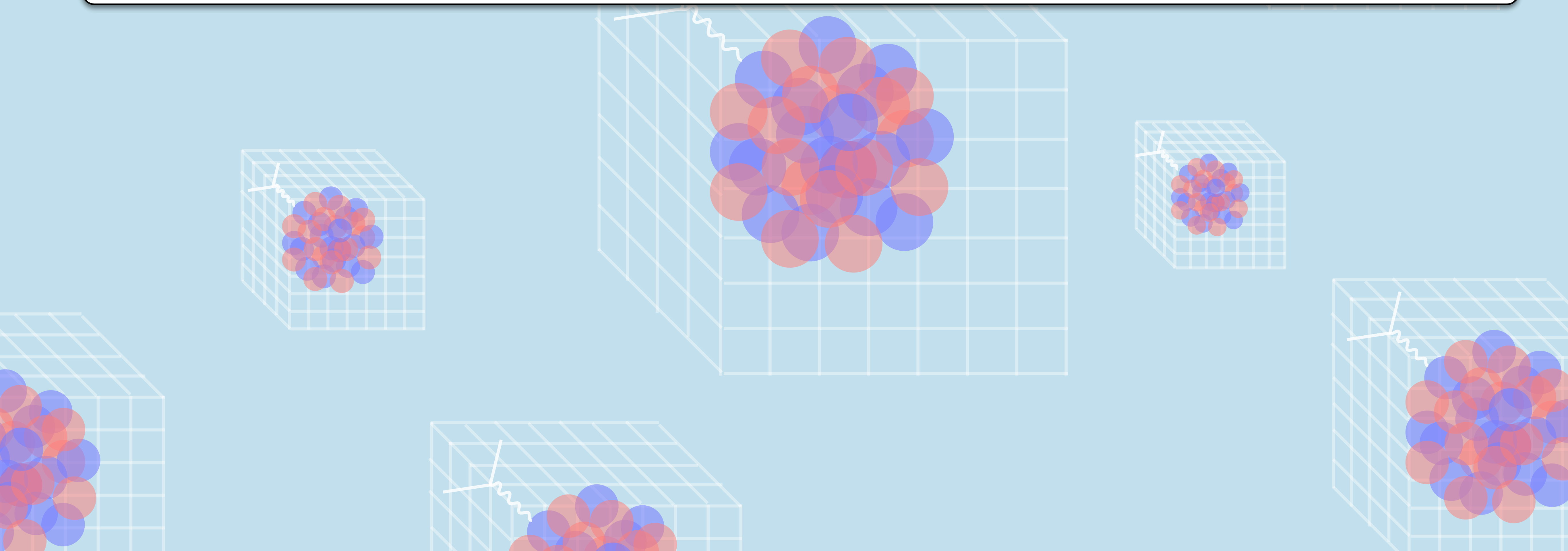
Quark kinetic and mass term

Quark/gluon interactions

$$\mathcal{L}_{QCD} = \sum_{f=1}^{N_f} \left[\bar{q}_f (i\gamma^\mu \partial_\mu - m_f) q_f - g A^i_\mu \bar{q}_f \gamma^\mu T^i q_f \right]$$

$$-\frac{1}{4} F^i_{\mu\nu} F^{i\mu\nu} + \frac{g}{2} f_{ijk} F^i_{\mu\nu} A^{i\mu} A^{j\nu} - \frac{g^2}{4} f_{ijk} f_{klm} A^j_\mu A^k_\nu A^{l\mu} A^{m\nu}$$

Gluons kinetic and interaction terms



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Gluons kinetic and interaction terms

Observe that:

i) There are only $1 + N_f$ input parameters plus QCD coupling. Fix them by a few quantities and all strongly-interacting aspects of nuclear physics is predicted (in principle)!

ii) QCD is asymptotically free such that: $\alpha_s(\mu') = \dfrac{1}{2b_0 \log \frac{\mu'}{\Lambda_{QCD}}}$

Positive constant for $N_f \leq 16$

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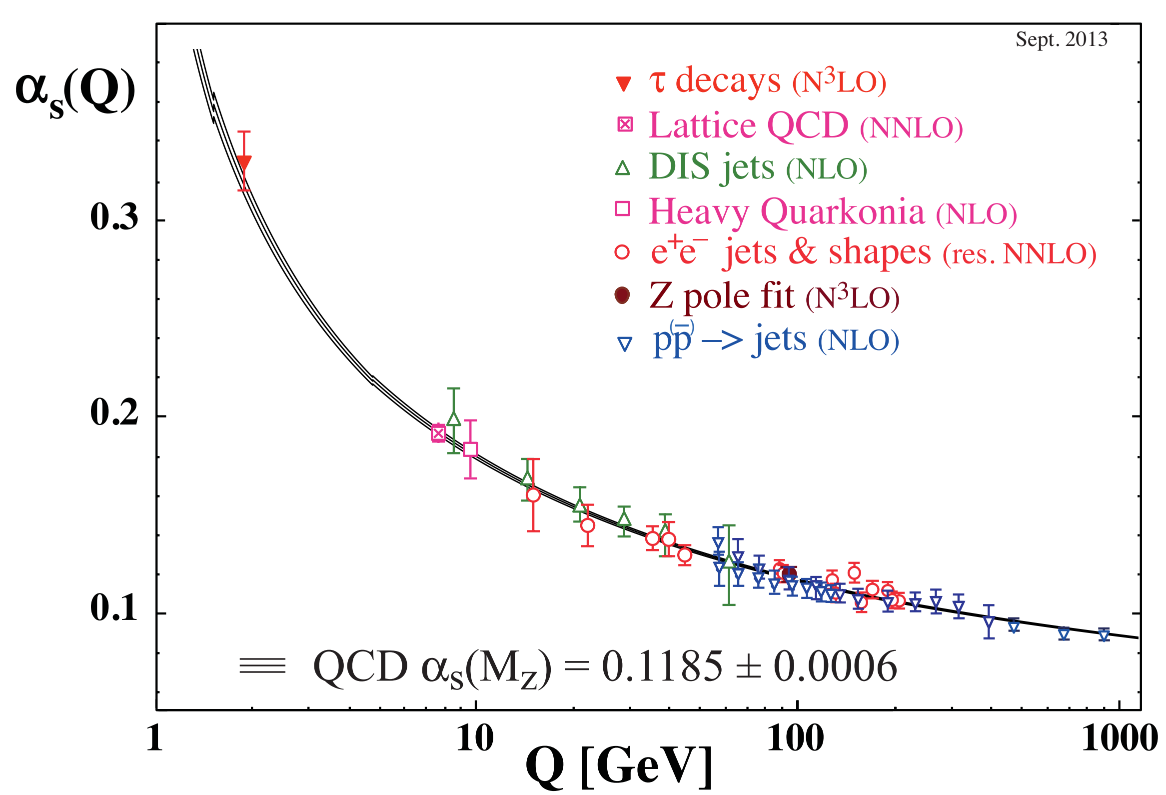

Let's enumerate the steps toward numerically simulating this theory nonperturbatively…

Step I: Discretize the QCD action in both space and time. Wick rotate to imaginary times. Consider a finite hypercubic lattice.

Step II: Generate a large sample of thermalized decorrelated vacuum configurations.

Step III: Form the correlation functions by contracting the quarks. Need to specify the interpolating operators for the state under study.

Step IV: Extract energies and matrix elements from correlation functions.

Step V: Make the connection to physical observables, such as scattering amplitudes, decay rates, etc.

`See e.g., ZD, arXiv:1409.1966 [hep-lat]`

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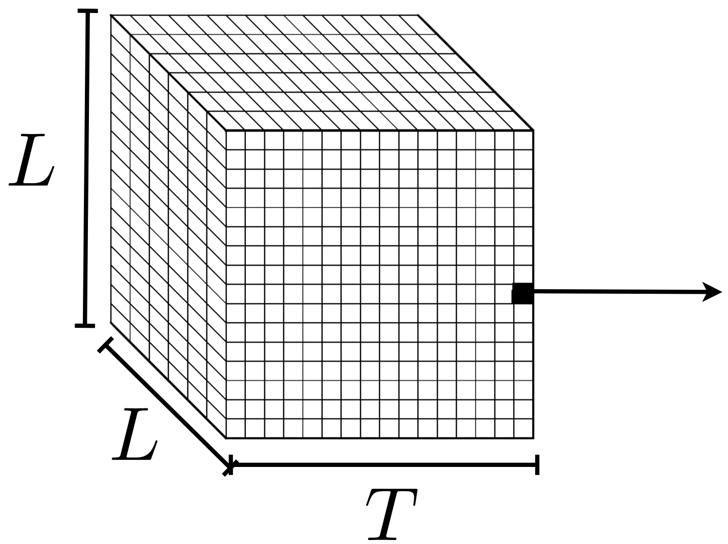

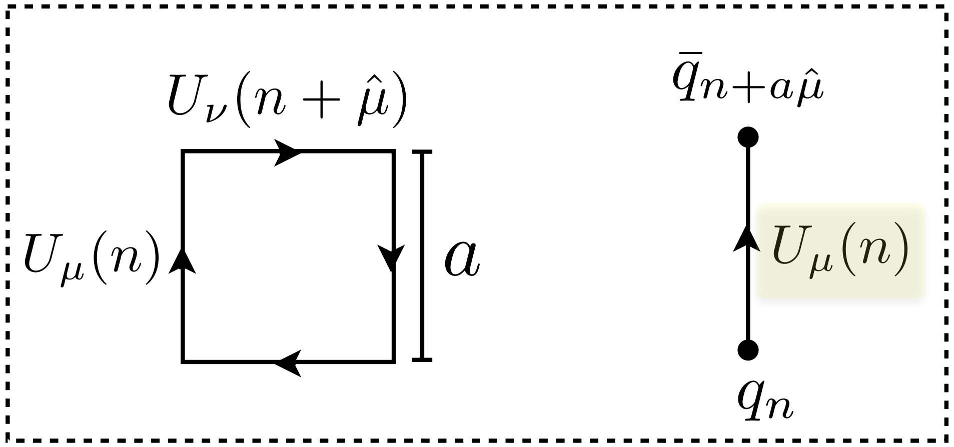

Plaquette

Link

$U_\mu(n) \equiv e^{igA_\mu(n)}$

Two conditions:

$T, L \gg m_\pi^{-1}$ $\qquad a \ll \Lambda_{QCD}^{-1}$

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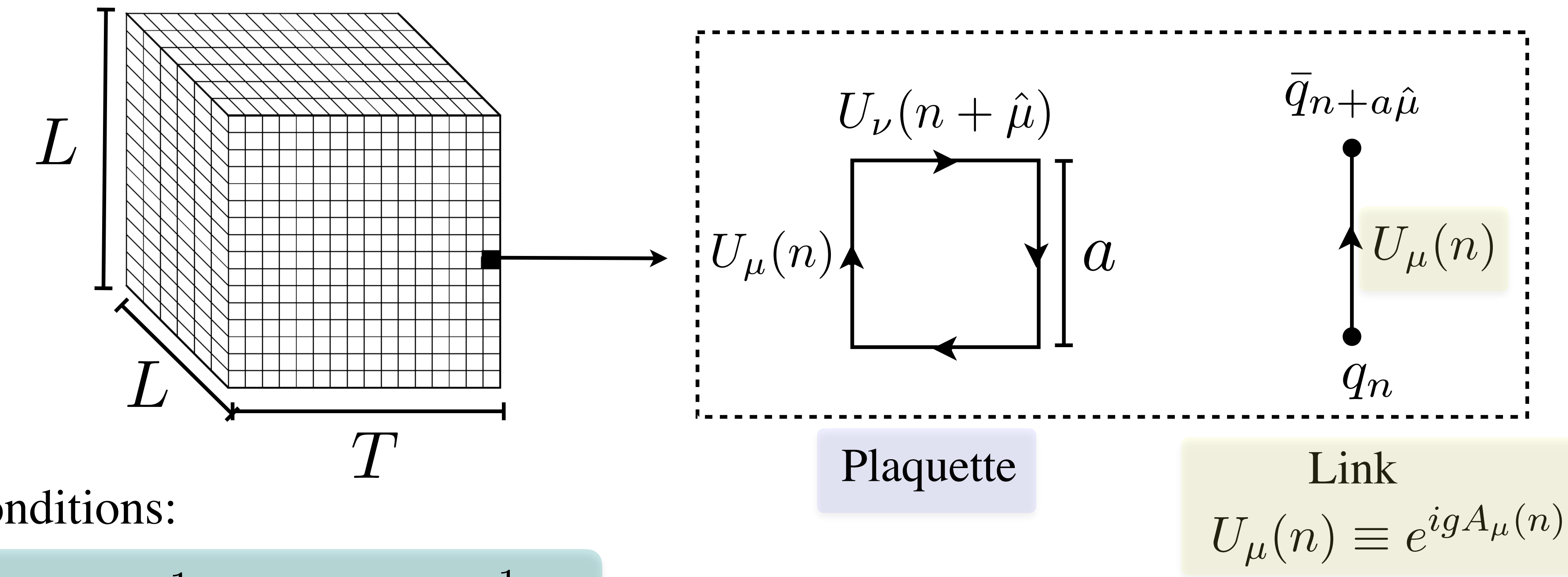

Two conditions:

$T, L \gg m_\pi^{-1}$ $\qquad$ $a \ll \Lambda_{QCD}^{-1}$

An example of a discretized action by K. Wilson:

$$S^{(E)}_{\text{Wilson}} = \frac{\beta}{N_c} \sum_n \sum_{\mu<\nu} \Re\text{Tr}[\mathbb{1} - P_{\mu\nu;n}] - \sum_n \bar{q}_n[\overline{m}^{(0)} + 4]q_n + \sum_n \sum_\mu \left[\bar{q}_n \frac{r - \gamma_\mu}{2} U_\mu(n) q_{n+\hat{\mu}} + \bar{q}_n \frac{r + \gamma_\mu}{2} U^\dagger_\mu(n-\hat{\mu}) q_{n-\hat{\mu}} \right]$$

$\beta = 2/g^2$

r: Wilson parameter. Gives the naive action if set to zero and has doublers problem.

`For discussions of actions consistent with chiral symmetry of continuum see: Kaplan, arXiv:0912.2560 [hep-lat].`

Step II: Generate a large sample of thermalized decorrelated vacuum configurations.

$$\langle \hat{\mathcal{O}} \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}U_\mu \mathcal{D}q\mathcal{D}\bar{q}\; e^{-S^{(G)}_{\text{lattice}}[U] - S^{(F)}_{\text{lattice}}[U,q,\bar{q}]}\; \hat{\mathcal{O}}[U,q,\bar{q}]$$

Quark part of expectation values

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Quark part of expectation values

Define: $\langle \hat{\mathcal{O}} \rangle_F = \frac{1}{\mathcal{Z}_F} \int \mathcal{D}q\mathcal{D}\bar{q}\; e^{-S^{(F)}_{\text{lattice}}[U,q,\bar{q}]} \mathcal{O}[q,\bar{q},U]$

$\mathcal{Z}_F = \int \mathcal{D}q\mathcal{D}\bar{q}\; e^{-S^{(F)}_{\text{lattice}}[U,q,\bar{q}]} = \prod_f \det D_f$ Dirac matrix

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⇓

$$\langle\hat{\mathcal{O}}\rangle = \frac{1}{N}\sum_i^N \langle\hat{\mathcal{O}}\rangle_F[U^{(i)}]$$

N number of $U^{(i)}$ sampled from the distribution: $\frac{1}{\mathcal{Z}}e^{-S^{(G)}_{\text{lattice}}[U]}\prod_f \det D_f$

Step III: Form the correlation functions by contracting the quarks. Need to specify the interpolating operators for the state under study.

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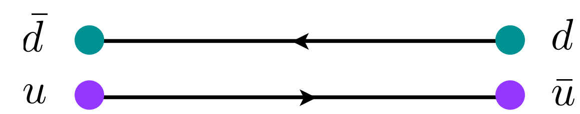
$$\Downarrow$$

e.g., $\boxed{\hat{O} = \bar{u}\gamma_5 d}$



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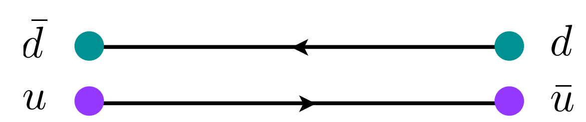
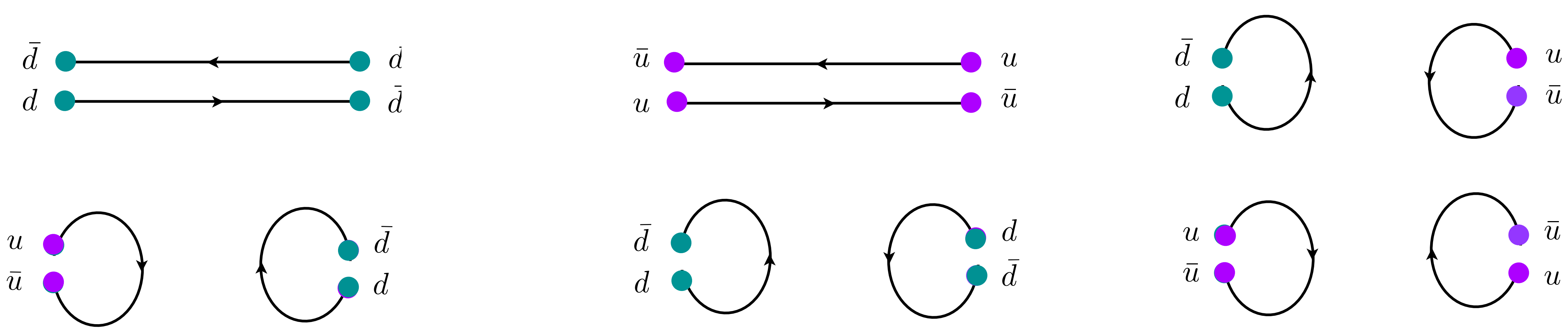


e.g., $\boxed{\hat{O} = \frac{1}{\sqrt{2}}(\overline{u}\gamma^5 u - \overline{d}\gamma^5 d)}$



Quark disconnected diagrams. Require expensive all-to-all propagators.

EXERCISE 1

Show that for the correlation function of the charged pion:

$$\langle \hat{O}^{\pi^+}(n)\hat{O}^{\pi^+\dagger}(0)\rangle_F = -\mathrm{Tr}\left[D_u^{-1}(n,0)D_d^{-1}(n,0)\right]$$

where D_u^{-1} and D_d^{-1} denote the the inverse Dirac matrix (the quark propagator) for the u and d quarks, respectively. Trace is over spin and color degrees of freedom.

BONUS EXERCISE 1

Show that for the correlation function of the neutral pion:

$$\begin{aligned}\langle \hat{O}^{\pi^0}(n)\hat{O}^{\pi^0\dagger}(0)\rangle_F = & -\frac{1}{2}\mathrm{Tr}\left[\gamma^5 D_u^{-1}(n,0)\gamma^5 D_u^{-1}(0,n)\right] \\ & +\frac{1}{2}\mathrm{Tr}\left[\gamma^5 D_u^{-1}(n,n)\right]\mathrm{Tr}\left[\gamma^5 D_u^{-1}(0,0)\right] \\ & -\frac{1}{2}\mathrm{Tr}\left[\gamma^5 D_u^{-1}(n,n)\right]\mathrm{Tr}\left[\gamma^5 D_d^{-1}(0,0)\right] + \{u \leftrightarrow d\}\end{aligned}$$

First on why steps II and III are expensive…

Example: Consider a lattice with: $L/a = 48,\ T/a = 256$

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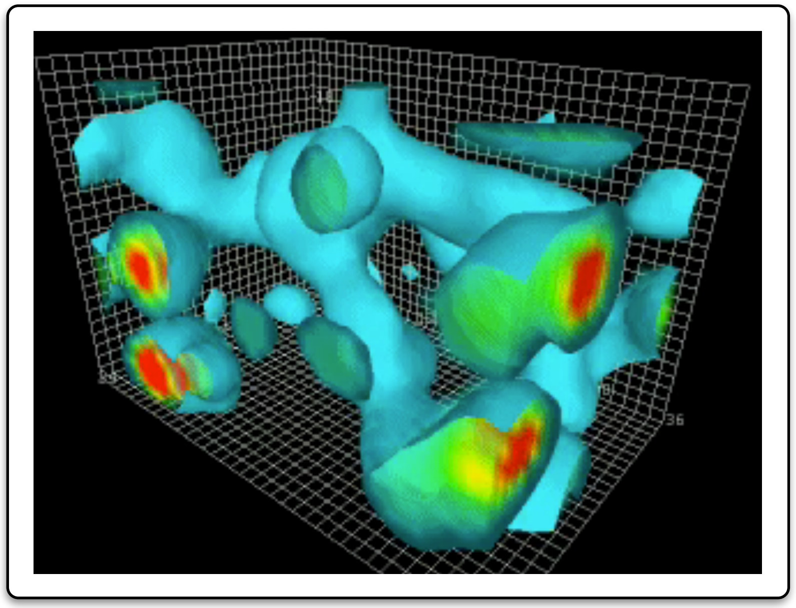
Sampling SU(3) matrices. Already for one sample requires storing

$$8 \times 48^3 \times 256 = 226,492,416$$

c-numbers in the computer!

Requires tens of thousands of uncorrelated samples. Molecular-dynamics-inspired hybrid Monte Carlo sampling algorithms often used.



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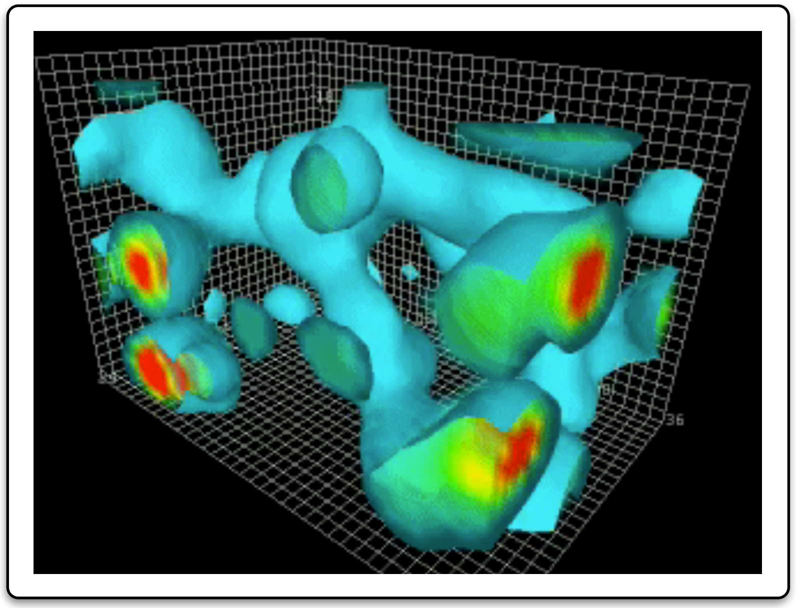


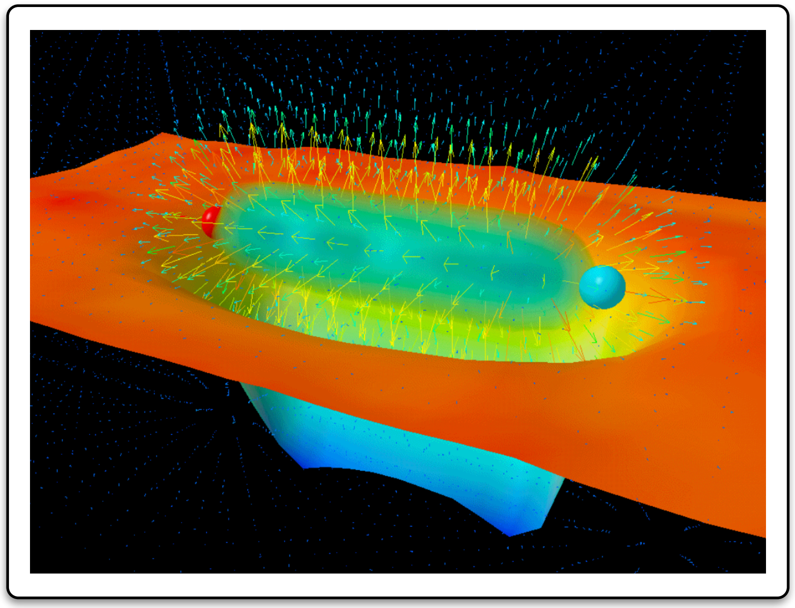
Solving

$$[D(U)]_{X,Y}[S(U)]_{Y,X_0} = G_{X,X_0}$$

Dirac matrix — Quark propagator — Source

Requires taking determinant and inverting a matrix with dimensions:

$$(4 \times 3 \times 48^3 \times 256)^2 = 339,738,624 \times 339,738,624$$



Step IV: Extract energies and matrix elements from correlation functions

$$C_{\hat{\mathcal{O}},\hat{\mathcal{O}}'}(\tau;\mathbf{d}) \;=\; \sum_{\mathbf{x}} e^{2\pi i \mathbf{d}\cdot\mathbf{x}/L}\langle 0|\hat{\mathcal{O}}'(\mathbf{x},\tau)\hat{\mathcal{O}}^\dagger(\mathbf{0},0)|0\rangle = \mathcal{Z}_0'\mathcal{Z}_0^\dagger e^{-E^{(0)}\tau} + \mathcal{Z}_1'\mathcal{Z}_1^\dagger e^{-E^{(1)}\tau} + \ldots$$

Ground state and a tower of excited states are, in principle, accessible!

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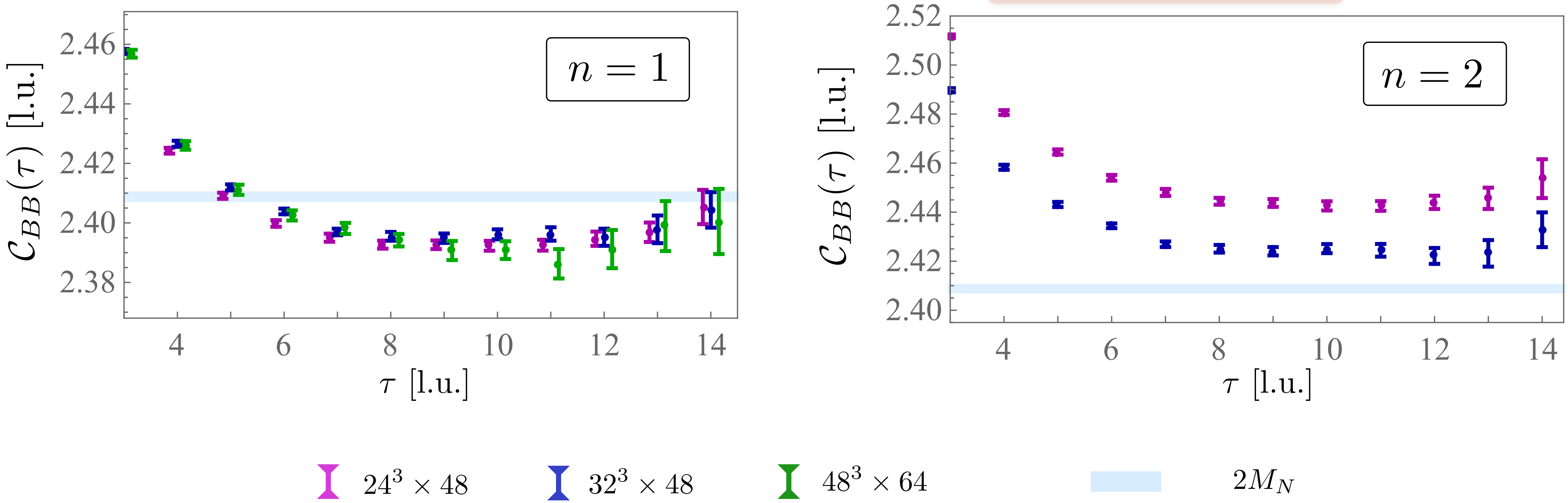
Example: $NN\ (^1S_0)$

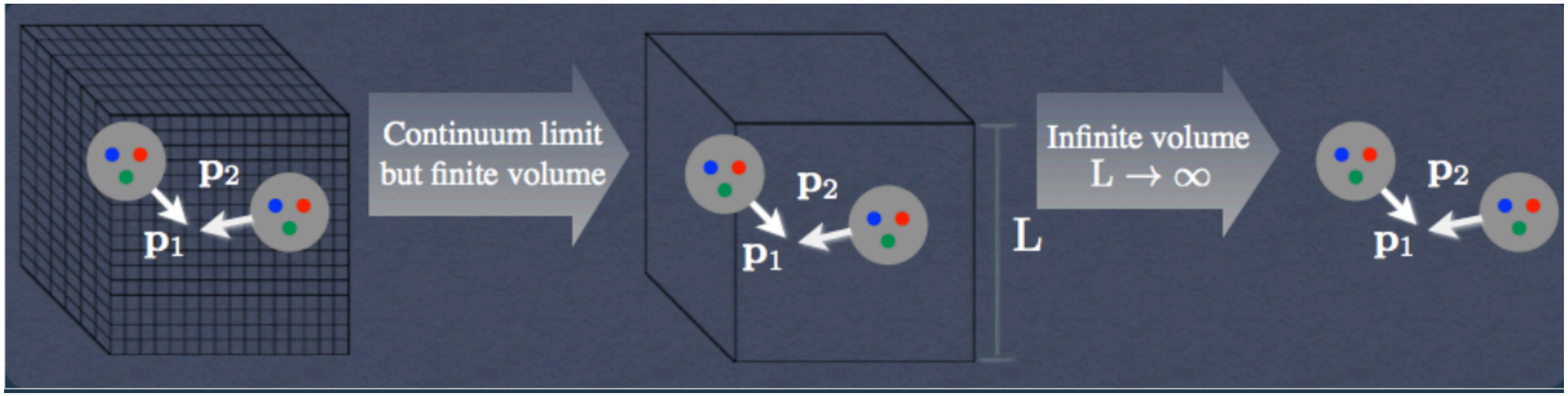
What should we make of the volume dependence?

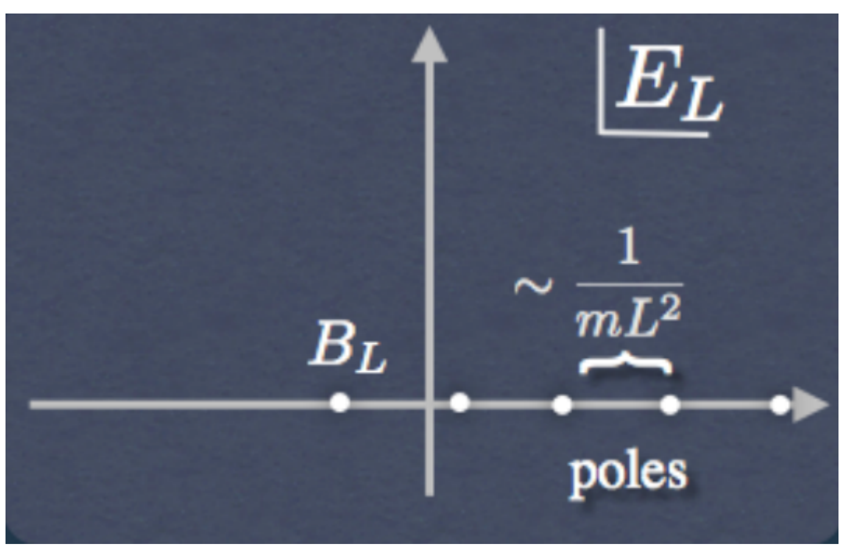


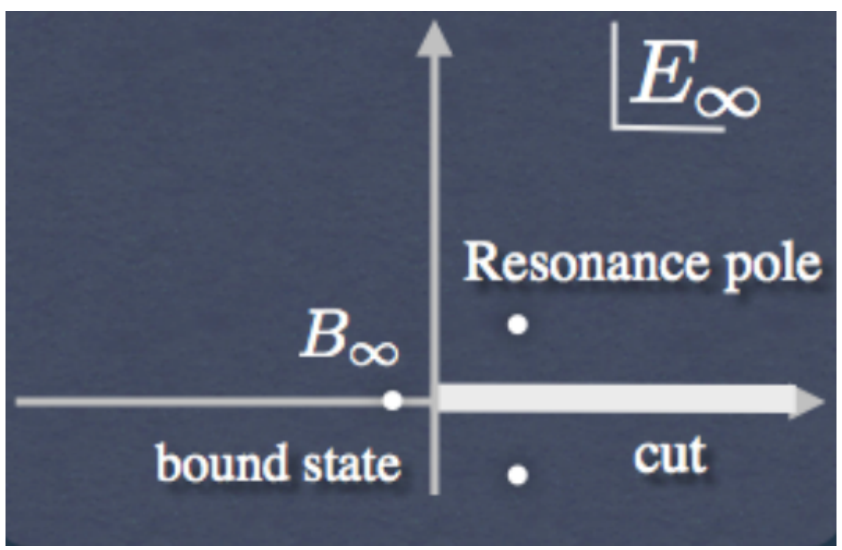
Beane et al (NPLQCD), arXiv:1705.09239, Wagman et al (NPLQCD), arXiv:1706.06550.

Step V: Make the connection to physical observables, such as scattering amplitudes, decay rates, etc. Still not fully developed and presents challenge in multi-hadron systems.

Example: two-hadron scattering







Let's discuss in greater depth step V:

Step V: make the connection to physical observables, such as scattering amplitudes, decay rates, etc.

i) Finite-volume effects in the single-hadron sector

ii) Finite-volume formalism for two-hadron elastic scattering

iii) Finite-volume formalism for coupled-channel two-hadron elastic scattering and resonances

iv) Finite-volume formalism for transition amplitudes and resonance form factors

v) Finite-volume formalism for three-hadron scattering and resonances

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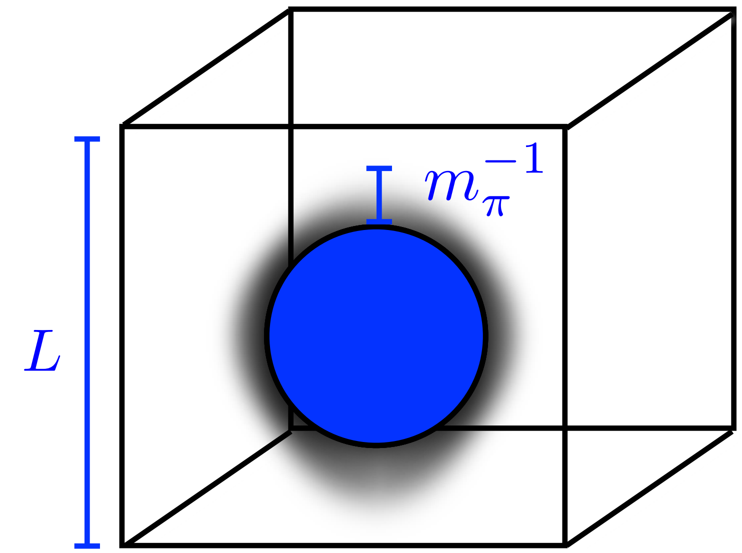
See e.g., ZD, arXiv:1409.1966 [hep-lat] and Briceno, Dudek and Young, RevModPhys.90.025001.

QCD finite-volume corrections to single-hadron observables are exponentially suppressed.

Example: The mass of the nucleon

Finite-volume effects are IR in nature. Can use an effective field theory description to characterize them.



Cubic volume with periodic boundary conditions

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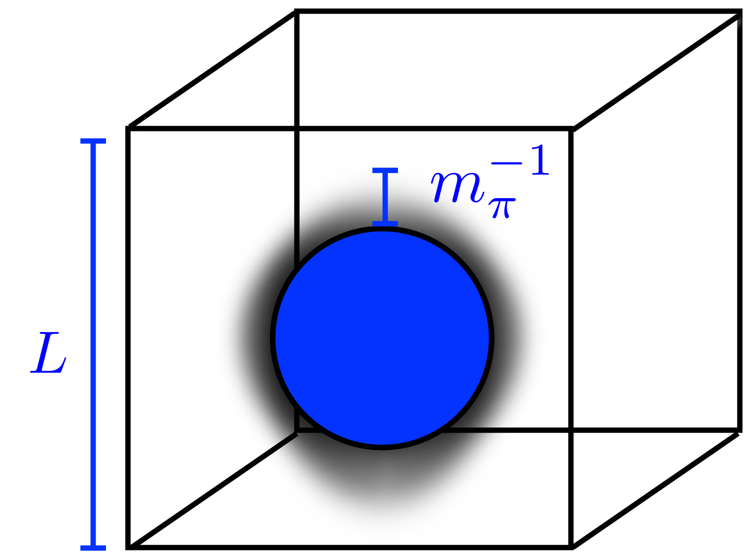
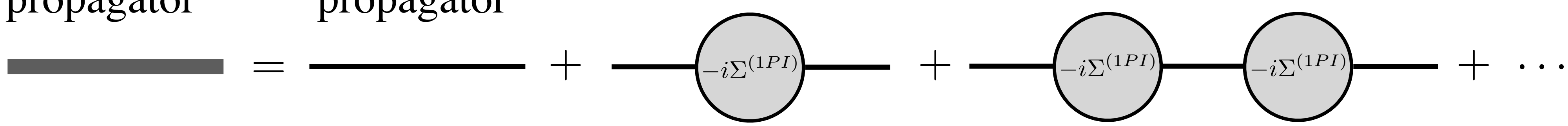


Cubic volume with periodic boundary conditions

Fully dressed propagator

Bare propagator

1-particle irreducible self energy



$$\mathcal{D}_{N_l} = \frac{i}{P\cdot v - M_N^{(0)} + i\epsilon}\left[1 - i\Sigma^{(1PI)}\frac{i}{P\cdot v - M_N^{(0)} + i\epsilon} + \left(-i\Sigma^{(1PI)}\frac{i}{P\cdot v - M_N^{(0)}}\right)^2 + \dots\right]$$

$$= \frac{i}{P\cdot v - M_N^{(0)} - \Sigma^{(1PI)} + i\epsilon} \equiv \frac{iZ_N}{P\cdot v - M_N + i\epsilon},$$

Bare mass

Velocity
(1,0,0,0) in the rest frame

True mass

$$M_N = M_N^{(0)} + \Sigma^{1PI}(P.v = M_N)$$

Example: The mass of the nucleon

Leading heavy baryon chiral perturbation theory Lagrangian:

$$\hat{\mathcal{L}}^{(1)}_{\pi N} = \overline{N}\left[i\partial_0 - \frac{1}{4f_\pi^2}\boldsymbol{\tau}\cdot(\boldsymbol{\pi}\times\partial_0\boldsymbol{\pi}) - \frac{g_A}{2f_\pi}\boldsymbol{\tau}\cdot(\boldsymbol{\sigma}\cdot\boldsymbol{\partial})\boldsymbol{\pi}\right]N$$

Nucleon axial charge

Pion decay constant

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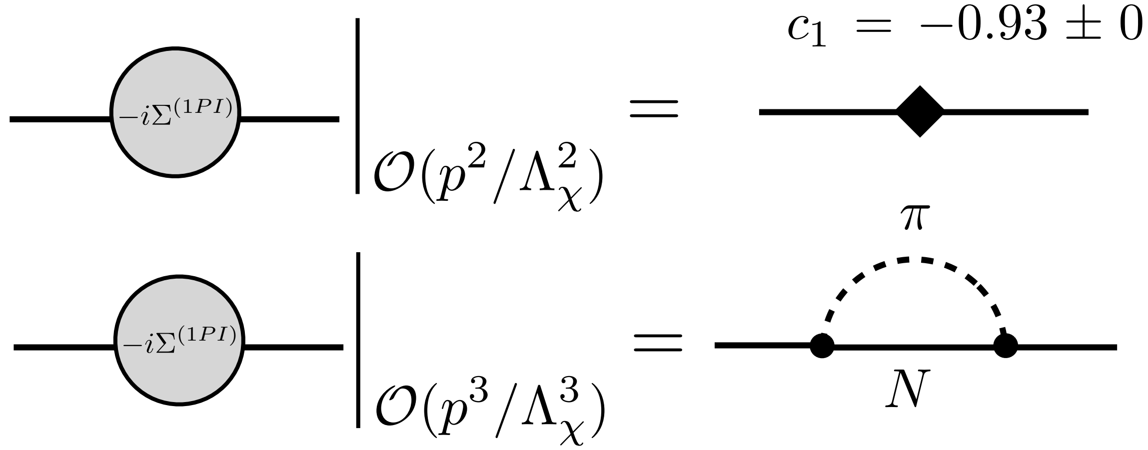
Leading corrections to nucleon self energy:

$c_1 = -0.93 \pm 0.10 \ \mathrm{GeV}^{-1}$



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Nucleon axial charge (g_A)

Pion decay constant (f_π)

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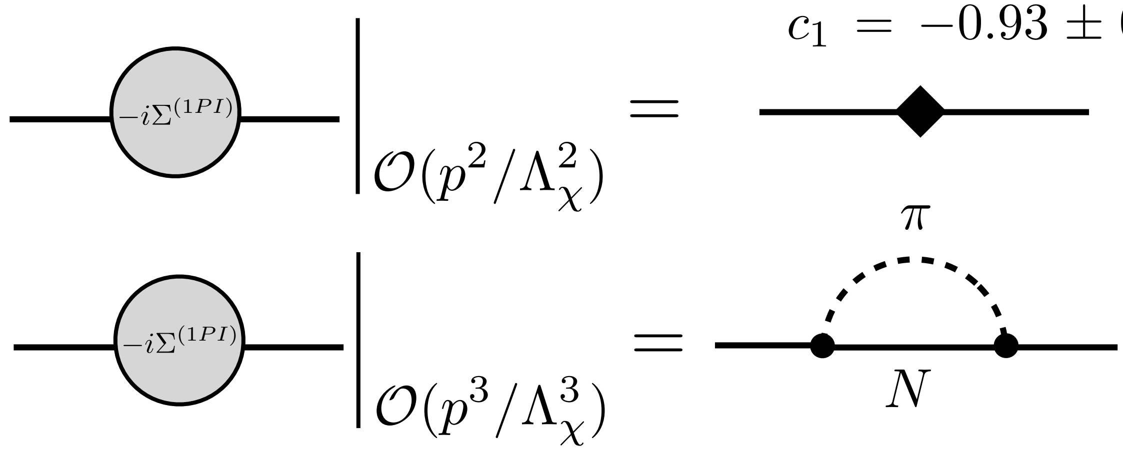

Which gives rise to:

$$T \to \infty, a \to 0$$

$$\delta_L M_N \equiv M_N(L) - M_N(\infty) = \frac{3g_A^2}{8\pi^2 f_\pi^2} \times \frac{\pi}{2} m_\pi^2 \sum_{\mathbf{n}\neq\mathbf{0}} \frac{e^{-|\mathbf{n}|m_\pi L}}{|\mathbf{n}|L}$$

`Beane, Phys.Rev.D70: 034507 (2004).`

EXERCISE 2

Plot the first few terms in the expression for the corrections to the nucleon mass as a function of the spatial extent of the volume. How large the volume must be such that correction to the mass of the nucleon are sub-percent?

BONUS EXERCISE 2

Drive the expression for the volume corrections to the mass of the nucleon at leading order in heavy-baryon chiral perturbation theory. The first step is to realize that the volume corrections arise from the loop integral where the integration over a continuous momentum is replaced by a summation over discretized momenta in a periodic cubic volume:

$$\int \frac{dk_0}{2\pi} \frac{d^3k}{(2\pi)^3} \to \int \frac{dk_0}{2\pi} \frac{1}{L^3} \sum_{\mathbf{k}=2\pi\mathbf{n}/L} \text{ with } \mathbf{n} \in \mathbb{Z}$$

The second step is to make use of Poisson resumption formula:

$$\frac{1}{L^3} \sum_{\boldsymbol{k}} f(\mathbf{k}) = \int \frac{d^3k}{(2\pi)^3} f(\mathbf{k}) + \sum_{\boldsymbol{m} \neq \mathbf{0}} \int \frac{d^3k}{(2\pi)^3} f(\mathbf{k}) e^{i\boldsymbol{k}.\boldsymbol{m}L}$$

where $\mathbf{m}$ is another integer three-vector.

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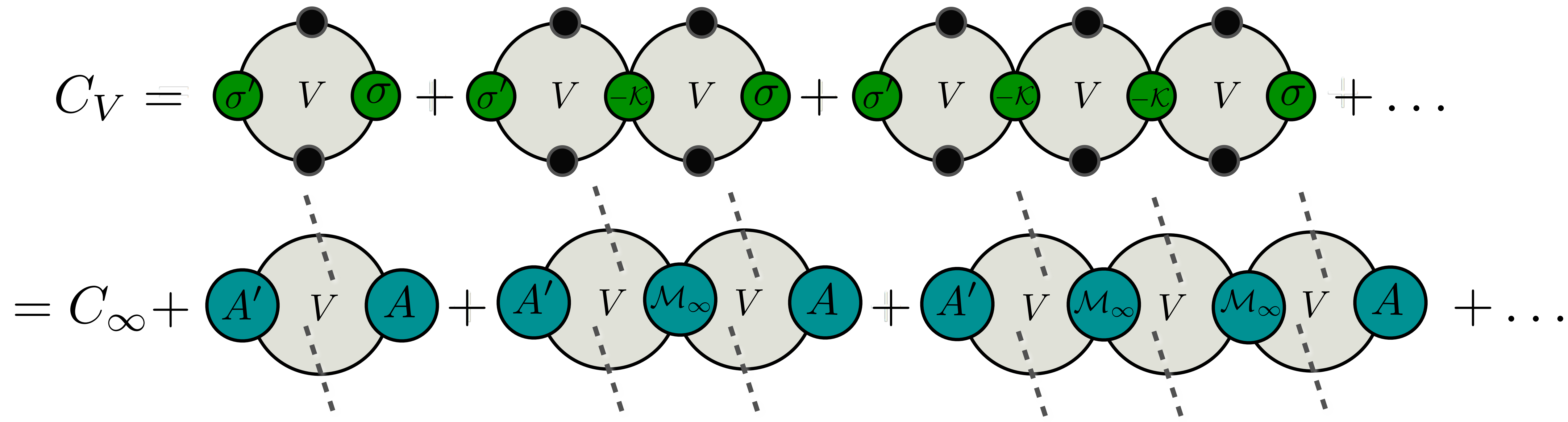
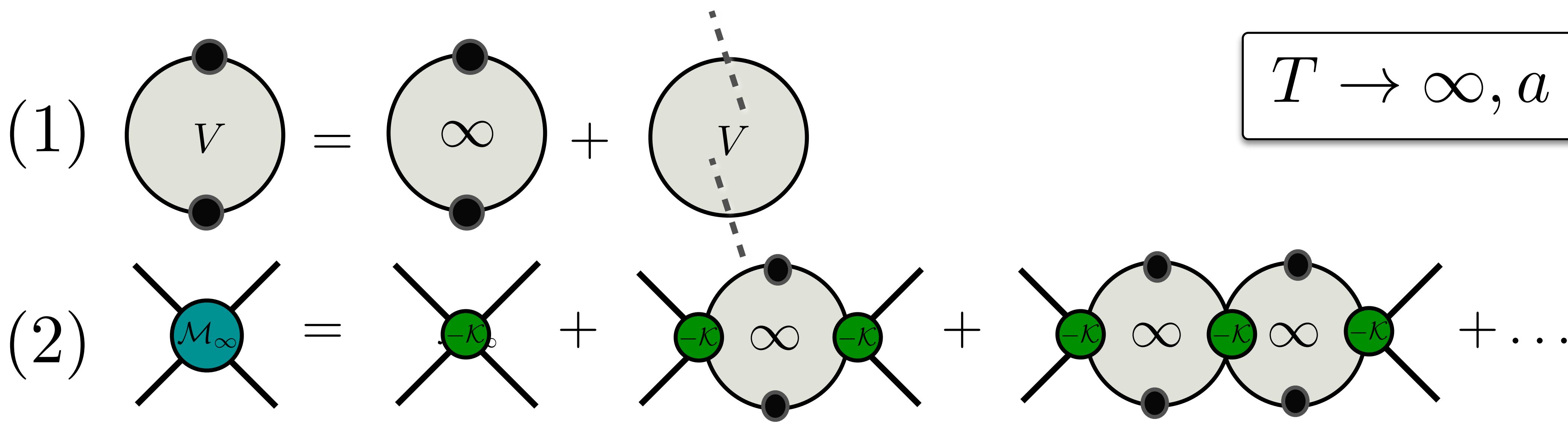
Let's derive the Luescher's formula first. A QFT derivation goes as follows:



Kim, Sachrajda and Sharpe, Nucl.Phys.B727(2005)218-243.

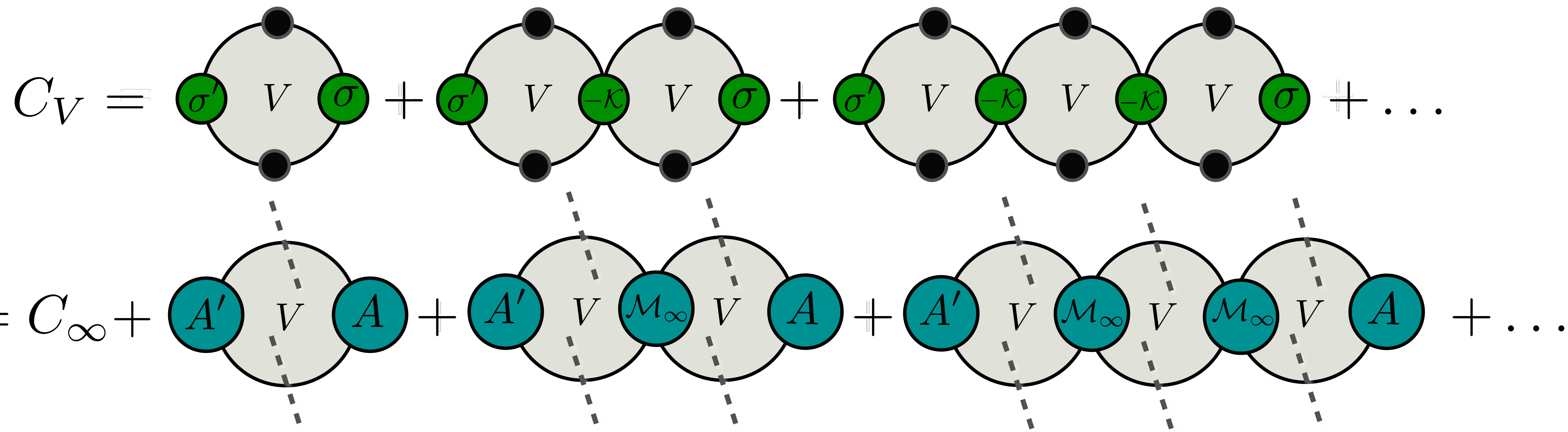


$$T \to \infty, a \to 0$$

EXERCISE 3

By rearranging the diagrams in C_V (the first line in the upper panel) using the relations in the lower panel, verify the expansion in the second line in the upper panel. What is the relation between $\sigma(\sigma')$ and $A(A')$?

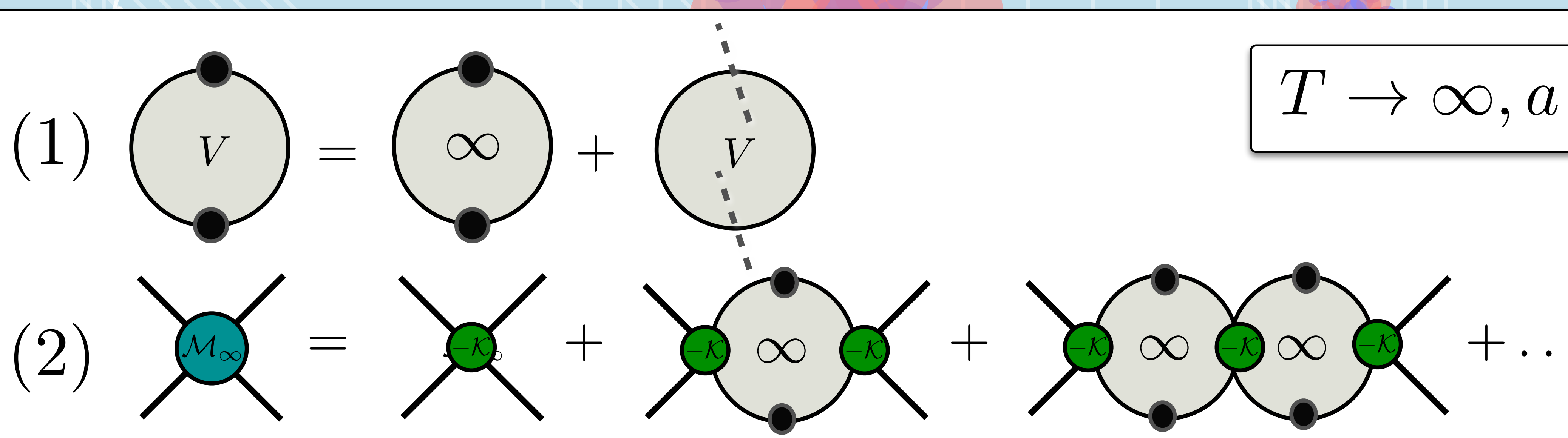
Let's derive the Luescher's formula first. A QFT derivation goes as follows:



$$\det\left[\delta\mathcal{G}^V(E^*) + \mathcal{M}^{-1}(E^*)\right] = 0$$

Kim, Sachrajda and Sharpe, Nucl.Phys.B727(2005)218-243.

Finite-volume function   Scattering amplitude



$$T \to \infty, a \to 0$$

Elastic amplitude more closely…

CM energy

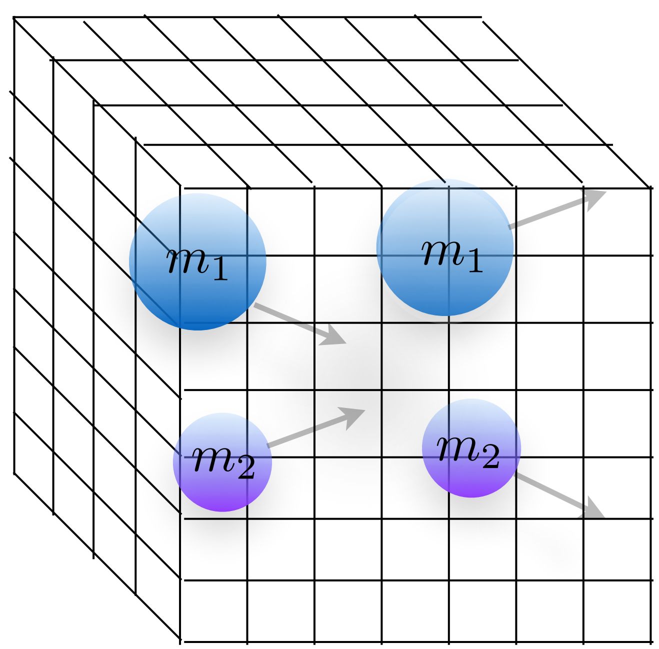
Phase shift

$$(\mathcal{M})_{l_1,m_1;l_2,m_2} = \delta_{l_1,l_2}\delta_{m_1,m_2}\frac{8\pi E^*}{nq^*}\frac{e^{2i\delta^{(l)}(q^*)}-1}{2i}$$

Symmetry factor

$$q^{*2} = \frac{1}{4}\left(E^{*2} - 2(m_1^2+m_2^2) + \frac{(m_1^2-m_2^2)^2}{E^{*2}}\right)$$

$$\det\left[\delta\mathcal{G}^V(E^*) + \mathcal{M}^{-1}(E^*)\right] = 0$$



Finite-volume function more closely…

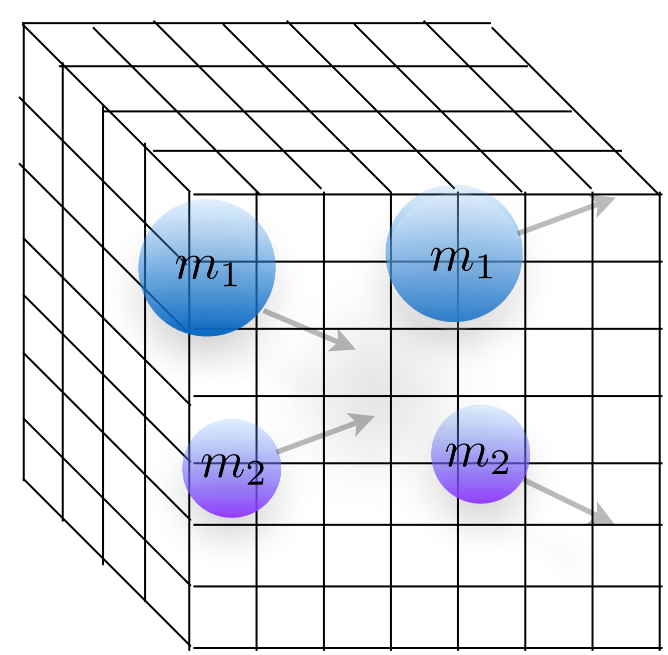
$$(\delta\mathcal{G}^V)_{l_1,m_1;l_2,m_2} = i\frac{q^* n}{8\pi E^*}\left(\delta_{l_1,l_2}\delta_{m_1,m_2} + i\frac{4\pi}{q^*}\sum_{l,m}\frac{\sqrt{4\pi}}{q^{*l}}c^{\mathbf{P}}_{lm}(q^{*2})\int d\Omega^* Y^*_{l_1m_1}Y^*_{lm}Y_{l_2m_2}\right)$$

$$c^{\mathbf{P}}_{lm}(x) = \frac{1}{\gamma}\left[\frac{1}{L^3}\sum_{\mathbf{k}} - \mathcal{P}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\right]\frac{\sqrt{4\pi}Y_{lm}(\hat{k}^*)\ k^{*l}}{k^{*2} - x}$$

$$\mathbf{k}^* = \gamma^{-1}\left[\mathbf{k}_\parallel - \frac{1}{2}(1+\frac{m_1^2-m_2^2}{E^{*2}})\mathbf{P}\right] + \mathbf{k}_\perp$$

ZD and Savage, Phys. Rev.D84,114502(2011).

$$\det\left[\delta\mathcal{G}^V(E^*) + \mathcal{M}^{-1}(E^*)\right] = 0$$



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$$\mathbf{k}^* = \gamma^{-1}\left[\mathbf{k}_{\parallel} - \frac{1}{2}\left(1 + \frac{m_1^2 - m_2^2}{{E^*}^2}\right)\mathbf{P}\right] + \mathbf{k}_{\perp}$$

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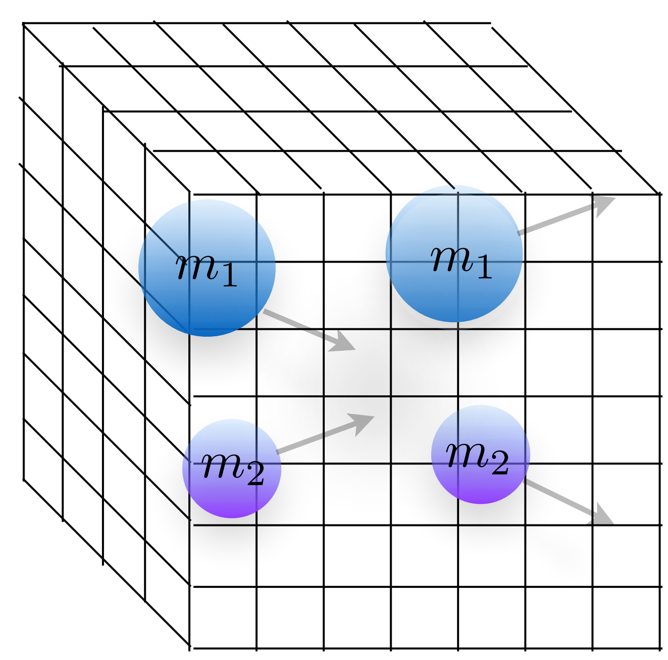


S-wave approximation, valid at low energies:

$$q^* \cot\delta^{(0)} = 4\pi c_{00}(q^{*2})$$

S-wave phase shift

EXERCISE 4

Derive the S-wave limit of Luescher's quantization condition from the master relation.

BONUS EXERCISE 3

Plot the S-wave finite-volume function C_{00} for a range of momenta q^{*2}, including negative values. At what values of q^{*2} do you observe singularities? What do these momenta correspond to?

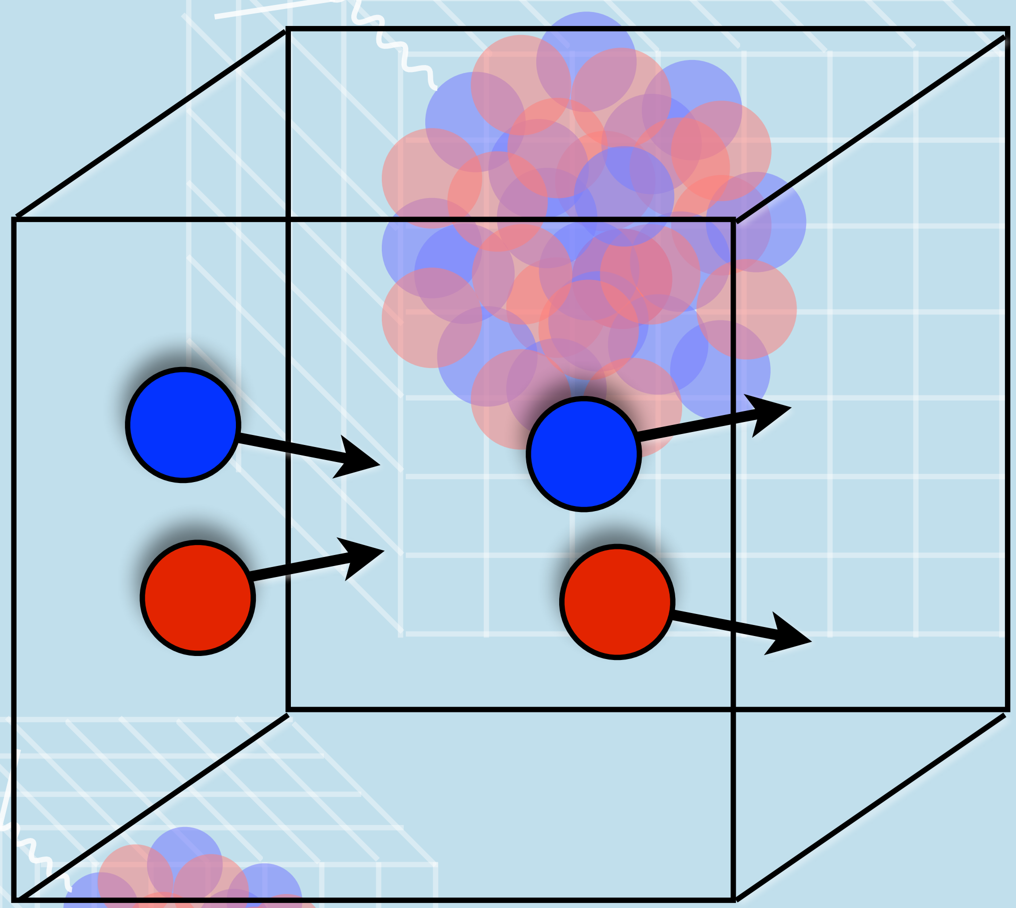
Now let's see an application of Luescher's method to obtain elastic scattering amplitudes of two hadrons from lattice QCD: Wagman et al.(NPLQCD), Phys.Rev.D 96,114510(2017).

Two-baryon states with SU(3) symmetry

$$\{n, p, \Sigma^+, \Sigma^0, \Sigma^-, \Xi^0, \Xi^+, \Lambda\}$$

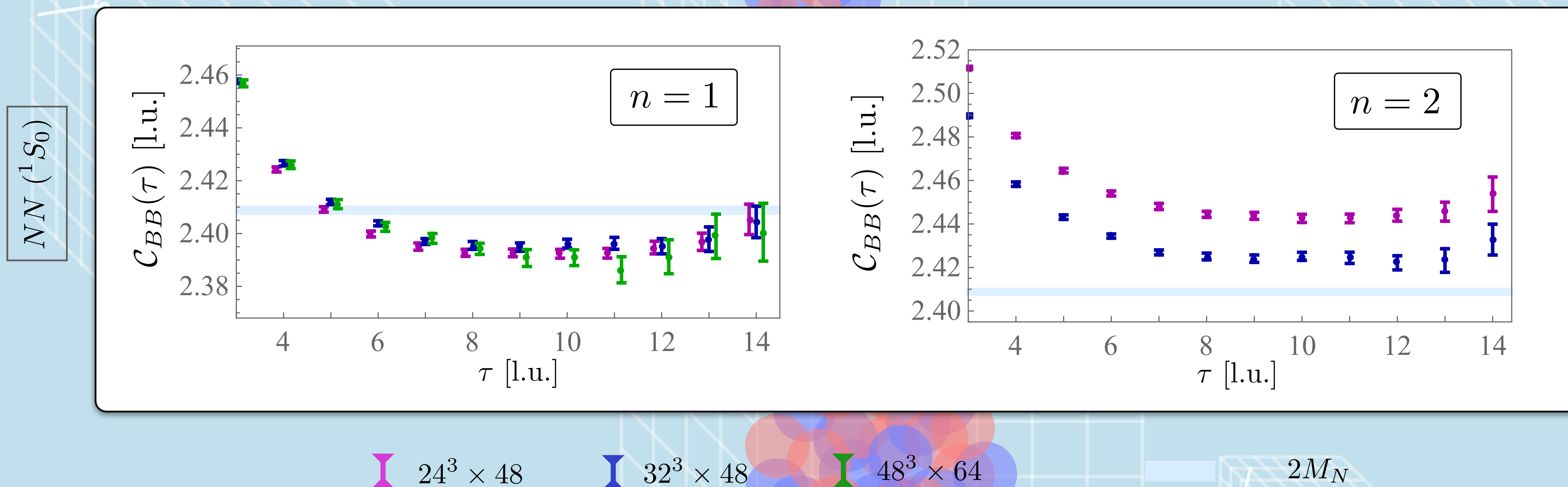
SU(3) decomposition of states: $8 \otimes 8 = 27 \oplus 10 \oplus \overline{10} \oplus 8_S \oplus 8_A \oplus 1$

Let's see what these states are…



Step 1: Obtain the lowest-lying spectra

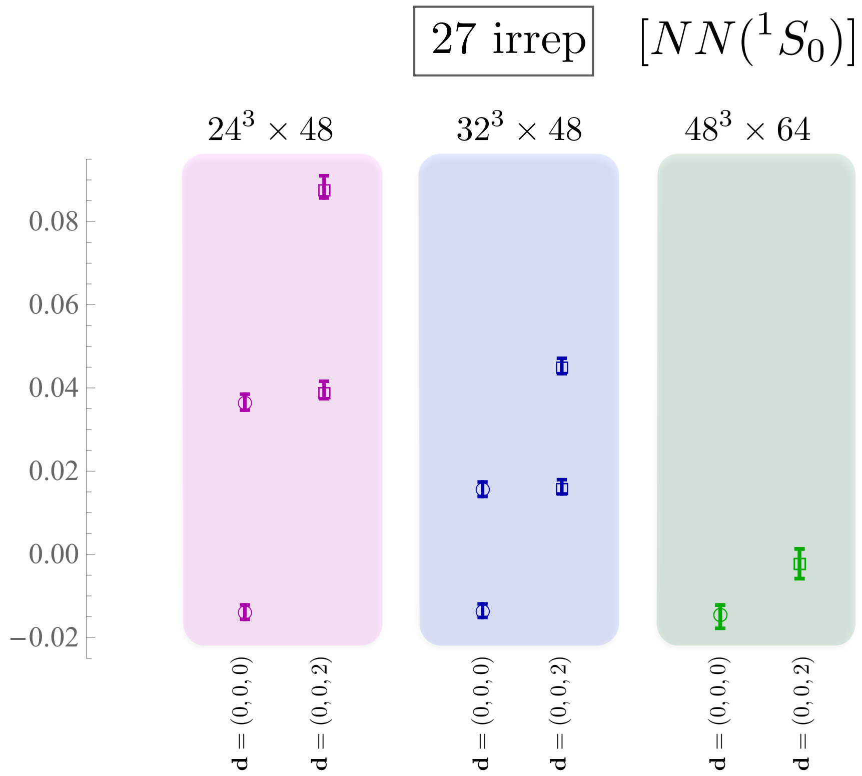
$N_f = 3,\ m_\pi = 0.806$ GeV, $a = 0.145(2)$ fm



$$C_{\hat{\mathcal{O}},\hat{\mathcal{O}}'}(\tau;\mathbf{d}) = \sum_{\mathbf{x}} e^{2\pi i \mathbf{d}\cdot\mathbf{x}/L} \langle 0|\hat{\mathcal{O}}'(\mathbf{x},\tau)\hat{\mathcal{O}}^\dagger(\mathbf{0},0)|0\rangle = \mathcal{Z}_0'\mathcal{Z}_0^\dagger e^{-E^{(0)}\tau} + \mathcal{Z}_1'\mathcal{Z}_1^\dagger e^{-E^{(1)}\tau} + \ldots$$

Beane et al (NPLQCD), arXiv:1705.09239, Wagman et al (NPLQCD), arXiv:1706.06550.

Step 1: Obtain the lowest-lying spectra



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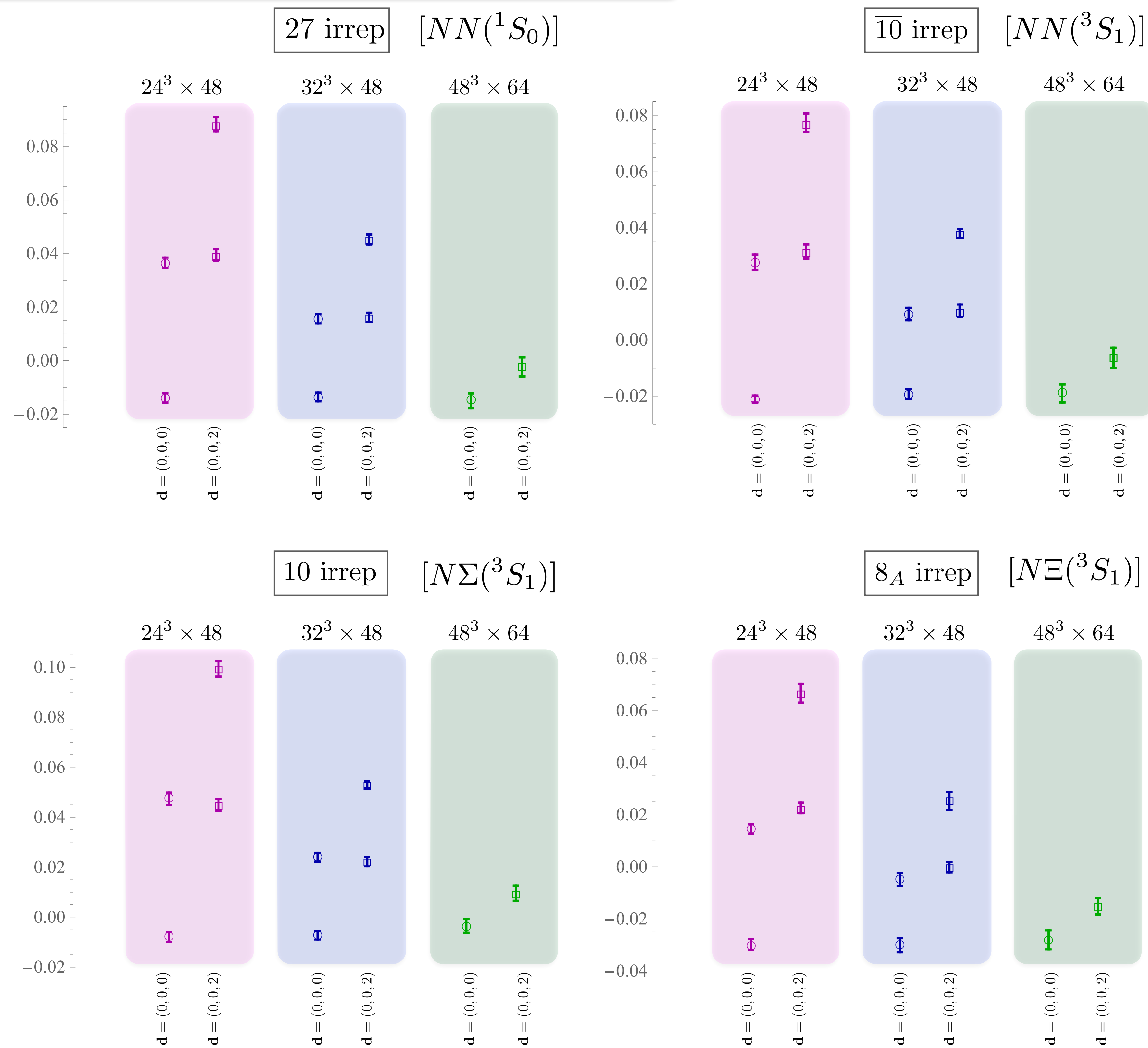
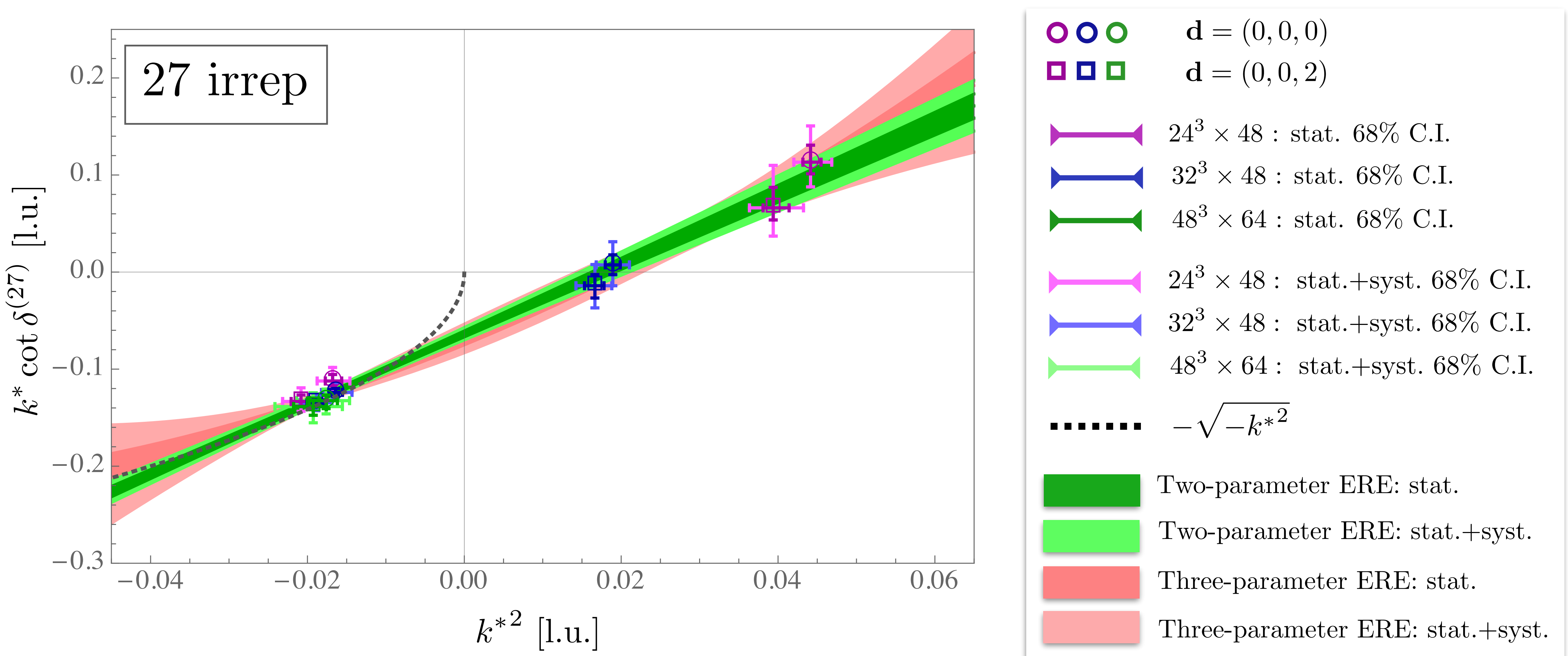


Step 2: Feed the energies to the Luescher's equation and obtain the S-wave scattering phase shifts.

$N_f = 3,\ m_\pi = 0.806$ GeV, $a = 0.145(2)$ fm



$$B = 20.6^{(+1.8)(+2.8)}_{(-2.4)(-1.6)}\ \texttt{MeV}$$

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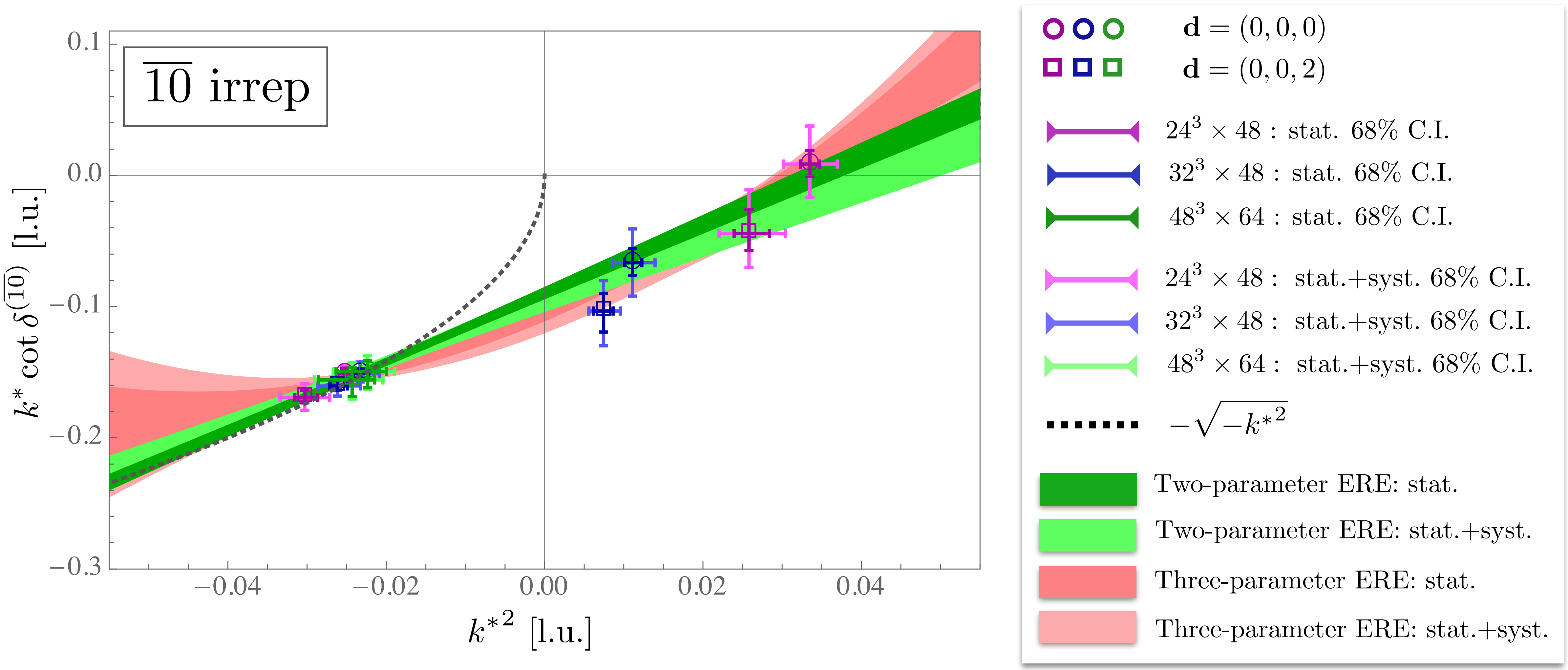
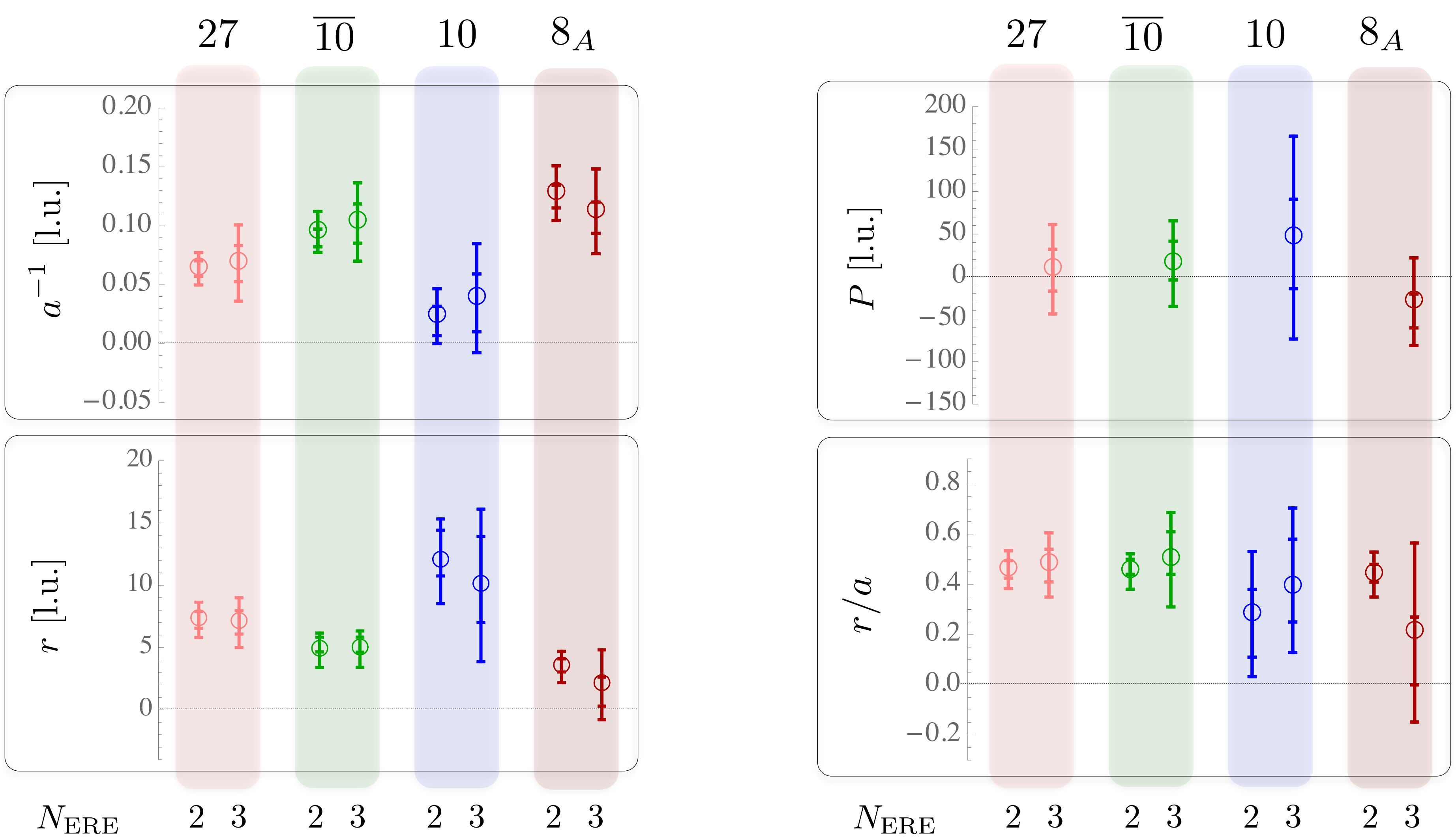


$$B = 27.9^{(+3.1)(+2.2)}_{(-2.3)(-1.4)}\ \texttt{MeV}$$

Step 3: Feed the energies to the Luescher's equation and obtain the S-wave scattering phase shifts.

$N_f = 3,\ m_\pi = 0.806\ \text{GeV},\ a = 0.145(2)\ \text{fm}$



Kaplan and Savage (1998).

$SU(N_f = 3) \longrightarrow SU(2N_f = 6) \longrightarrow SU(16)$

This is in fact a prediction of QCD with a large number of colors for nuclear and hyper nuclear interactions.

Let's discuss in greater depth step V:

Step V: make the connection to physical observables, such as scattering amplitudes, decay rates, etc.

i) Finite-volume effects in the single-hadron sector

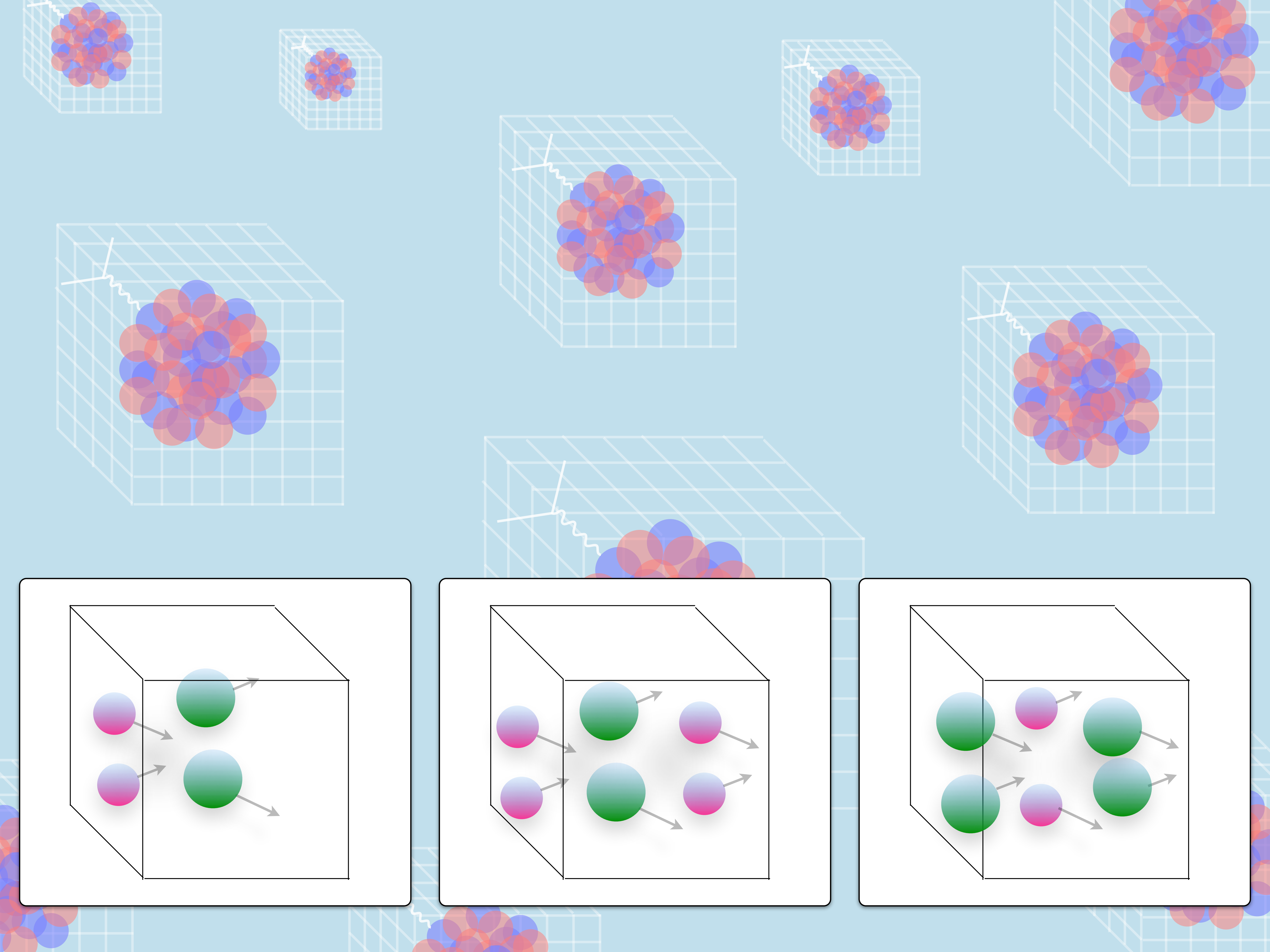
ii) Finite-volume formalism for two-hadron elastic scattering

iii) Finite-volume formalism for coupled-channel two-hadron elastic scattering and resonances

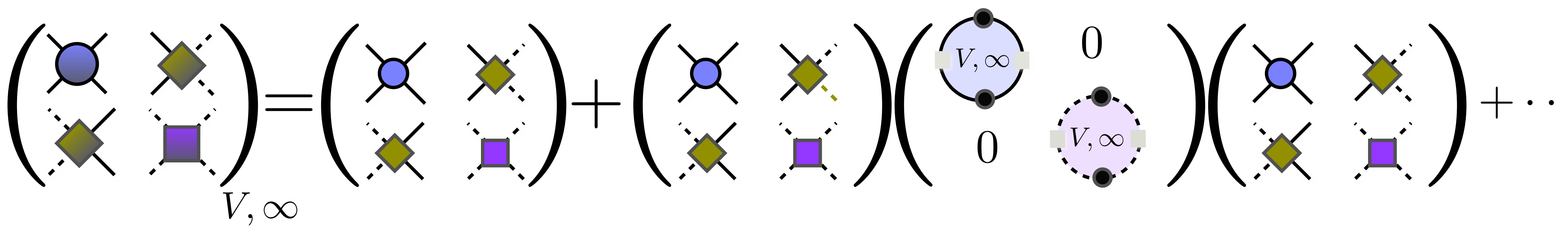
iv) Finite-volume formalism for transition amplitudes and resonance form factors

v) Finite-volume formalism for three-hadron scattering and resonances

vi) Finite-volume effects in lattice QED+QCD studies of hadrons

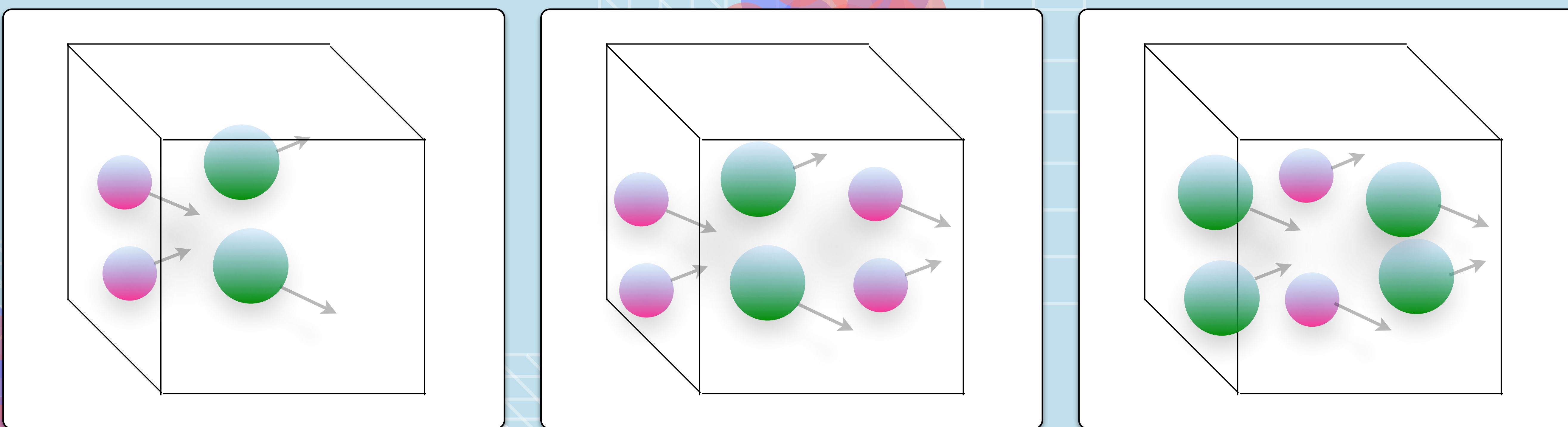


Coupled-channel generalization of luescher's formula is straightforward. Requires upgrading amplitudes and finite-volume functions to matrices in the channel space:



Briceno and ZD, Phys. Rev.D88,094507(2013).

Hansen and Sharpe, Phys. Rev. D86, 016007(2012).



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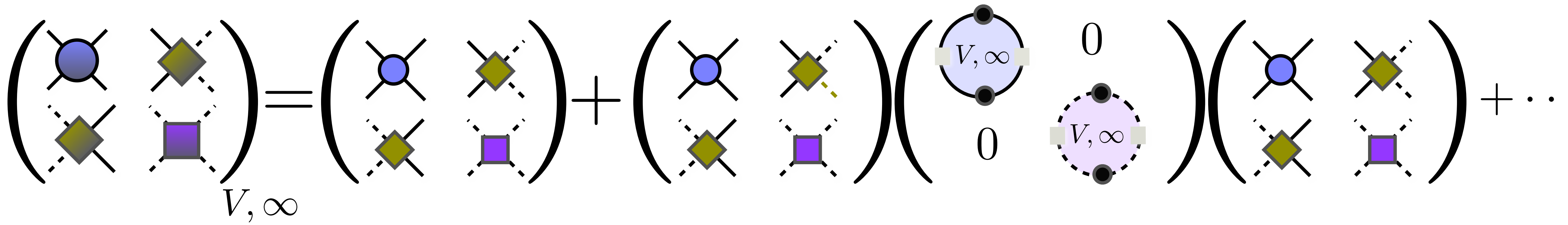

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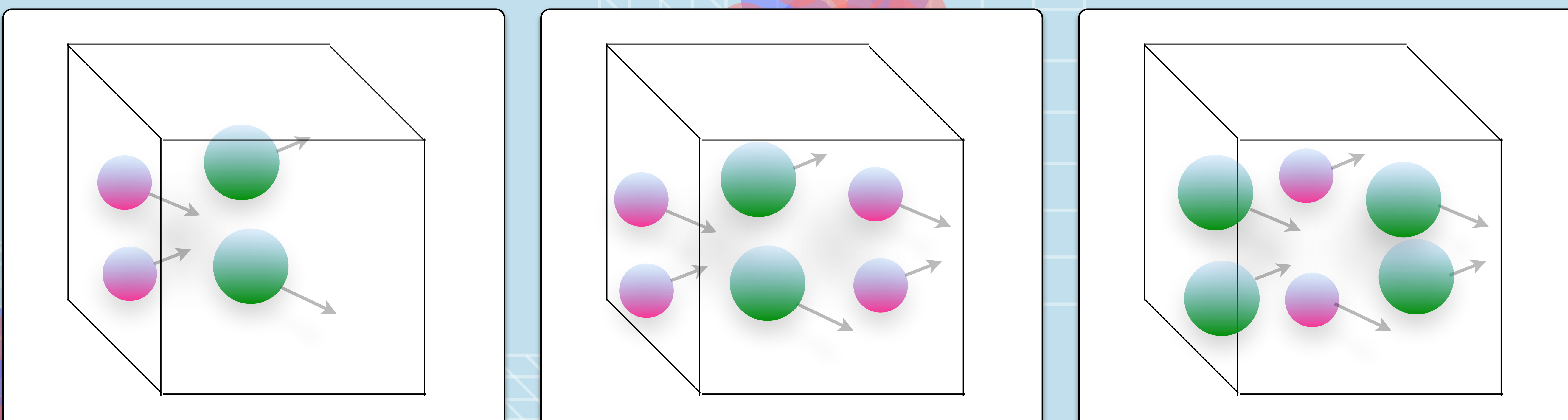

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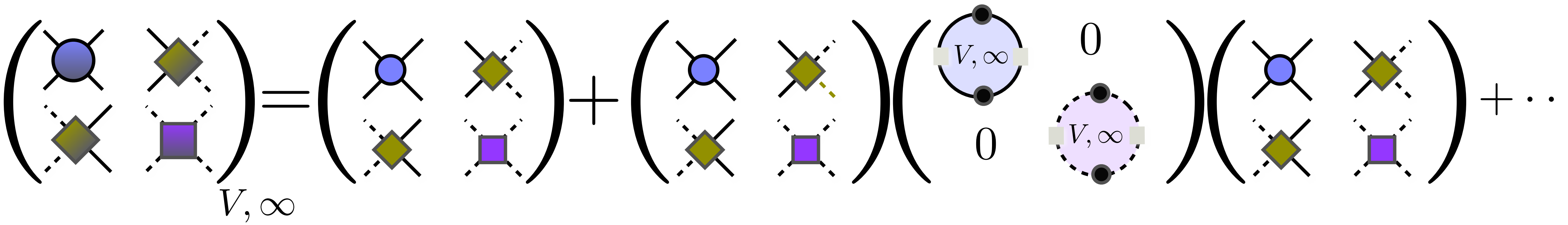

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$$(\mathcal{M}_{i,i})_{l_1,m_1;l_2,m_2}=\delta_{l_1,l_2}\delta_{m_1,m_2}\frac{8\pi E^*}{n_i q_i^*}\frac{\cos(2\bar{\epsilon})e^{2i\delta_i^{(l_1)}(q_i^*)}-1}{2i},$$

Channel index I or II

$$(\mathcal{M}_{I,II})_{l_1,m_1;l_2,m_2}=\delta_{l_1,l_2}\delta_{m_1,m_2}\frac{8\pi E^*}{\sqrt{n_I n_{II} q_I^* q_{II}^*}}\sin(2\bar{\epsilon})\frac{e^{i(\delta_I^{(l_1)}(q_I^*)+\delta_{II}^{(l_1)}(q_{II}^*))}}{2}$$

Mixing angle between two channels

EXERCISE 5

Derive the manifestly real form of a coupled two-channel scattering in the S-wave limit:

$$\cos 2\bar{\epsilon} \cos\left(\phi_1^P + \delta_1 - \phi_2^P - \delta_2\right) = \cos\left(\phi_1^P + \delta_1 + \phi_2^P + \delta_2\right)$$

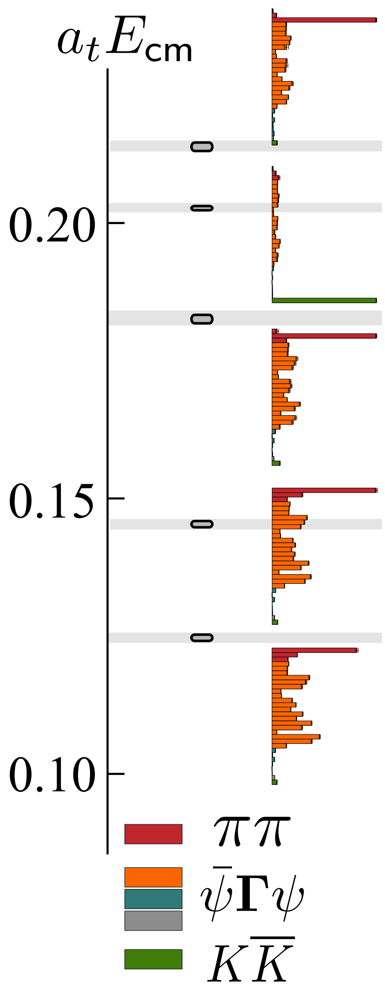
Here 1 and 2 indices refer to the two channels and superscript (0) is removed from the S-wave phase shifts for brevity. The finite-volume phase function ϕ_i^P is defined as:

$$q_i^* \cot(\phi_i^P) \equiv -4\pi c_{00}^P(q_i^{*2})$$

for *i*=1,2. This is a generic result: Luescher's "quantization condition" is a real condition.

Now let's see an application of the coupled-channel formalism: Hunting resonances using lattice QCD in the P-wave coupled $\pi\pi - K\overline{K}$ channel

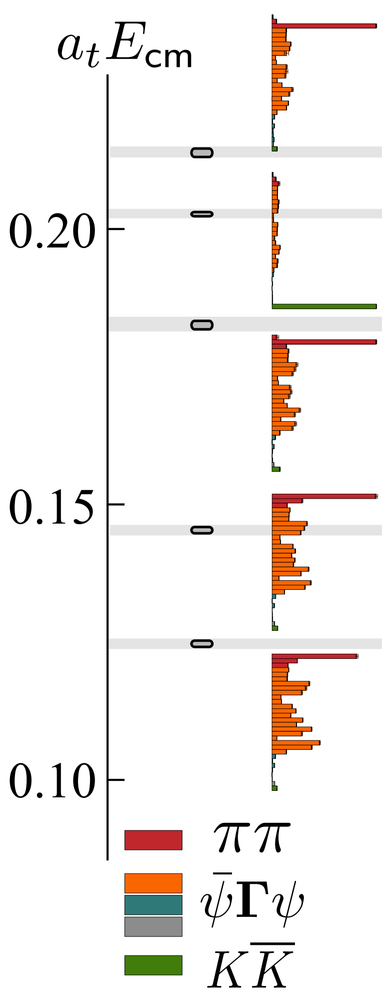
Wilson et al.(HadSpec), Phys.Rev. D92 (2015), 094502



Example: T1 irrep energies

$N_f = 2+1, m_\pi = 236$ MeV, $V \approx (4\text{ fm})^3$

Wilson et al.(HadSpec),
Phys.Rev. D92 (2015), 094502



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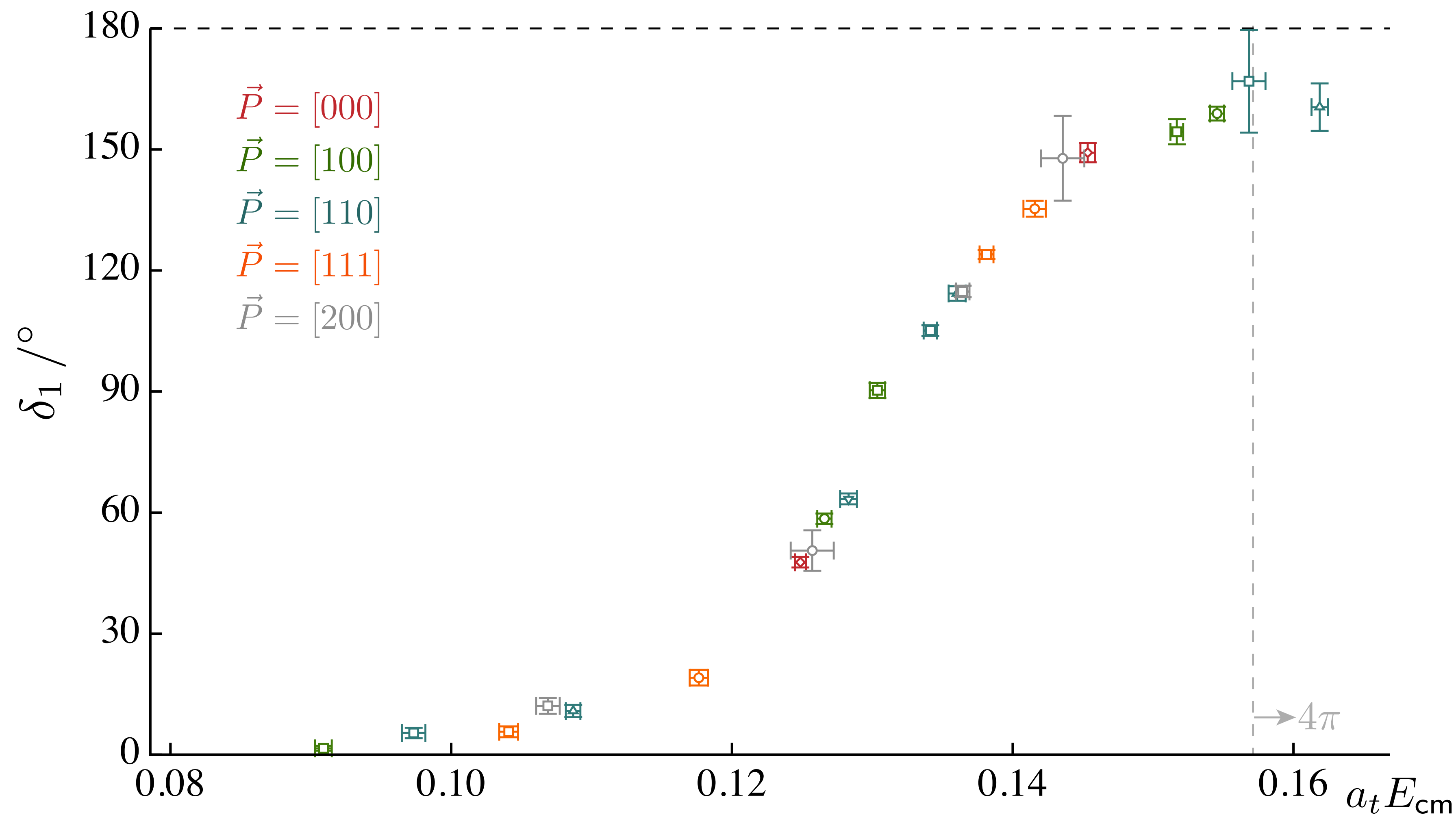
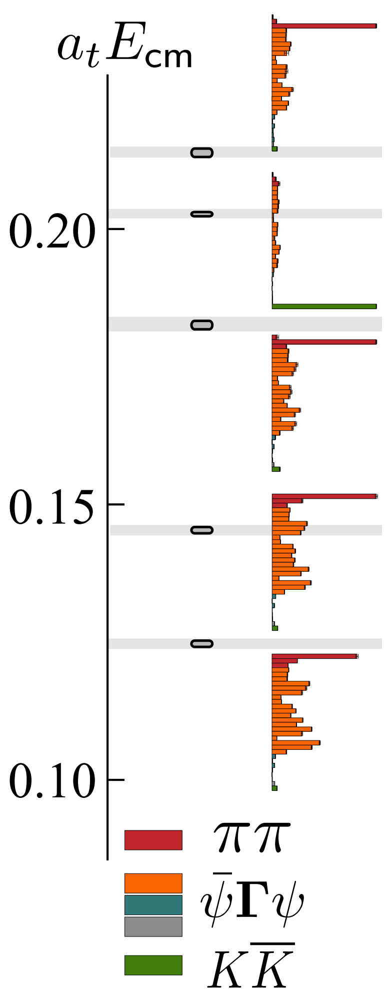


P-wave $\pi\pi$ phase shift as a function of energy

$N_f = 2+1, m_\pi = 236\ \text{MeV},\ V \approx (4\ \text{fm})^3$

Wilson et al.(HadSpec), Phys.Rev. D92 (2015), 094502



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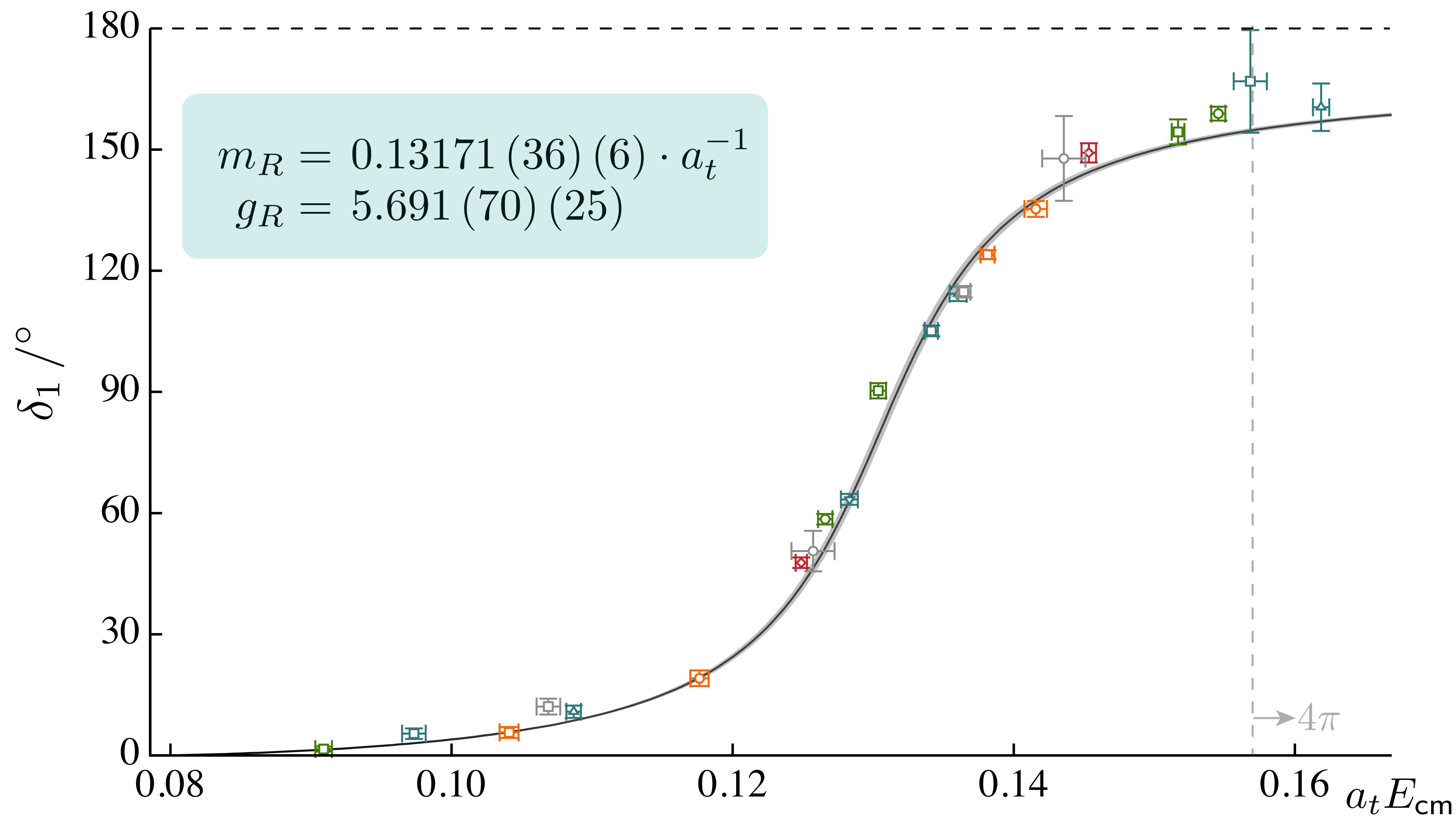


Fit to a Breit-Wigner form

$$\mathcal{M}(s) = \frac{1}{\rho(s)} \frac{\sqrt{s}\,\Gamma(s)}{m_R^2 - s - i\sqrt{s}\,\Gamma(s)}$$

$$\rho_i(E_{\mathrm{cm}}) = 2k_i/E_{\mathrm{cm}}$$

$$s = E_{\mathrm{cm}}^2$$

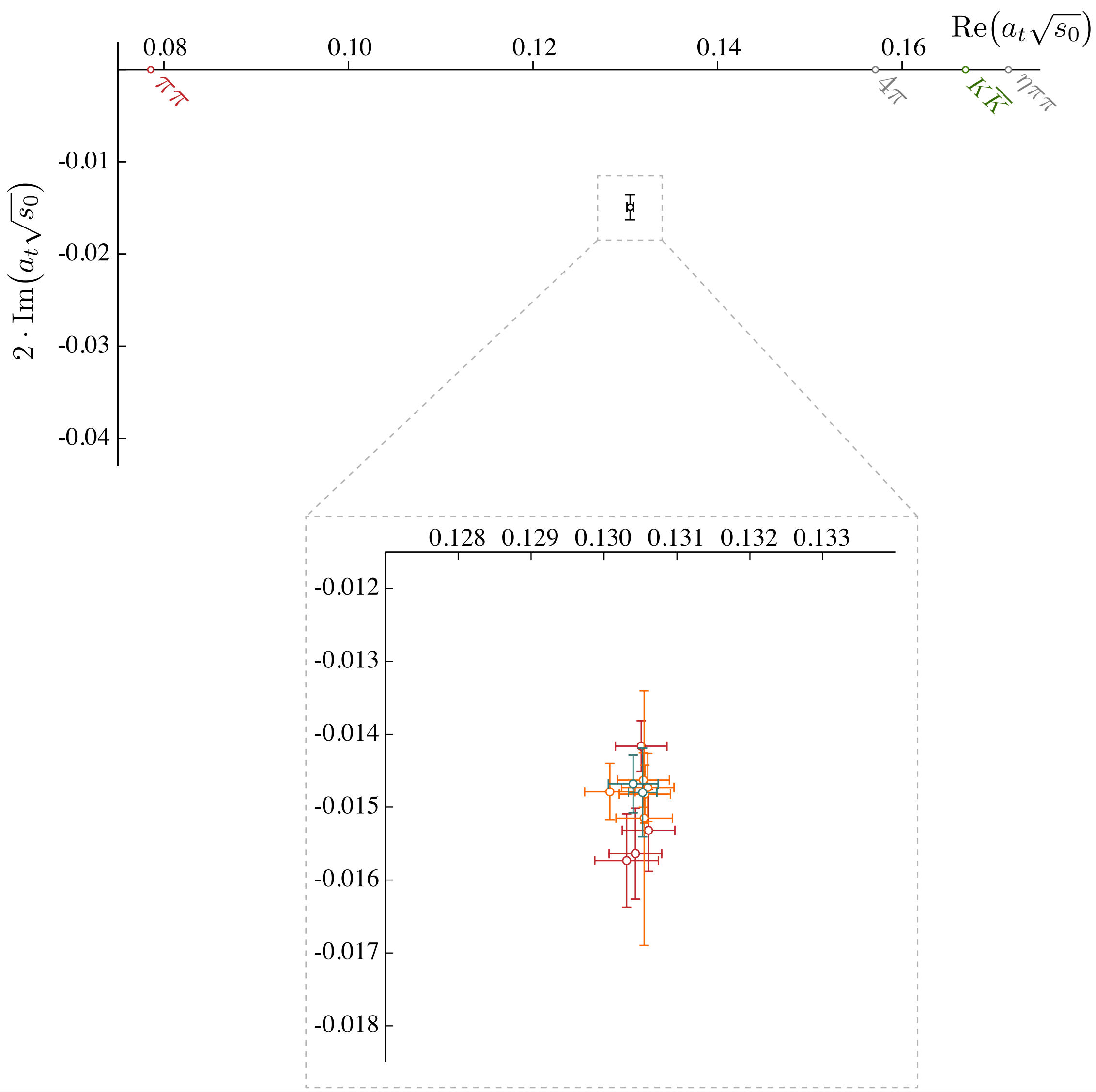
$$\Gamma(s) = \frac{g_R^2}{6\pi}\frac{k^3}{s}$$

$$N_f = 2+1, m_\pi = 236\ \mathrm{MeV},\ V \approx (4\ \mathrm{fm})^3$$

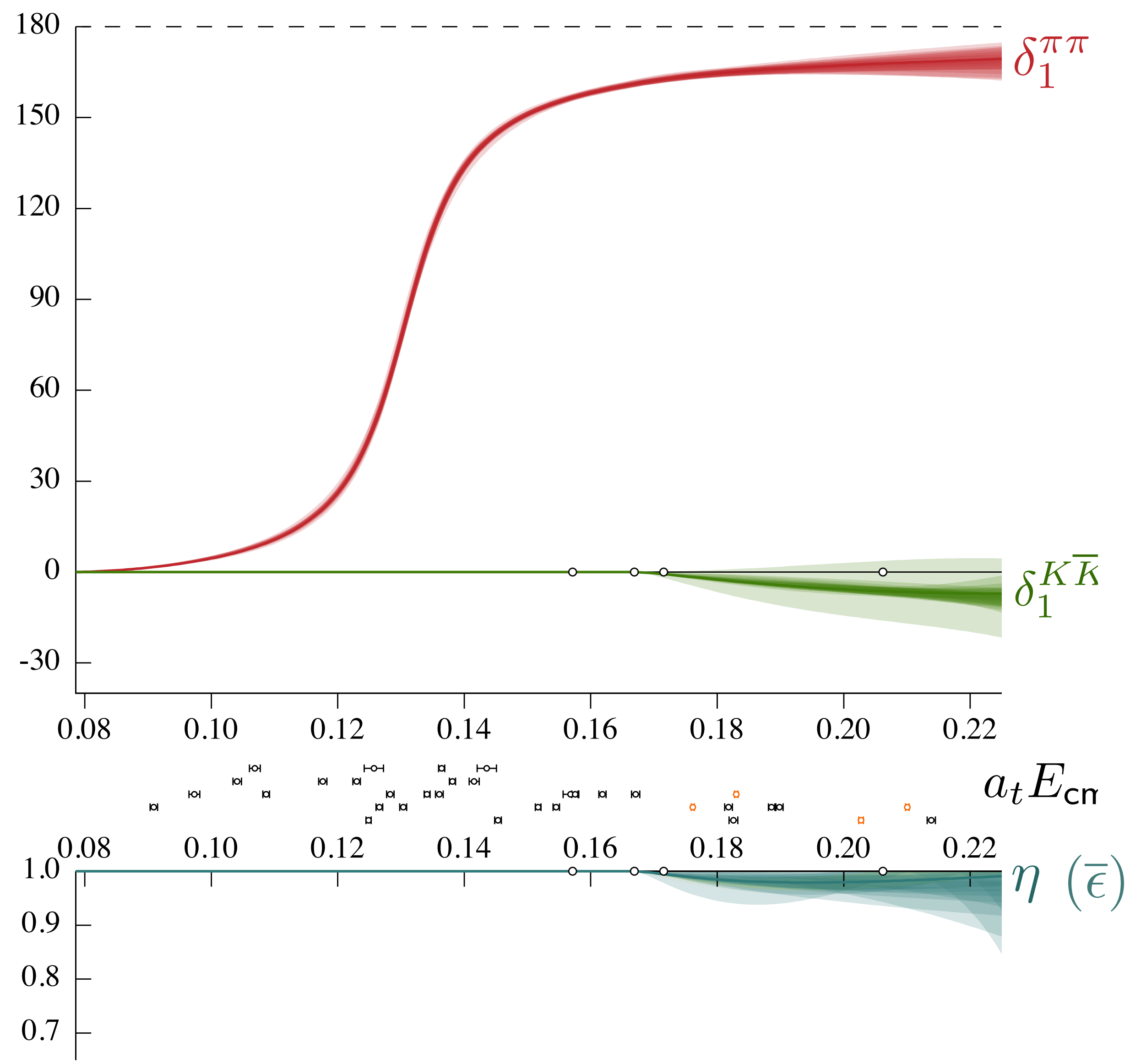
Wilson et al.(HadSpec),
Phys.Rev. D92 (2015), 094502

Using a range of parametrizations:

Pole position:



All three scattering parameters:



$N_f = 2+1, m_\pi = 236\ \mathrm{MeV},\ V \approx (4\ \mathrm{fm})^3$

SUMMARY OF LECTURE I

Lattice QCD workflow

GENERATE A SAMPLE OF VACUUM CONFIGURATIONS → COMPUTE EUCLIDEAN CORRELATION FUNCTIONS → ANALYZE CORRELATION FUNCTIONS: NUMERICS AND ANALYTICAL WORK

GENERATE A SAMPLE OF VACUUM CONFIGURATIONS

— Hybrid Monte Carlo to sample gauge configurations
— Determinant of a high-dimensional matrix required

COMPUTE EUCLIDEAN CORRELATION FUNCTIONS

— Quark contractions
— Inverting a high-dimensional matrix required (to get the quark propagators)

ANALYZE CORRELATION FUNCTIONS: NUMERICS AND ANALYTICAL WORK

— Assess stat. and sys. uncertainties (take the continuum and infinite-volume limits)
— Connect to physical observables

LECTURE II: STRUCTURE QUANTITIES FROM LATTICE QCD…

LECTUTE II:
NUCLEON STRUCTURE FROM LATTICE QCD

Let’s enumerate some of the methods that give access to structure quantities in general:

Three(four)-point functions

For e.g., form factors, moments of structure functions, Compton amplitude, transition amplitudes

Background-field methods

For e.g., EM moments and polarizabilities, charge radius, form factors and transition amplitudes.

Feynman-Hellmann inspired methods

Similar to background fields. For e.g., axial charge, form factors, EM moments, transition amplitudes

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A three-point (3pt) function:

$$C_{\widetilde{\chi}\mathcal{O}\chi}(x', y, x) \equiv \langle \chi(x')\mathcal{O}(y)\widetilde{\chi}(x)\rangle$$

Annihilate the state

Insert the operator

Create the state

Chambers, http://inspirehep.net/record/1744874/

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Annihilate the state

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Create the state

Spectral decomposition of the 3pt function in Euclidean spacetime

$$G_{\chi\mathcal{O}\widetilde{\chi}}(\mathbf{p}', \mathbf{p}; t', \tau, t) = \sum_{\mathrm{X,Y}} \frac{e^{-E_{\mathrm{X}}(\mathbf{p}')(t'-\tau)}}{2E_{\mathrm{X}}(\mathbf{p}')} \frac{e^{-E_{\mathrm{Y}}(\mathbf{p})(\tau-t)}}{2E_{\mathrm{Y}}(\mathbf{p})} \langle\Omega|\chi(0)|\mathrm{X}(\mathbf{p}')\rangle \langle\mathrm{X}(\mathbf{p}')|\mathcal{O}(0)|\mathrm{Y}(\mathbf{p})\rangle \langle\mathrm{Y}(\mathbf{p})|\widetilde{\chi}(0)|\Omega\rangle$$

A complete set of states

Another complete set of states

A three-point (3pt) function:

$$C_{\tilde{\chi}\mathcal{O}\chi}(x', y, x) \equiv \langle \chi(x') \mathcal{O}(y) \tilde{\chi}(x) \rangle$$

Annihilate the state

Insert the operator

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Spectral decomposition of the 3pt function in Euclidean spacetime

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A complete set of states

Another complete set of states

Long-separation behavior dominated by ground states

$$G_{\chi\mathcal{O}\tilde{\chi}}(\mathbf{p}', \mathbf{p}; t', \tau, t) \xrightarrow{\text{large } t'-\tau, \tau-t} \frac{e^{-E_{\mathrm{X}_0}(\mathbf{p}')(t'-\tau)}}{2E_{\mathrm{X}_0}(\mathbf{p}')} \frac{e^{-E_{\mathrm{X}_0}(\mathbf{p})(\tau-t)}}{2E_{\mathrm{X}_0}(\mathbf{p})} \sum_{r',r} \langle \Omega | \chi(0) | \mathrm{X}_0(\mathbf{p}', r') \rangle \, \langle \mathrm{X}_0(\mathbf{p}', r') | \mathcal{O}(0) | \mathrm{X}_0(\mathbf{p}, r) \rangle \, \langle \mathrm{X}_0(\mathbf{p}, r) | \tilde{\chi}(0) | \Omega \rangle$$

If there are degenerate ground states

Desired ground state to ground state matrix element (unrenormalized and in a finite volume)

A three-point (3pt) function:

$$C_{\widetilde{\chi}\mathcal{O}\chi}(x', y, x) \equiv \langle \chi(x') \mathcal{O}(y) \widetilde{\chi}(x) \rangle$$

Annihilate the state

Insert the operator

Create the state

Spectral decomposition of the 3pt function in Euclidean spacetime

$$G_{\chi\mathcal{O}\widetilde{\chi}}(\mathbf{p}', \mathbf{p}; t', \tau, t) = \sum_{\mathrm{X,Y}} \frac{e^{-E_{\mathrm{X}}(\mathbf{p}')(t'-\tau)}}{2E_{\mathrm{X}}(\mathbf{p}')} \frac{e^{-E_{\mathrm{Y}}(\mathbf{p})(\tau-t)}}{2E_{\mathrm{Y}}(\mathbf{p})} \langle \Omega | \chi(0) | \mathrm{X}(\mathbf{p}') \rangle \, \langle \mathrm{X}(\mathbf{p}') | \mathcal{O}(0) | \mathrm{Y}(\mathbf{p}) \rangle \, \langle \mathrm{Y}(\mathbf{p}) | \widetilde{\chi}(0) | \Omega \rangle$$

A complete set of states

Another complete set of states

Long-separation behavior dominated by ground states

$$G_{\chi\mathcal{O}\widetilde{\chi}}(\mathbf{p}', \mathbf{p}; t', \tau, t) \xrightarrow{\text{large } t'-\tau, \tau-t} \frac{e^{-E_{\mathrm{X}_0}(\mathbf{p}')(t'-\tau)}}{2E_{\mathrm{X}_0}(\mathbf{p}')} \frac{e^{-E_{\mathrm{X}_0}(\mathbf{p})(\tau-t)}}{2E_{\mathrm{X}_0}(\mathbf{p})} \sum_{r', r} \langle \Omega | \chi(0) | \mathrm{X}_0(\mathbf{p}', r') \rangle \, \langle \mathrm{X}_0(\mathbf{p}', r') | \mathcal{O}(0) | \mathrm{X}_0(\mathbf{p}, r) \rangle \, \langle \mathrm{X}_0(\mathbf{p}, r) | \widetilde{\chi}(0) | \Omega \rangle$$

If there are degenerate ground states

Desired ground state to ground state matrix element (unrenormalized and in a finite volume)

Taking a proper ratio to 2pt functions

$$R_{\chi\mathcal{O}\widetilde{\chi}}(\mathbf{p}', \mathbf{p}; t', \tau, t) \overset{\text{large } t'-\tau, \tau-t}{\propto} \langle \mathrm{X}_0(\mathbf{p}', r') | \mathcal{O}(0) | \mathrm{X}_0(\mathbf{p}, r) \rangle$$

EXERCISE 6

If the computational resources do not allow large source, operator and sink time separations to be achieved, one should worry about the effect of excited states. One way to have more confidence over the extracted ground state to ground state matrix element is to perform a multi-exponential fits to the ratio of 3pt to 2pt functions as a function of both the source-sink and the source-operator separations. Assume that both the ground state and the first excited states contribute significantly to such a ratio. Write down a generic form for such a multi-exponential function.

BONUS EXERCISE 4

In the above exercise, sum over the time insertions of the operator and write down a new form for the ratio of 3pt to 2pt functions, which now is only a function of the source-sink time separation. This is referred to as the summation method in literature.

Example: The application of 3pt function method to obtain the axial charge/form factors of the nucleon

Constantinou, arXiv:1411.0078 [hep-lat].

$$\langle N(p',s')|\bar{\psi}(x)\,\gamma_\mu\,\gamma_5\,\psi(x)|N(p,s)\rangle = i\left(\frac{m_N^2}{E_N(\mathbf{p}')E_N(\mathbf{p})}\right)^{1/2}\bar{u}_N(p',s')\left[G_A(q^2)\gamma_\mu\gamma_5 + \frac{q_\mu\gamma_5}{2m_N}G_p(q^2)\right]u_N(p,s)$$

Axial-vector current

Nucleon spinor

Axial and pseudo scalar form factors

$G_A(0) = g_A$

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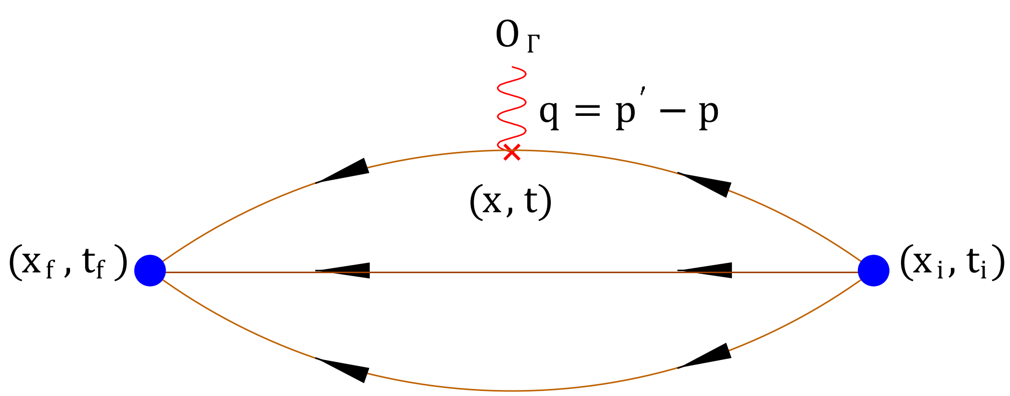
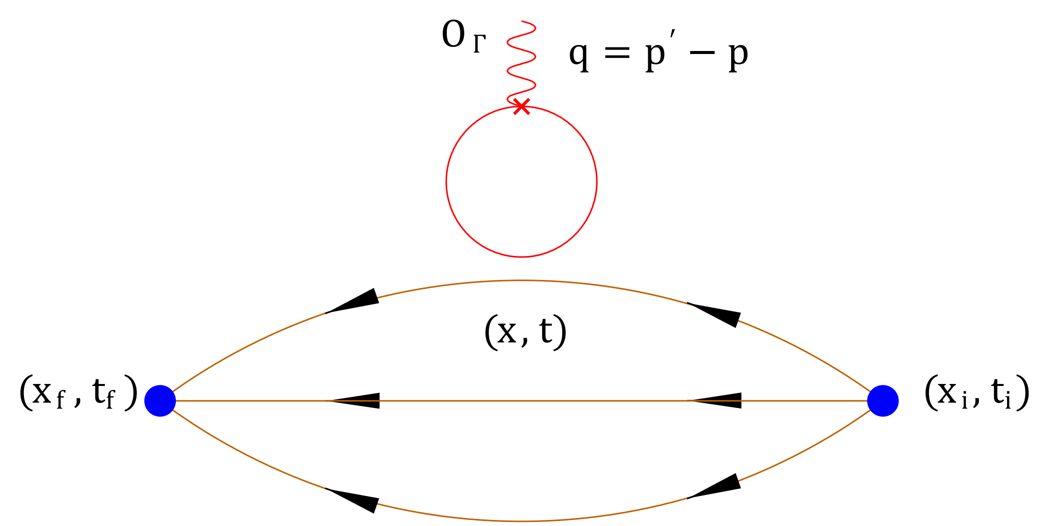


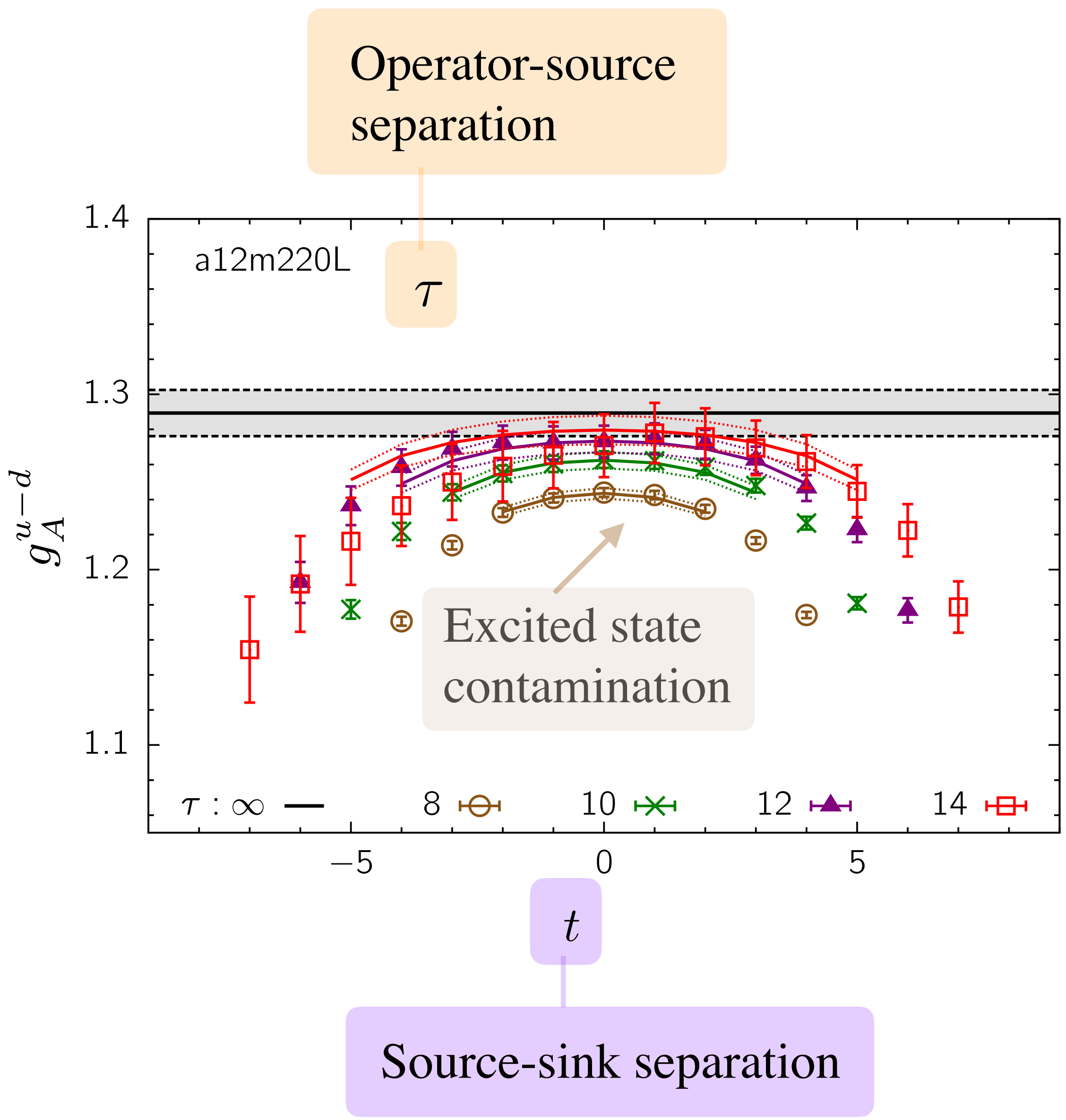
Connected contribution



Disconnected contribution
(vanishes at isospin limit for isovector quantities)

Example: The application of 3pt function method to obtain the axial charge/form factors of the nucleon

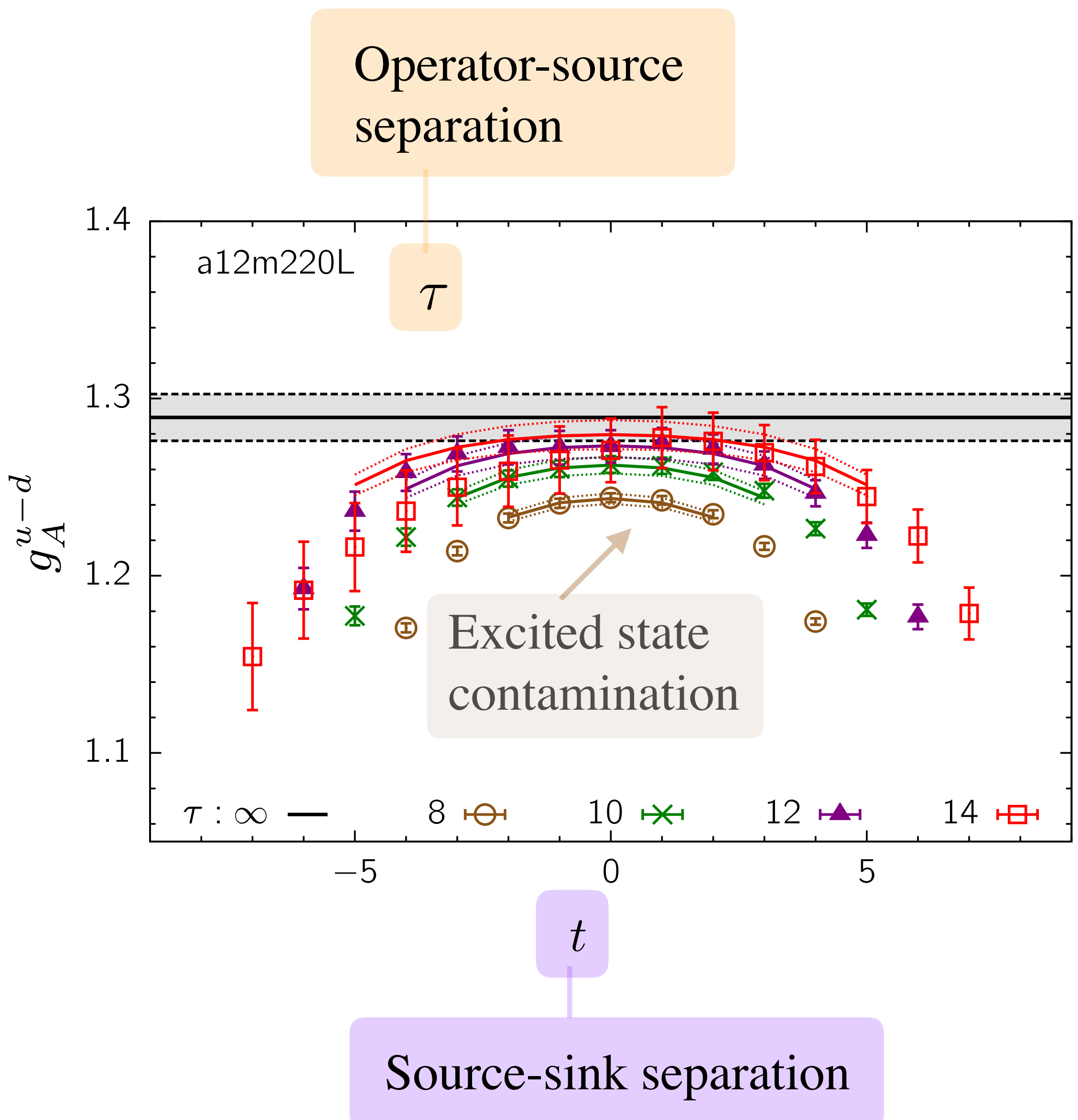
Gupta et al (PNDME), Phys. Rev. D 98, 034503 (2018)



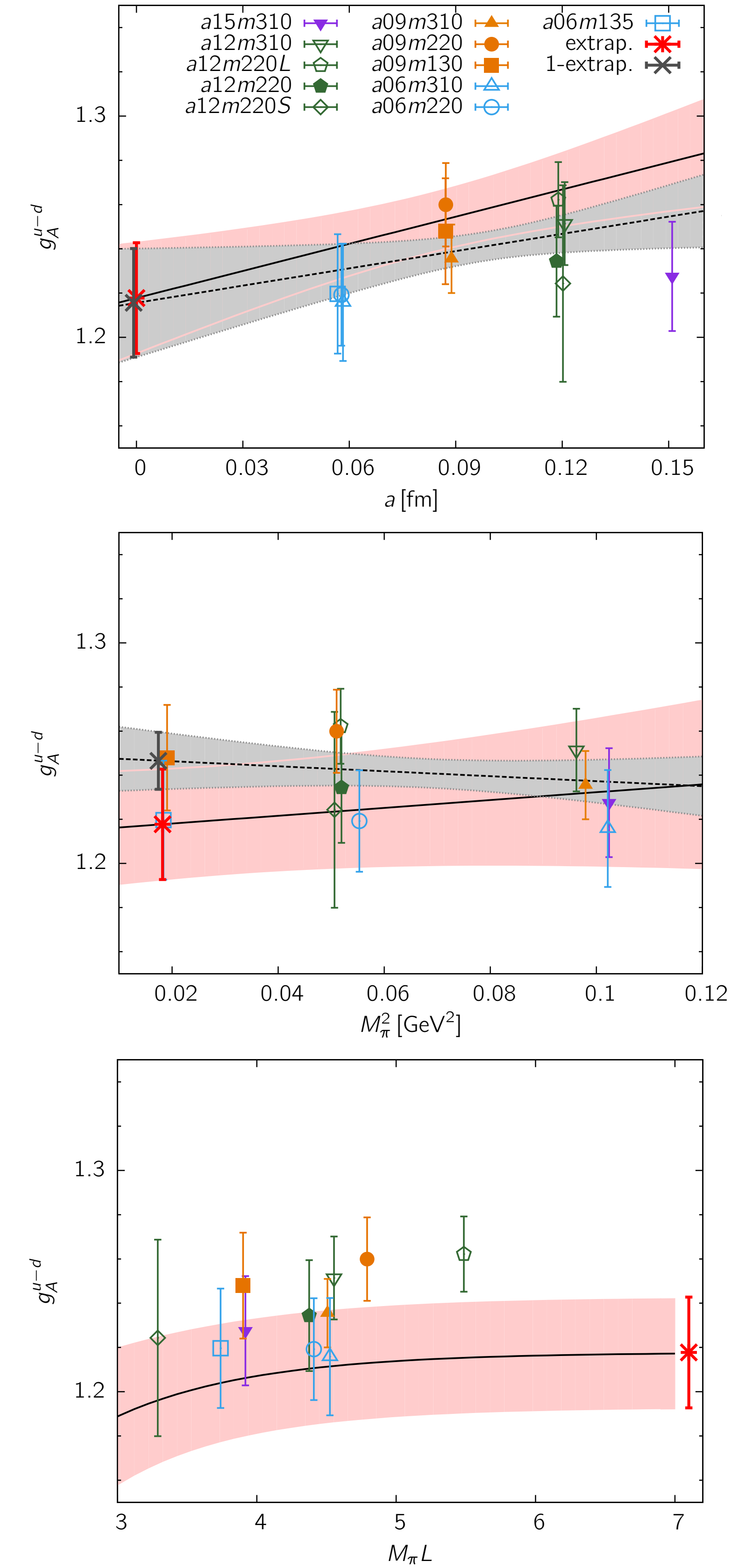
3pt function for a single lattice spacing, volume and quark masses

Example: The application of 3pt function method to obtain the axial charge/form factors of the nucleon

Gupta et al (PNDME), Phys. Rev. D 98, 034503 (2018)



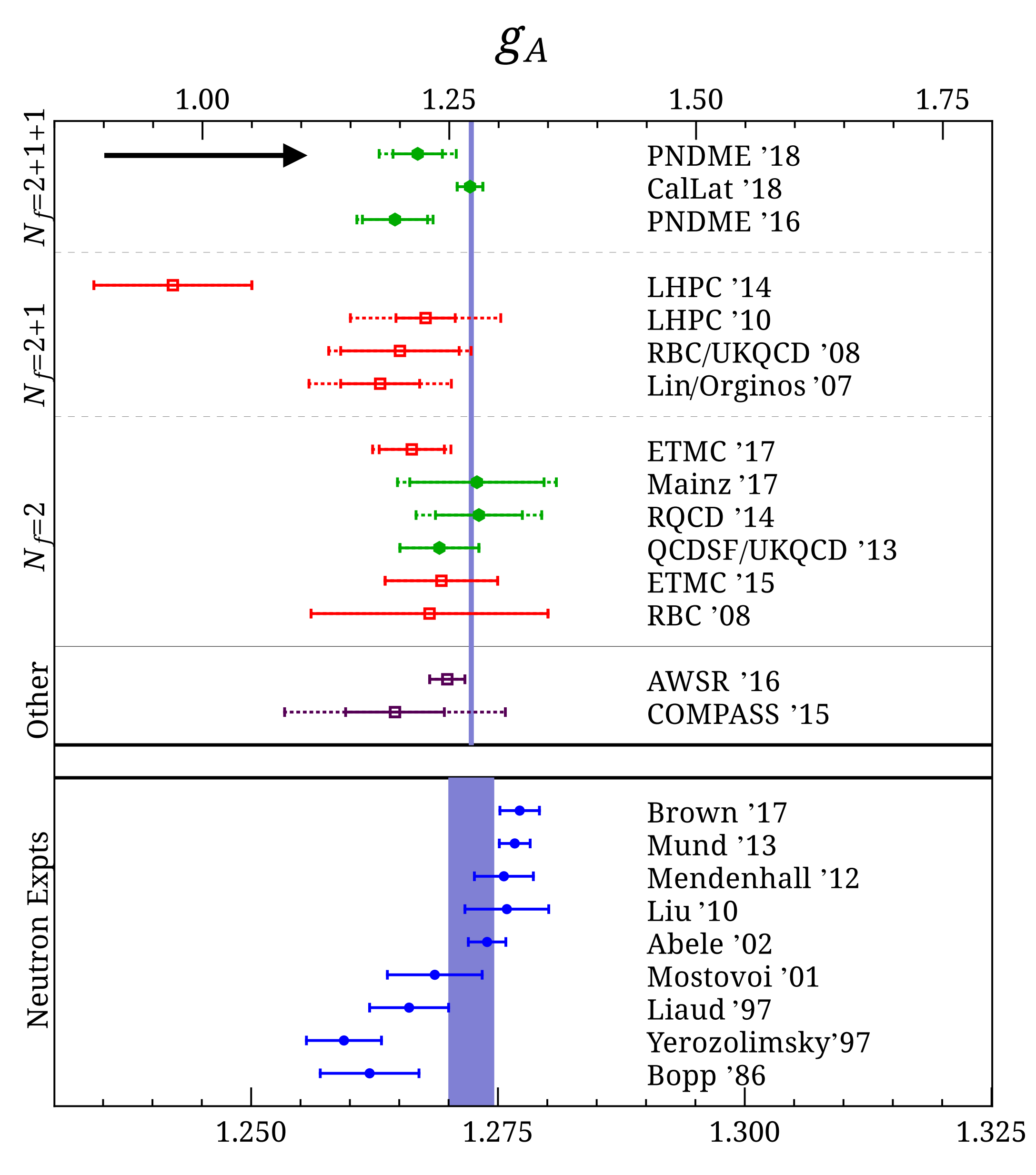
3pt function for a single lattice spacing, volume and quark masses



Extrapolation to continuum, infinite volume and physical quark masses

Example: The application of 3pt function method to obtain the axial charge/form factors of the nucleon

Gupta et al (PNDME), Phys. Rev. D 98, 034503 (2018)

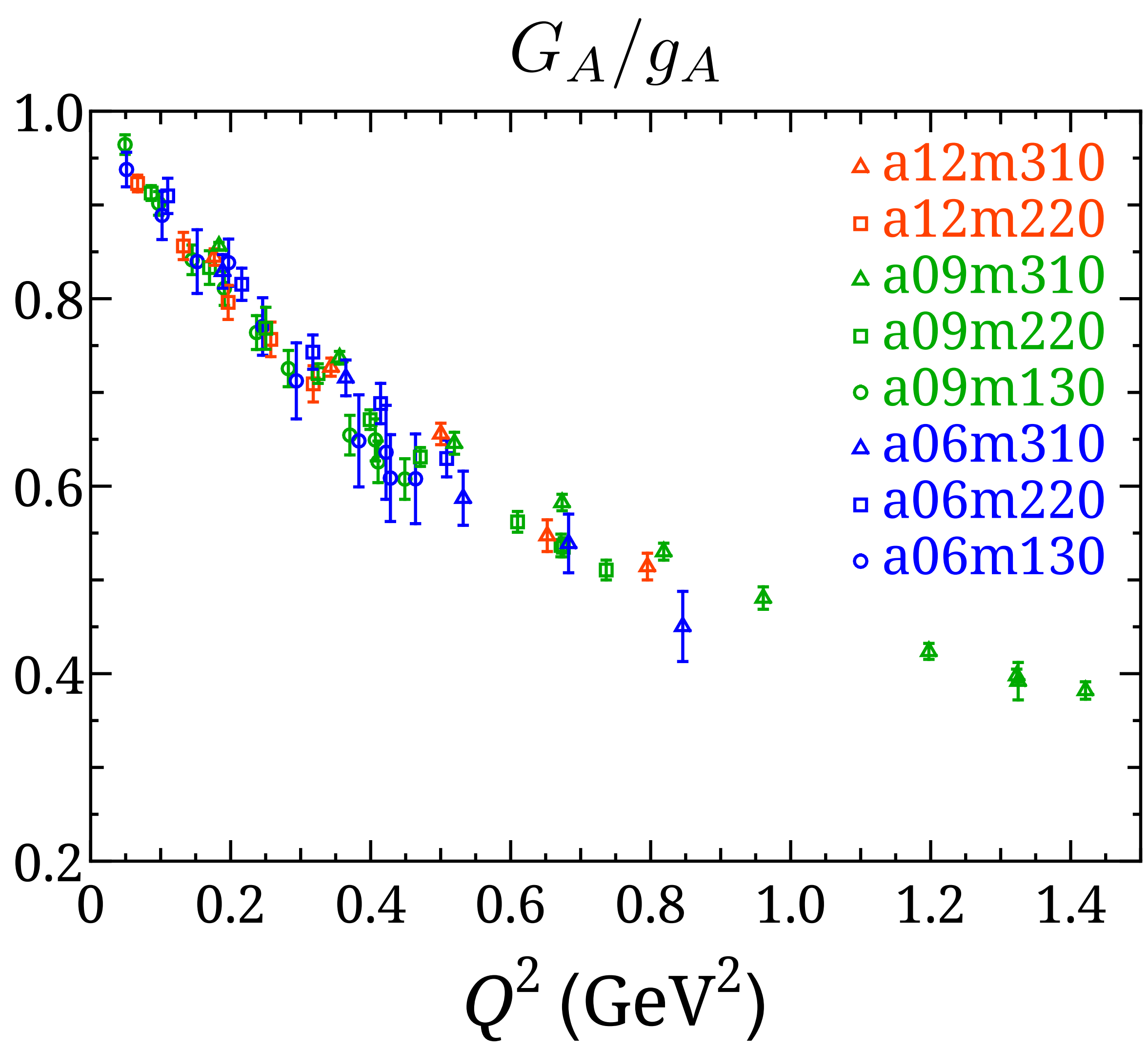


Compilation of results

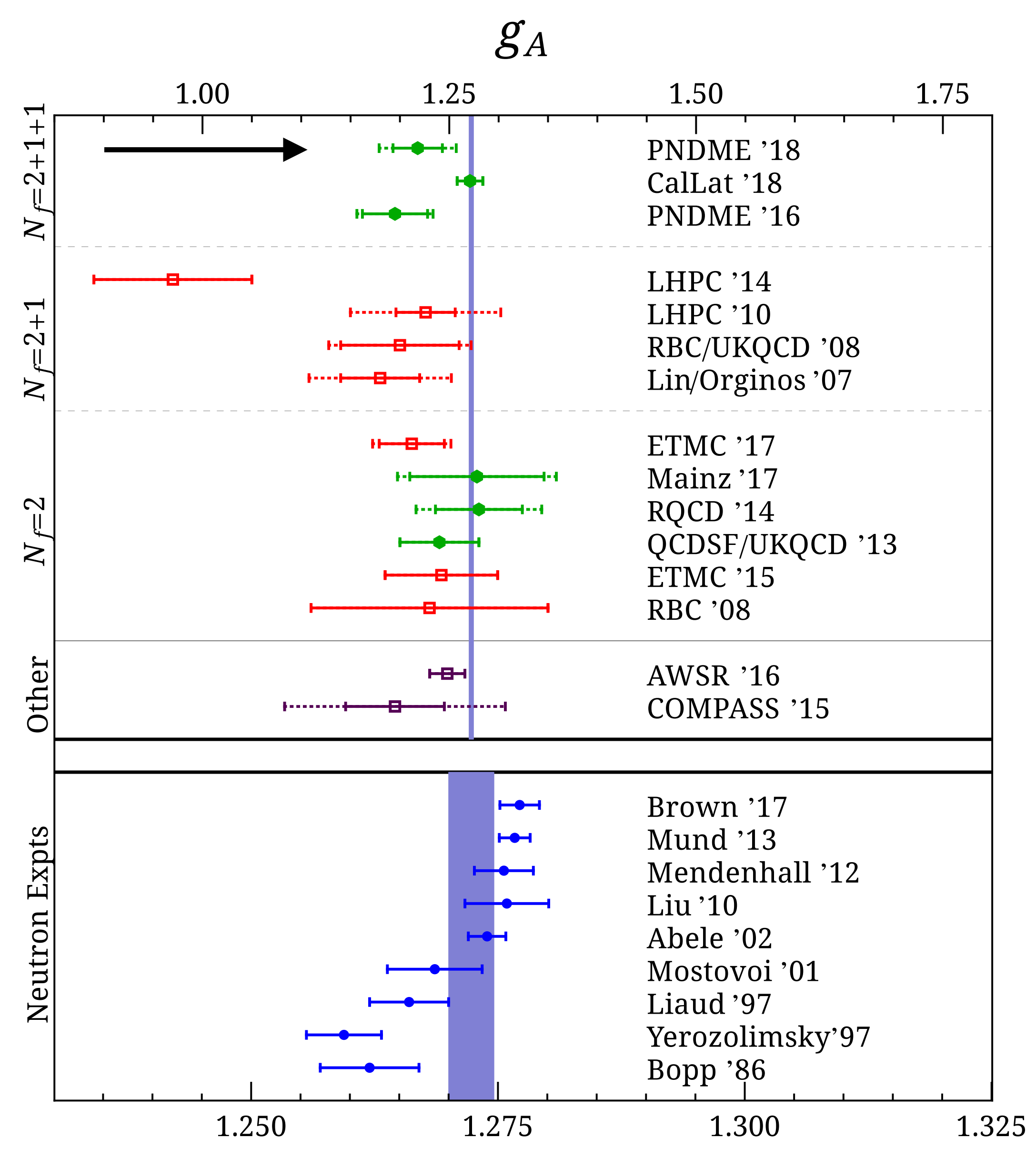
Example: The application of 3pt function method to obtain the axial charge/form factors of the nucleon

Gupta et al (PNDME), Phys. Rev. D 98, 034503 (2018)

Jang et al, EPJ Web Conf. 175, 06033 (2018)



Axial form factor results



Compilation of results

Example: The spin decomposition of the nucleon

Alexandru, Phys. Rev. Lett. 119, 142002 (2017).

$$J_N = \sum_{q=u,d,s,c\cdots} \left(\frac{1}{2}\Delta\Sigma_q + L_q \right) + J_g$$

- $\frac{1}{2}\Delta\Sigma_q$: Quark spin
- L_q: quark orbital angular momentum
- J_g: Total gluon angular momentum

Ji, Phys. Rev. Lett. 78, 610 (1997).

Matrix elements needed

$$\langle N(p,s')|\mathcal{O}^{\mu}_{A}|N(p,s)\rangle = \bar{u}_N(p,s')\left[g^q_A \gamma^\mu \gamma_5\right] u_N(p,s)$$

$$\langle N(p',s')|\mathcal{O}^{\mu\nu}_{V}|N(p,s)\rangle = \bar{u}_N(p',s')\Lambda^q_{\mu\nu}(Q^2) u_N(p,s)$$

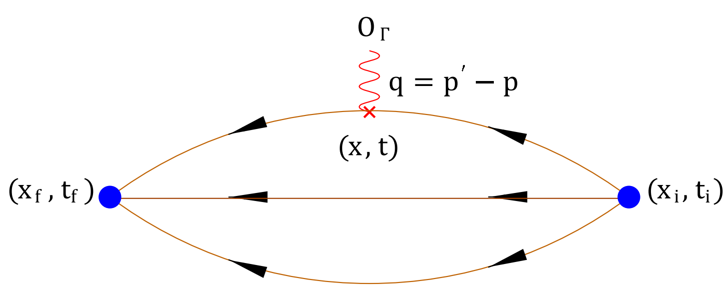
$$\langle N(p',s')|\mathcal{O}^{\mu\nu}_{g}|N(p,s)\rangle = \bar{u}_N(p',s')\Lambda^g_{\mu\nu}(Q^2) u_N(p,s)$$

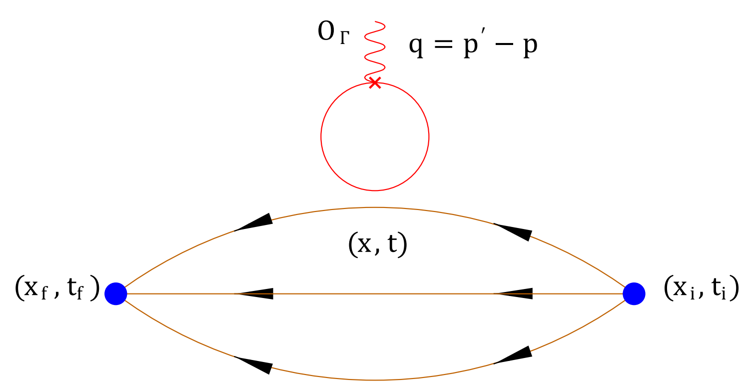
With operators

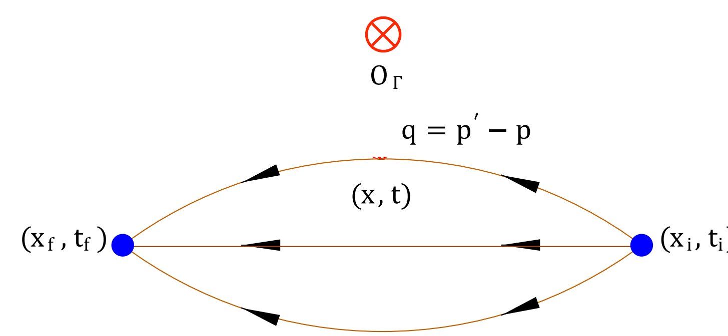
$$\mathcal{O}^{\mu}_{A} = \bar{q}\gamma_\mu\gamma_5 q$$

$$\mathcal{O}^{\mu\nu}_{V} = \bar{q}\gamma^{\{\mu}\overleftrightarrow{D}^{\nu\}} q$$

$$\mathcal{O}^{\mu\nu}_{g} = 2\mathrm{Tr}[G_{\mu\sigma}G_{\nu\sigma}]$$





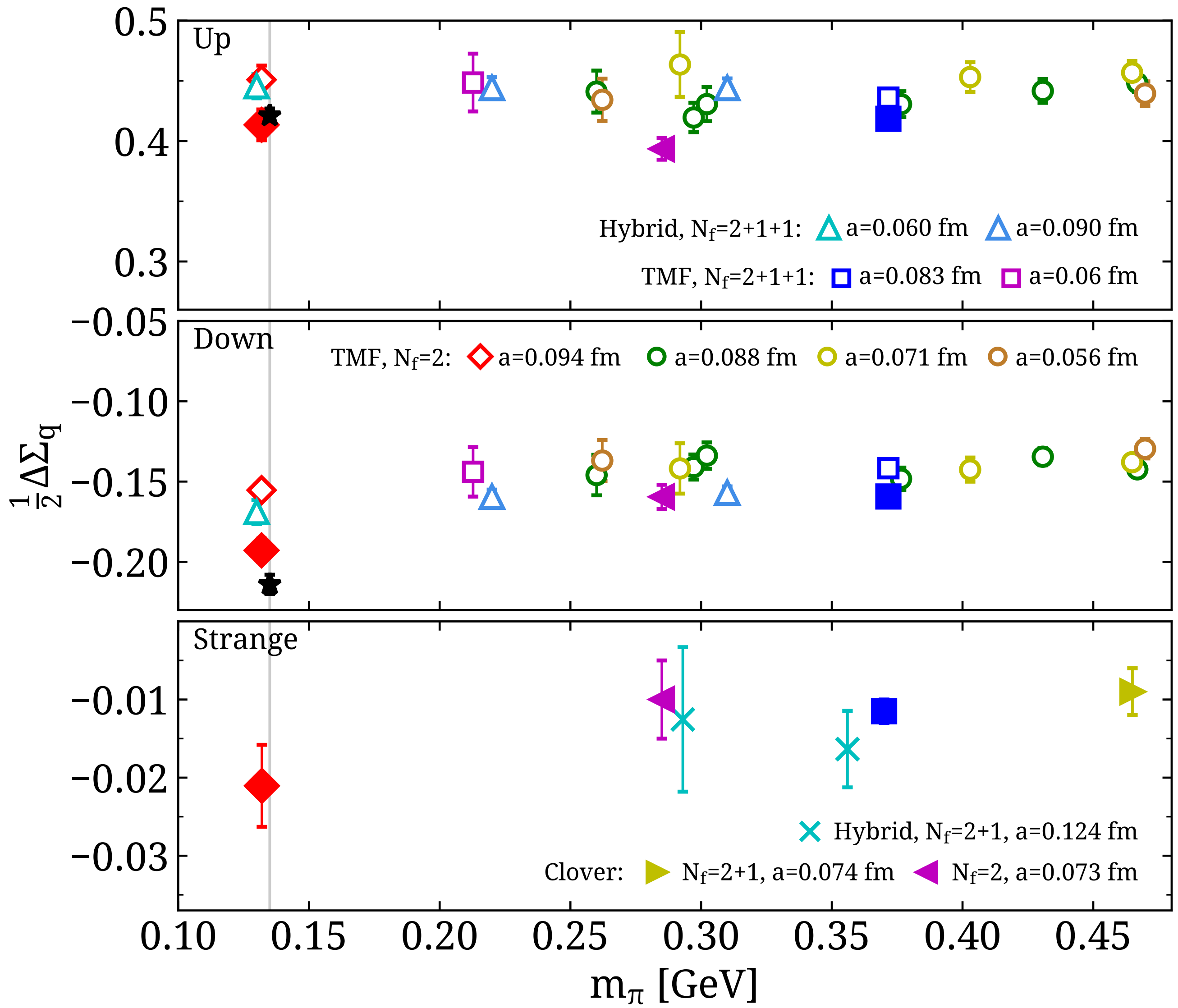


Quark contributions

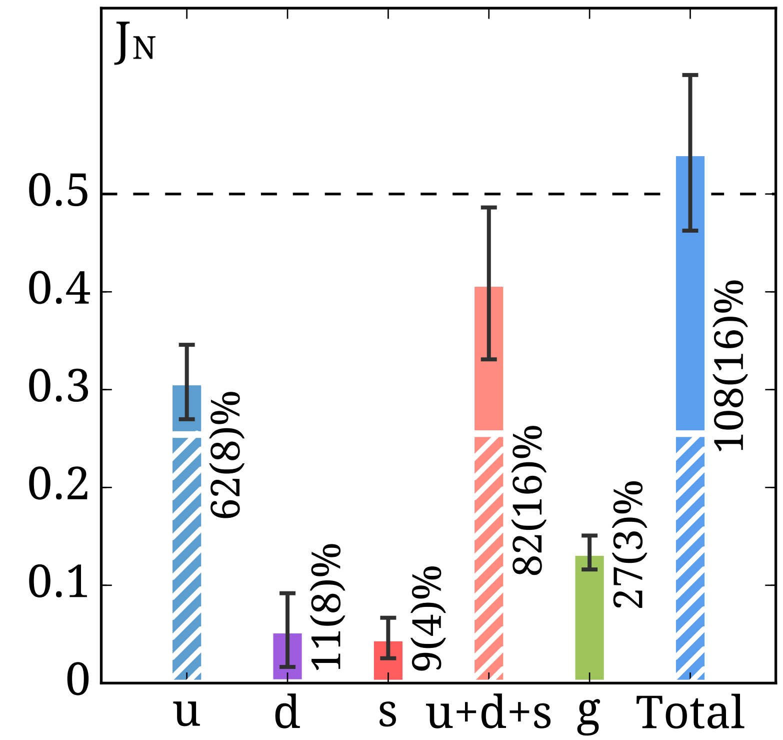
Qluonic contribution

Example: The spin decomposition of the nucleon

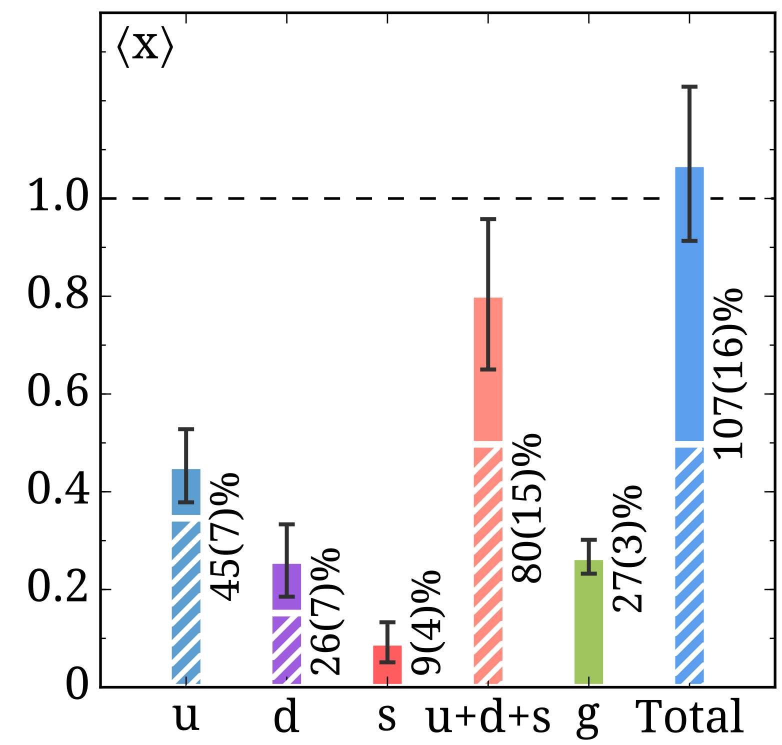
Alexandru, Phys. Rev. Lett. 119, 142002 (2017).



Quark spin contributions



Nucleon spin decomposition



Longitudinal momentum decomposition

Let’s enumerate a some of the methods that give access to structure quantities in general:

Three(four)-point functions

For e.g., form factors, moments of structure functions, Compton amplitude, transition amplitudes

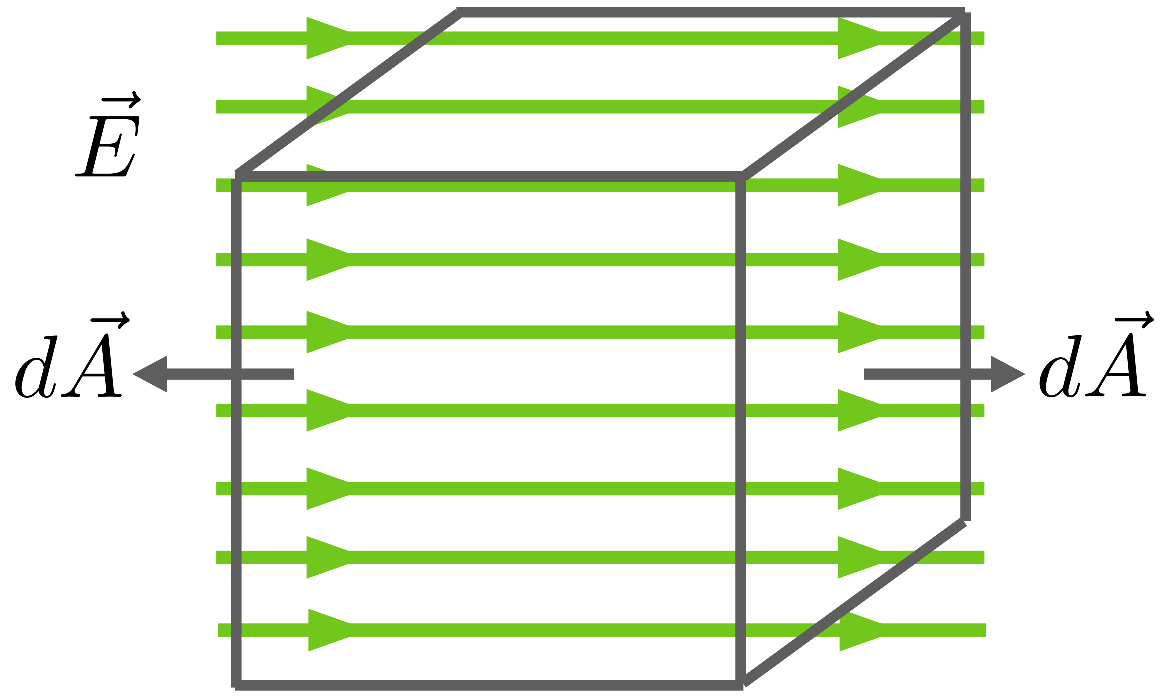
Background-field methods

For e.g., EM moments and polarizabilities, charge radius, form factors and transition amplitudes.

Feynman-Hellmann inspired methods

Similar to background fields. For e.g., axial charge, form factors, EM moments, transition amplitudes

Background fields are non-dynamical, i.e., there will be no pair creation and annihilation in vacuum with a classical EM background field. This mean the photon zero mode is no problem: it is absent in the calculation!



$$U^{(\mathrm{QCD})} \to U^{(\mathrm{QCD})} \times U^{(\mathrm{QED})}$$

Modify the links when forming the quark propagators.

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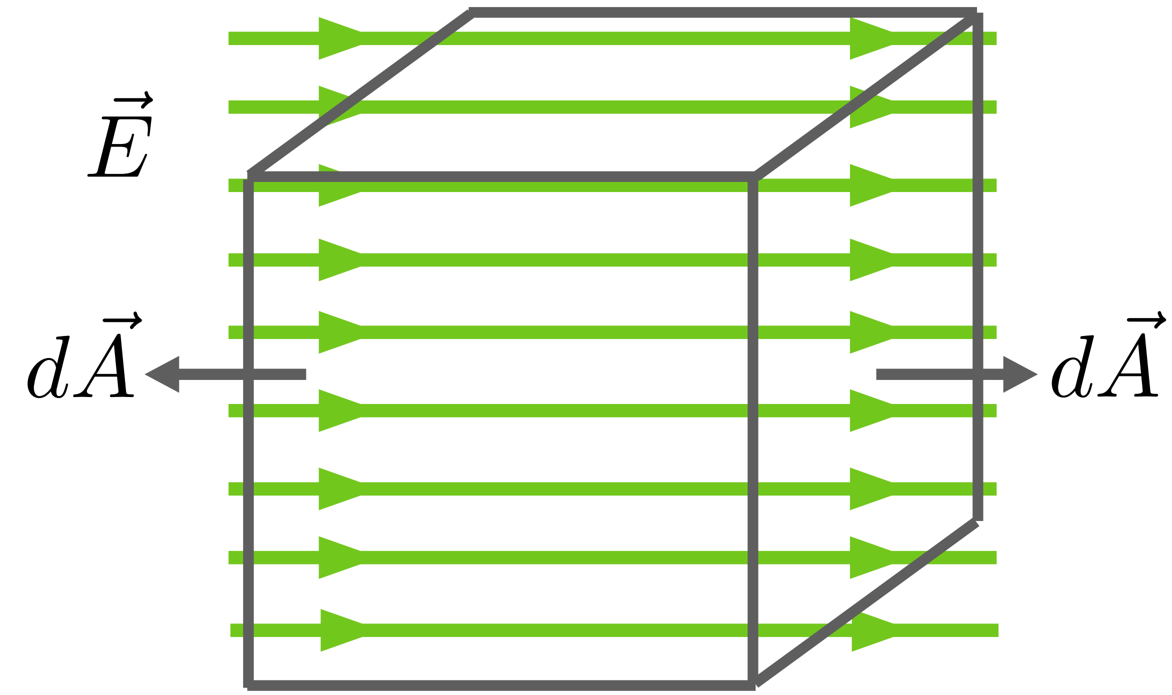
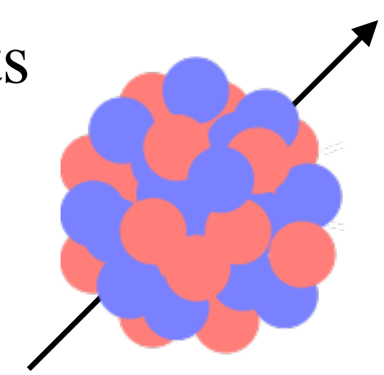


$$U^{(\mathrm{QCD})} \rightarrow U^{(\mathrm{QCD})} \times U^{(\mathrm{QED})}$$

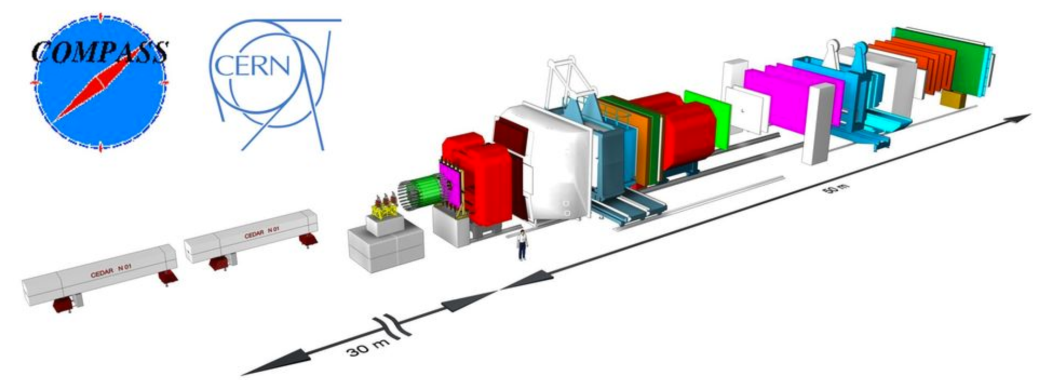
Modify the links when forming the quark propagators.

Traditionally they are used for constraining the response of hadrons/nuclei to external probes:

Magnetic moments



Electric and magnetic polarizabilities



See e.g., BEANE et al (NPLQCD), Phys.Rev.Lett. 113 (2014) 25, 252001 and Phys.Rev. D92 (2015) 11, 114502. for nuclear-physics calculations.

What does the requirement of periodicity impose on background fields? Let's consider a uniform background field.

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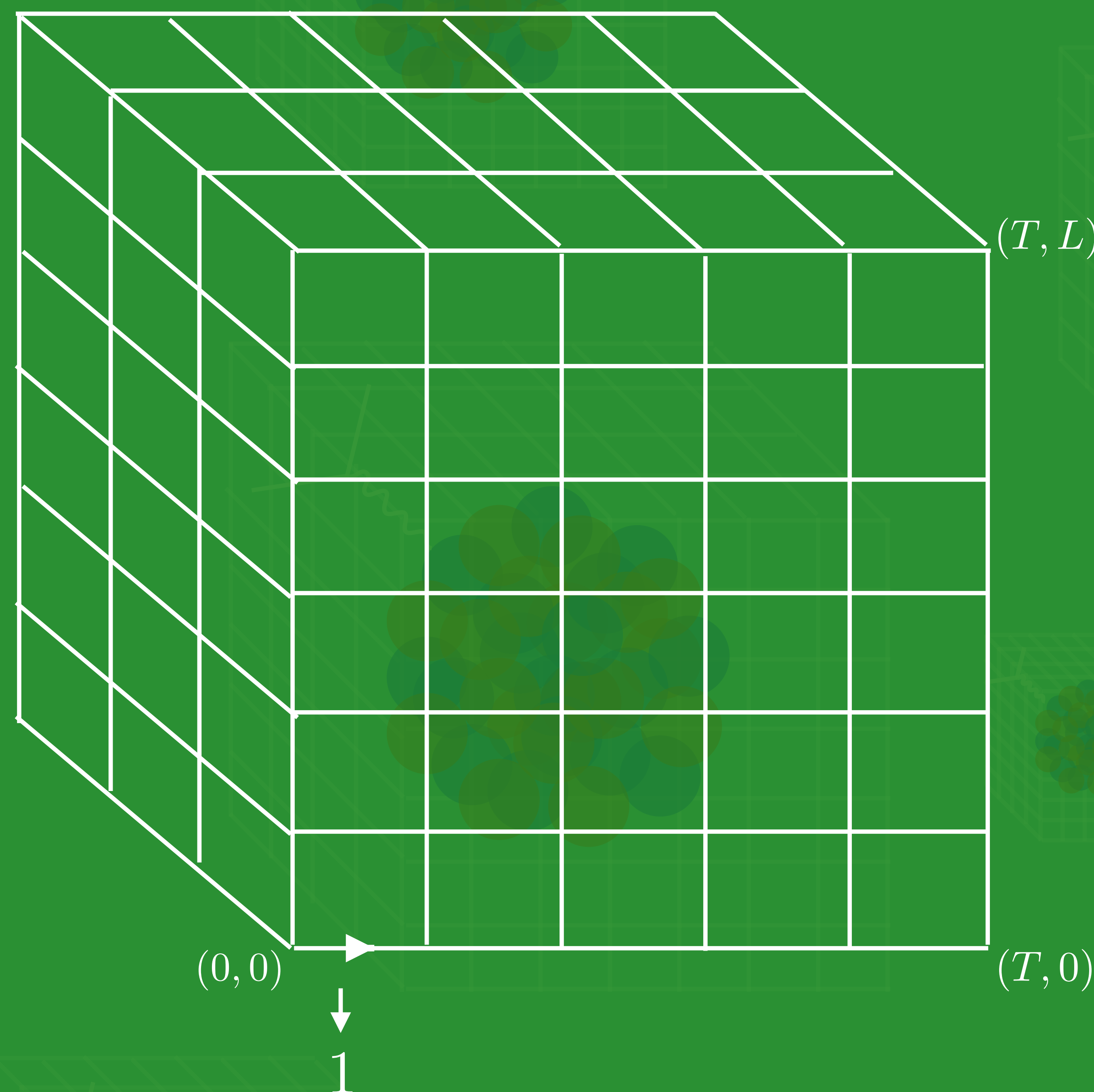
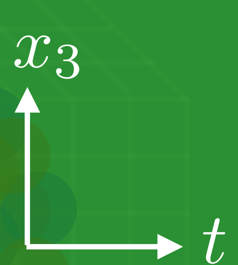
$$A_\mu = \left(-E \times \left(\mathbf{x}_3 - \left[\frac{\mathbf{x}_3}{L}\right] L\right), \mathbf{0}\right) \qquad \rightarrow \qquad \mathbf{E} = E\hat{\mathbf{x}}_3$$

$$U = e^{ie\hat{Q}\int A_\mu(z)dz_\mu}$$





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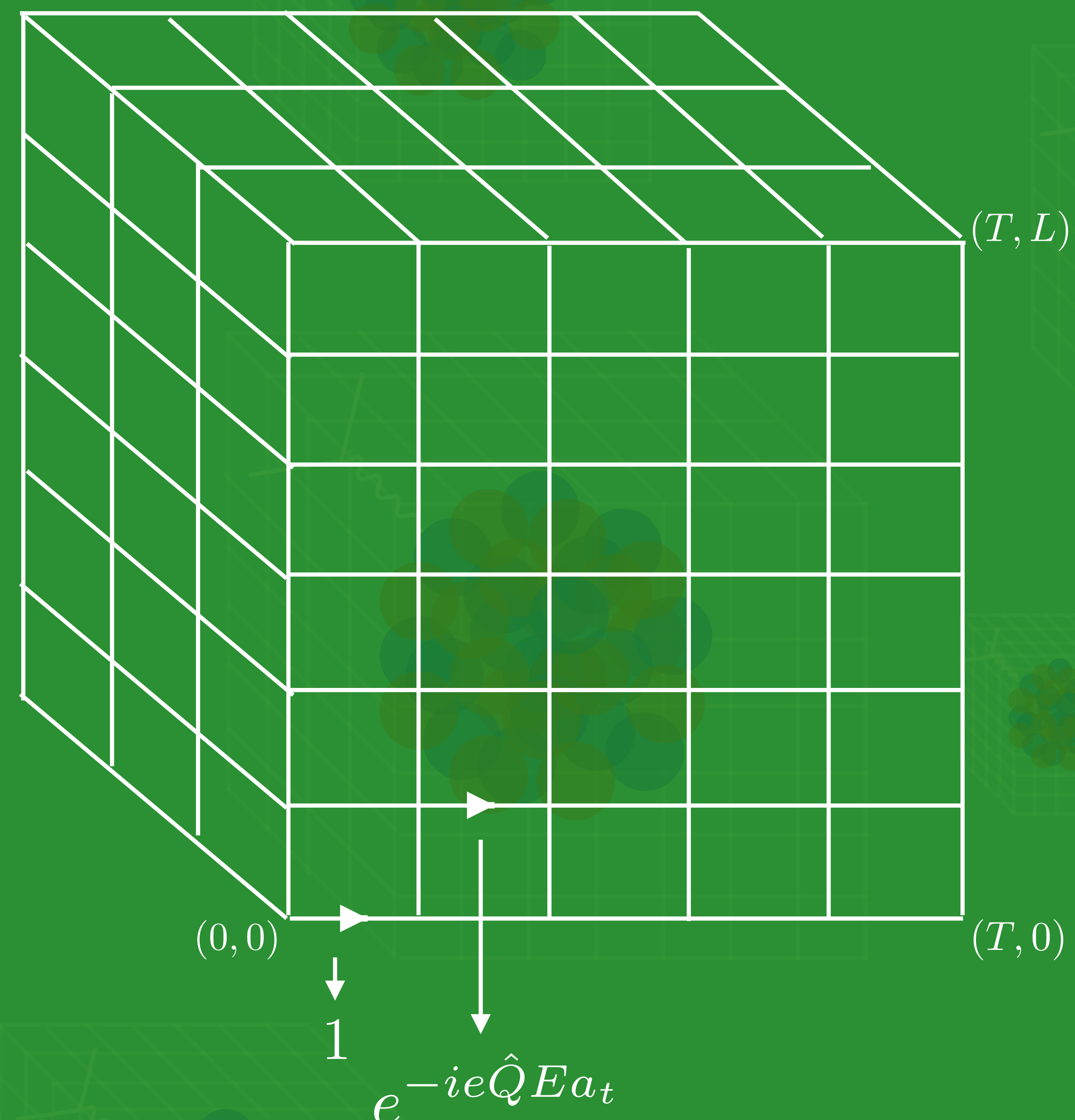

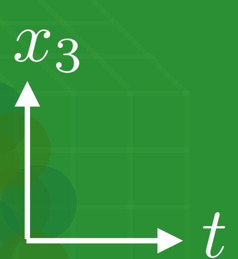

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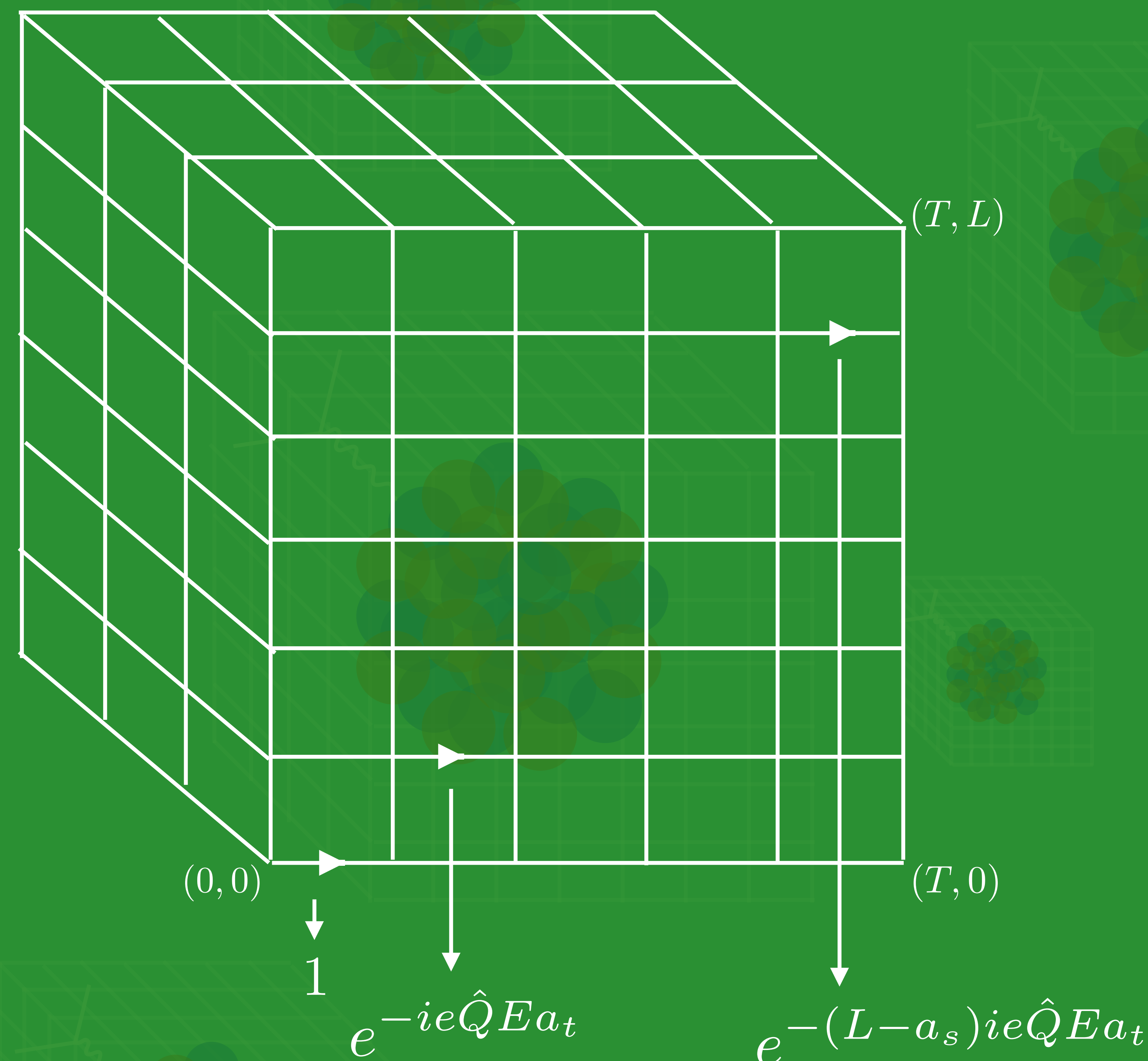

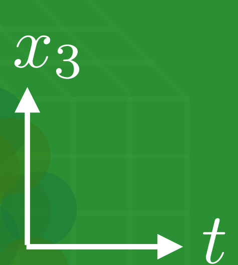

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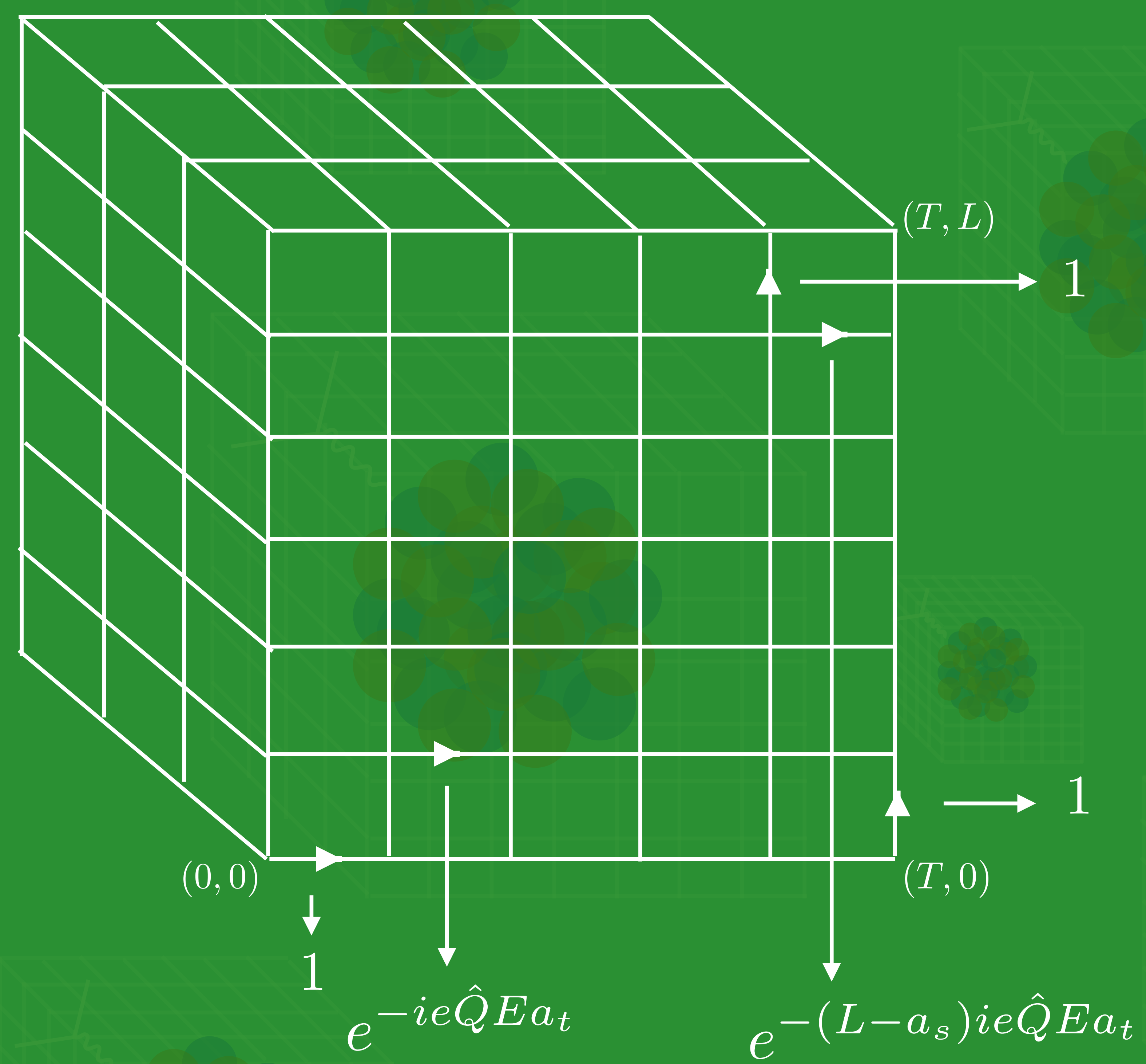

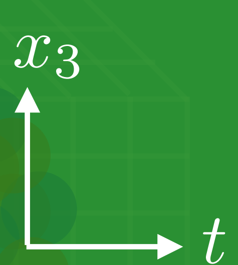

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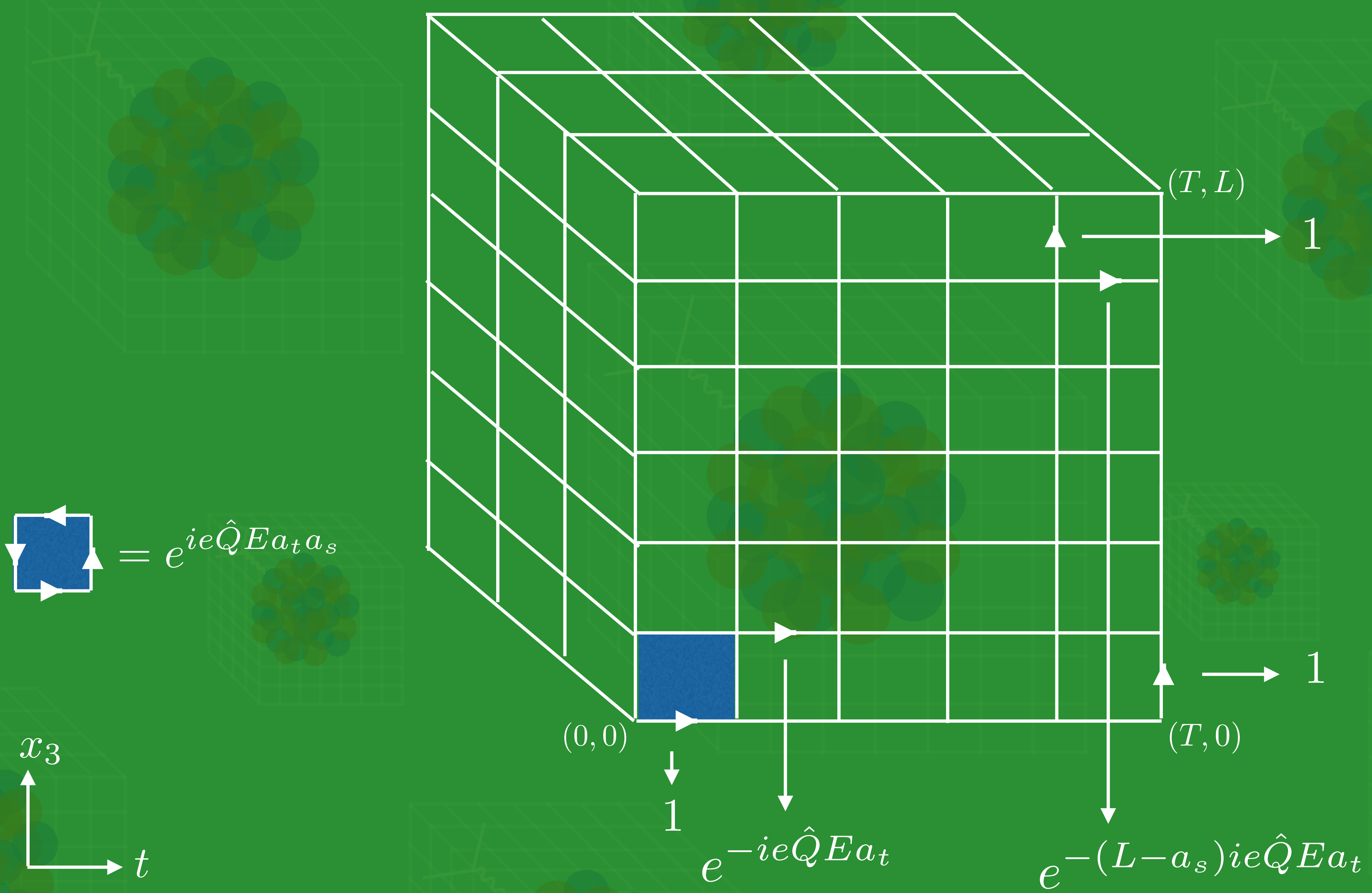

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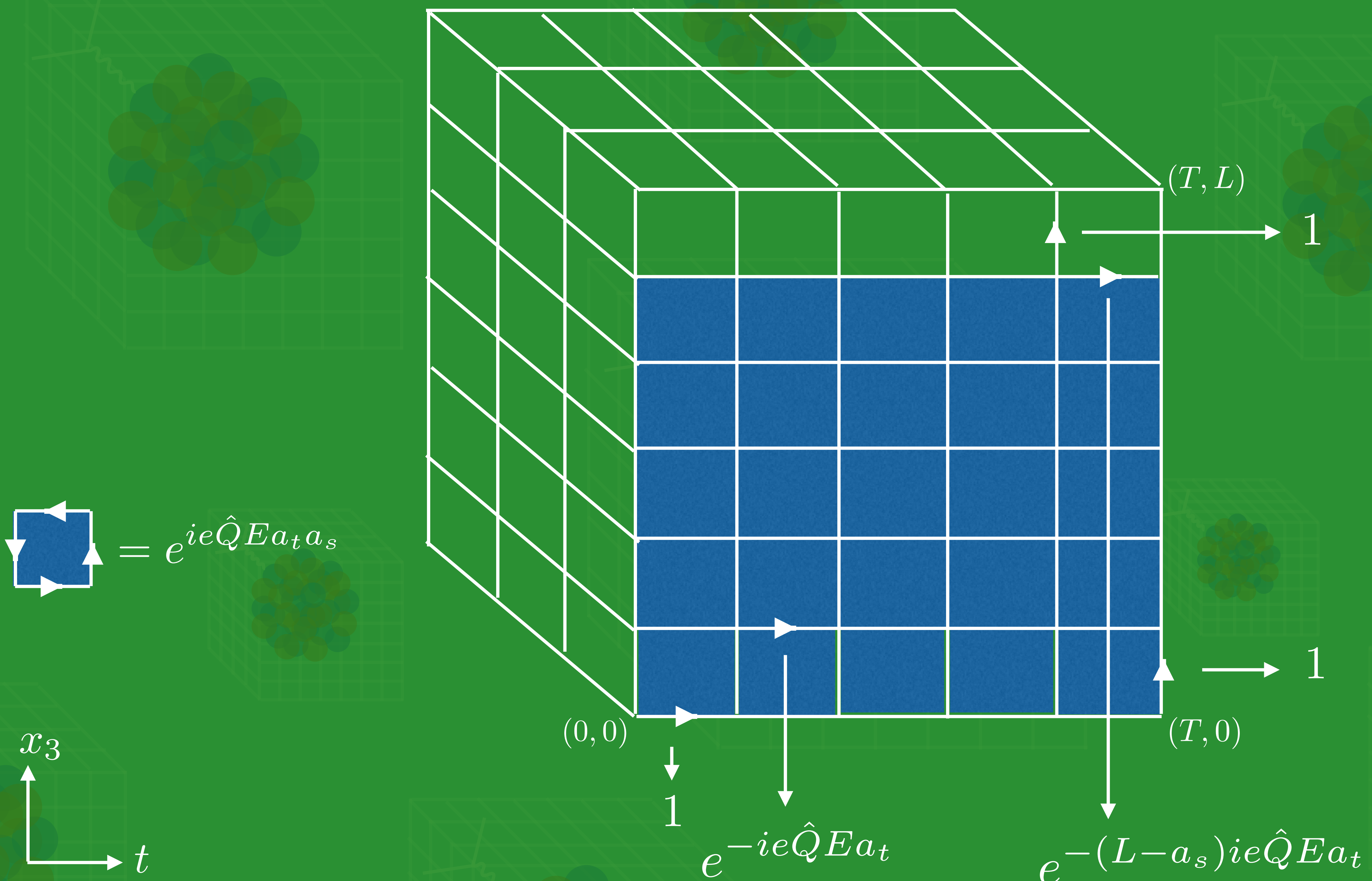

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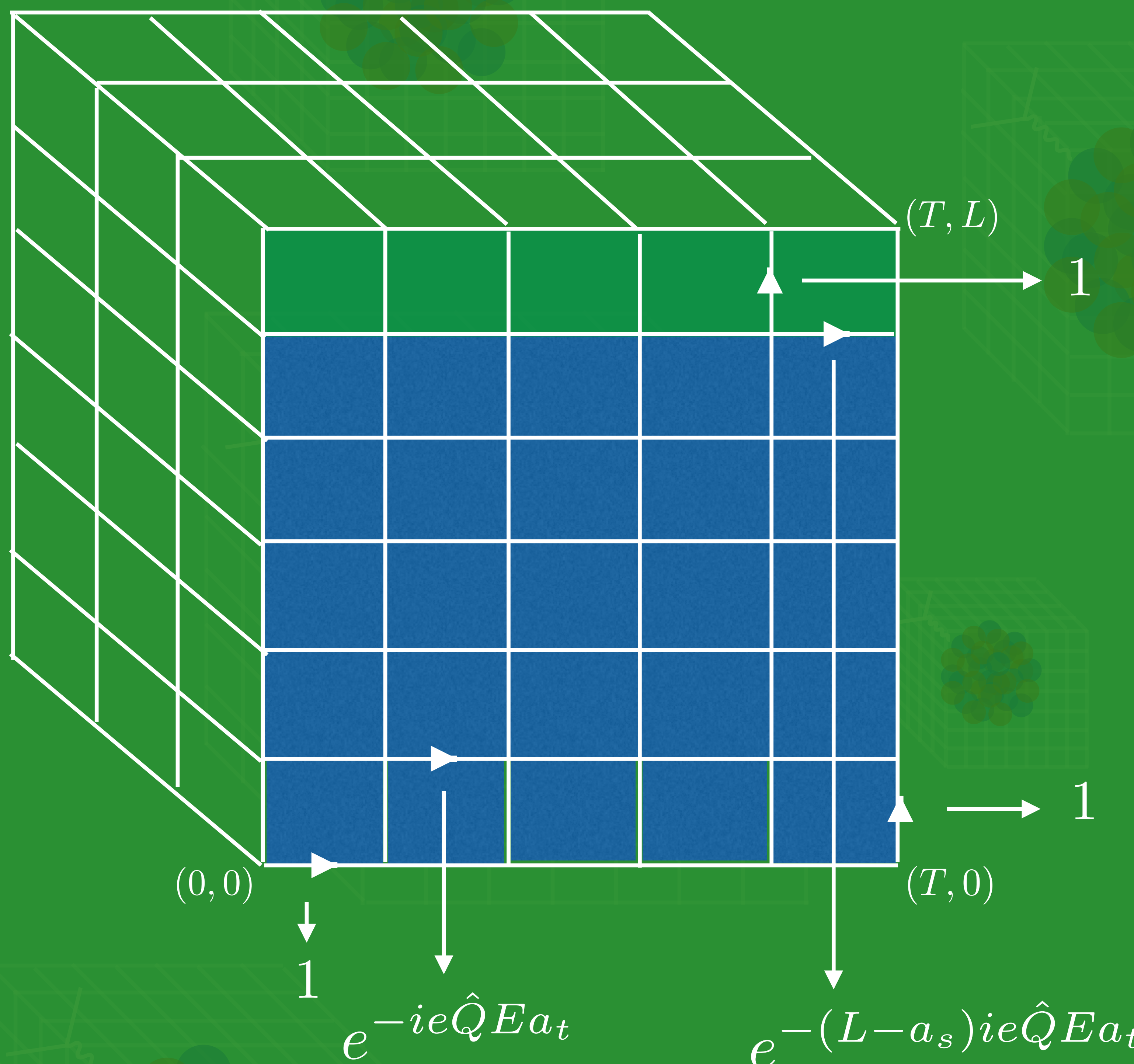
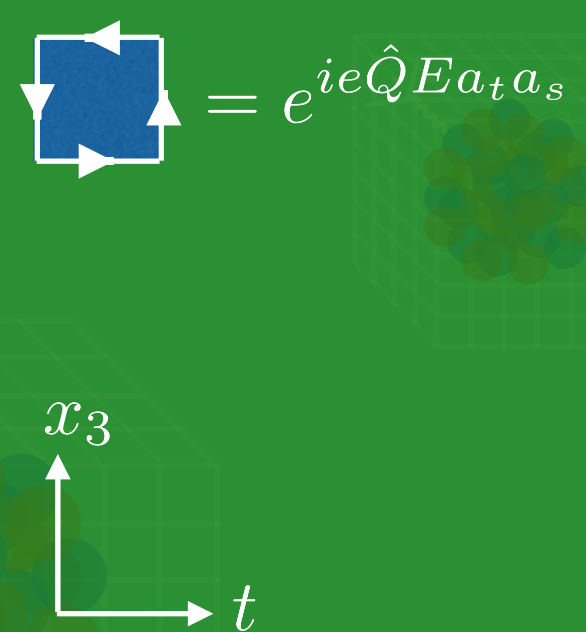
PERIODIC BC





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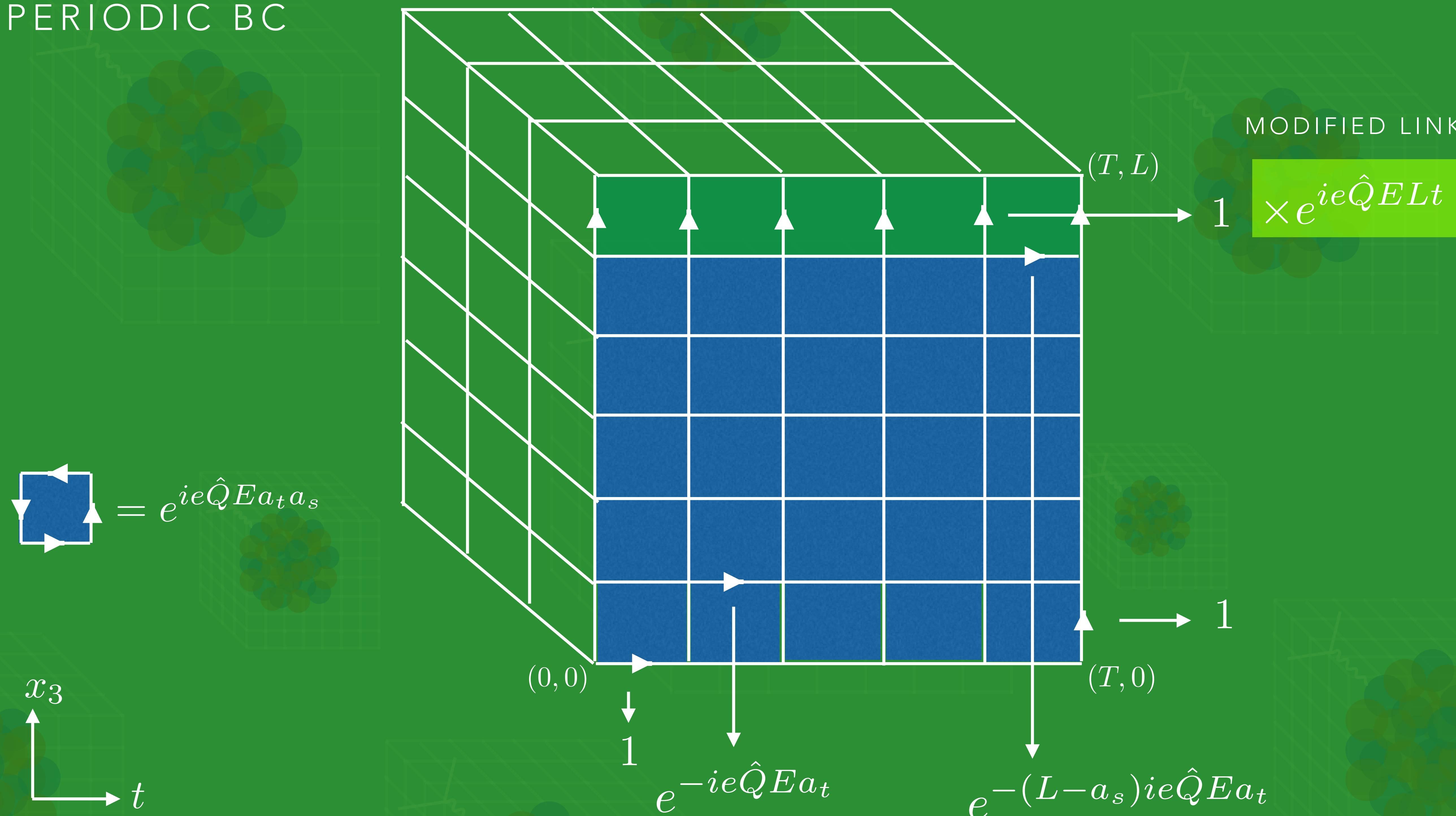

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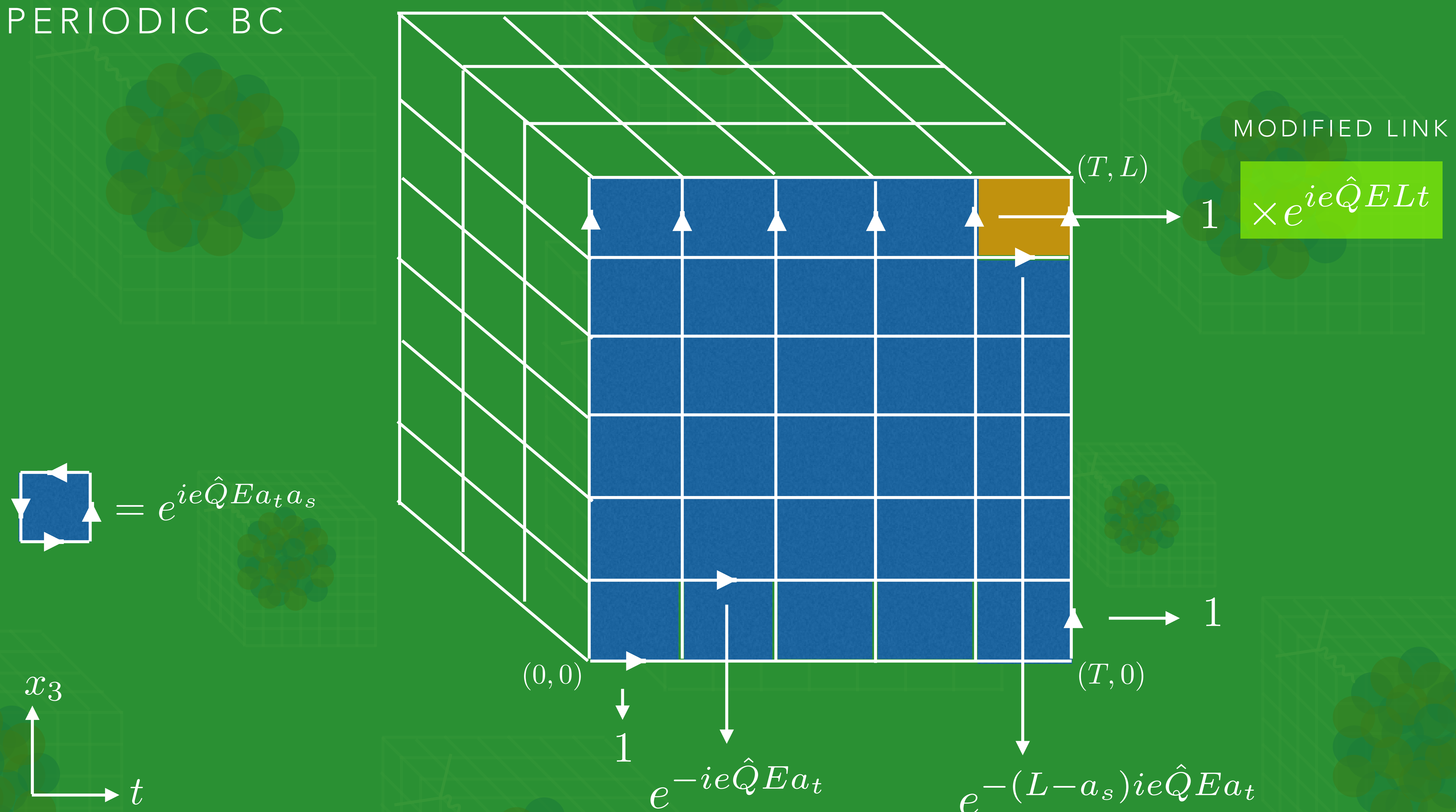
PERIODIC BC



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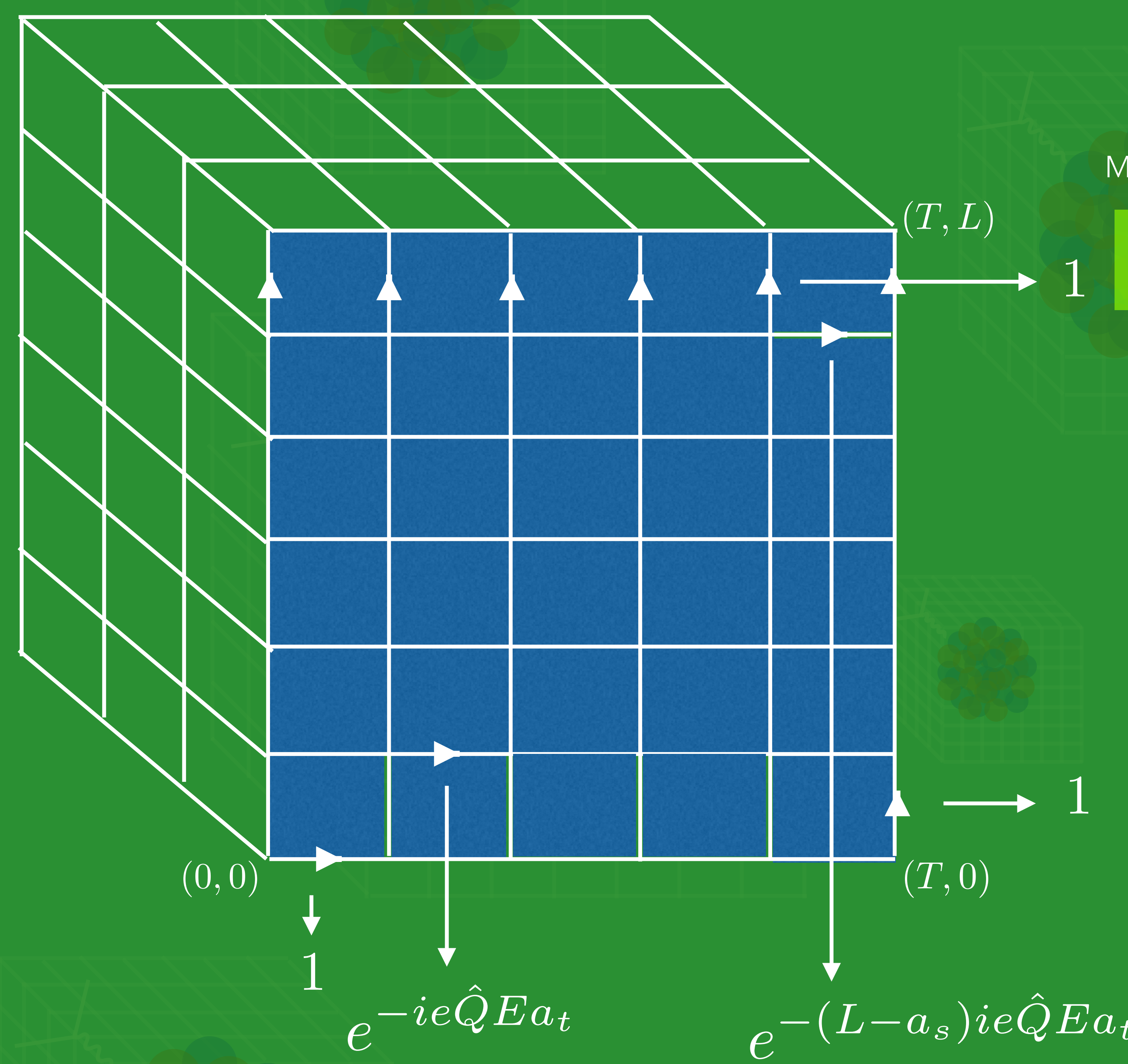
PERIODIC BC

$$e^{ie\hat{Q}ELT} = 1$$

$$E = \frac{2\pi n}{e\hat{Q}TL}$$

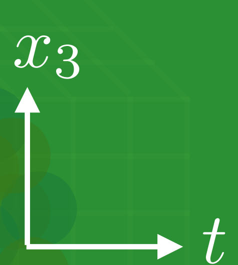
'T HOOFT QUANTIZATION CONDITION

$$= e^{ie\hat{Q}Ea_t a_s}$$

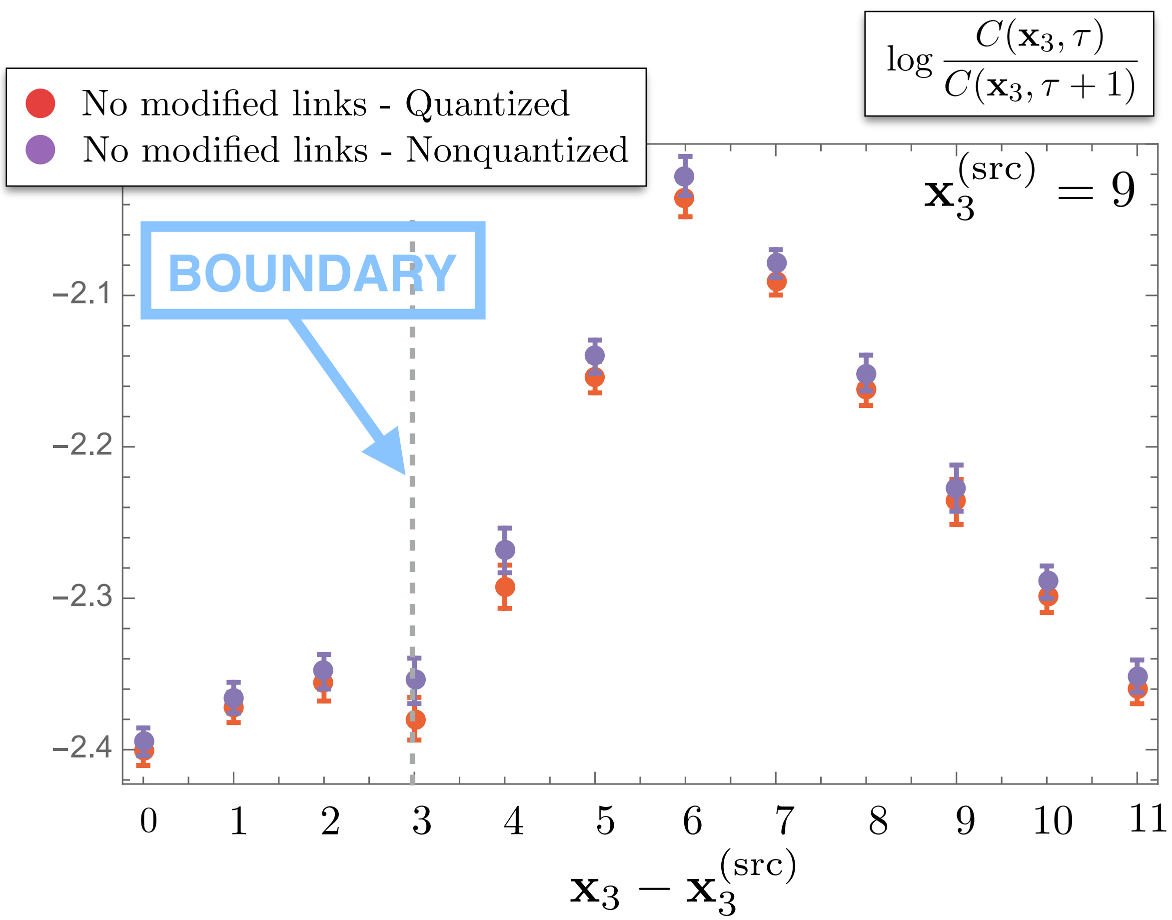


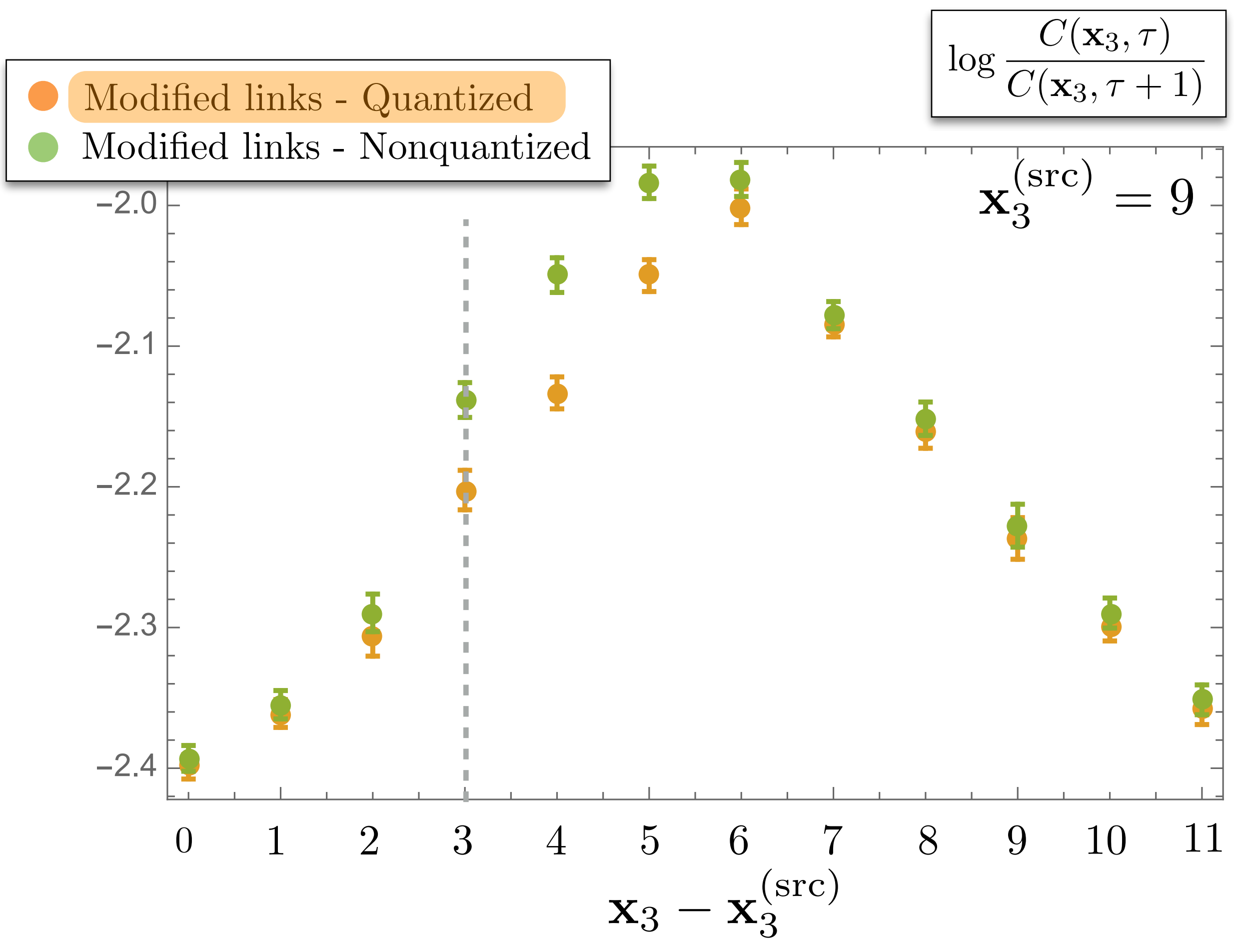
MODIFIED LINK

$$\times e^{ie\hat{Q}ELt}$$



An example: A neutral pion correlation function with $\mathbf{E} = E\mathbf{x}_3$





ZD and Detmold, Phys. Rev. D 92, 074506 (2015).

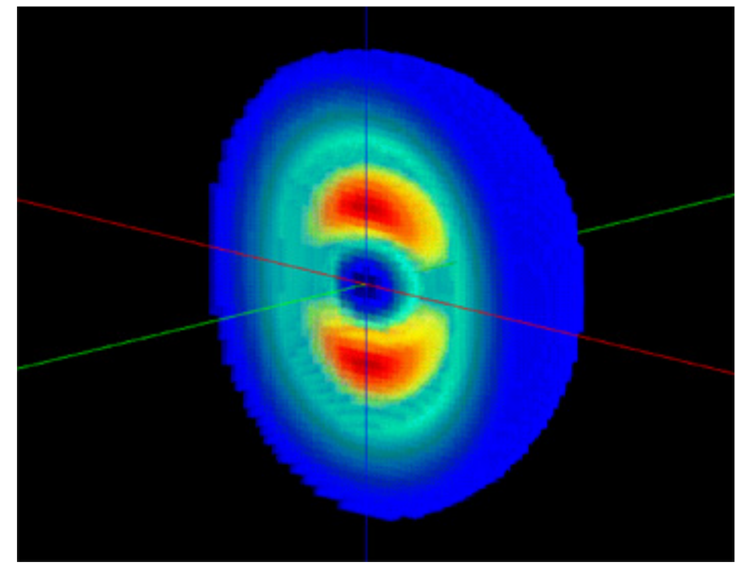
Various other structure properties of hadrons and nuclei, as well as their transitions, can be studied using more complex background fields:

1) EM charge radius

ZD and Detmold, Phys. Rev. D 93, 014509 (2016).

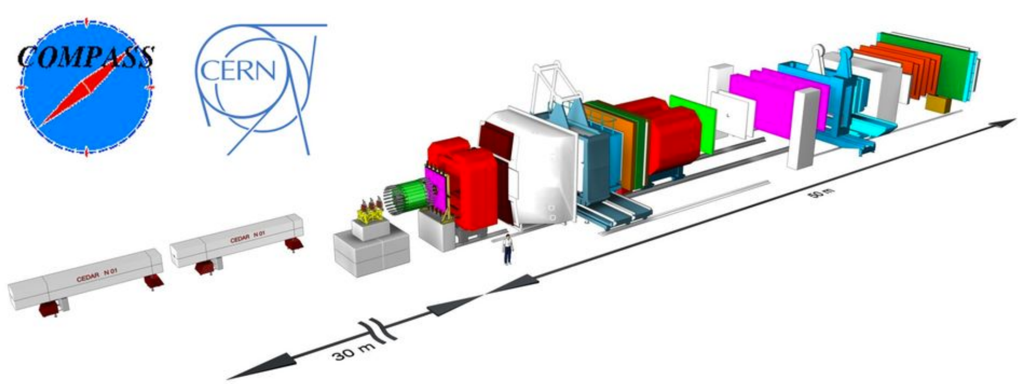
2) Electric quadrupole moment

ZD and Detmold, Phys. Rev. D 93, 014509 (2016).



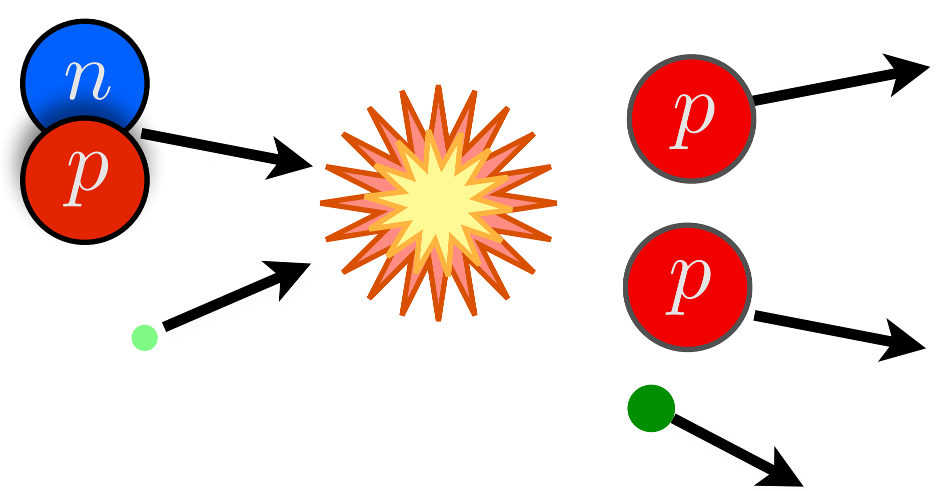
3) Form factors

Detmold, Phys.Rev. D71, 054506 (2005).



4) Axial background fields

Beane at al, Phys.Rev. Lett, 115 132001 (2015).



EXERCISE 7

Consider a non-uniform background electric field that is produced by a background gauge potential

$$A_\mu = \left(-\frac{E_0}{2}(\mathbf{x}_3 - R - \left[\frac{\mathbf{x}_3}{L}\right] L)^2, \mathbf{0} \right)$$

where $0 \leq R < L$. Derive the prescription for the modified links as well as a quantization condition on the slope of the electric field strength, E_0, along direction $\hat{\mathbf{x}}_3$.

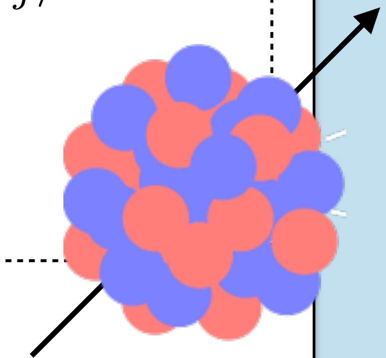
Here's an application of the background-field technique to obtain magnetic moment and polarizabilities of the nucleon:

$$E_{h;j_z}(\mathbf{B}) = \sqrt{M_h^2 + P_\parallel^2 + (2n_L+1)|Q_h e\mathbf{B}|} - \boldsymbol{\mu}_h \cdot \mathbf{B} - 2\pi\beta_h^{(M0)}|\mathbf{B}|^2 - 2\pi\beta_h^{(M2)}\langle \hat{T}_{ij} B_i B_j\rangle + \ldots$$

Landau levels for charged particles

Magnetic moment

Magnetic polarizabilities



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Landau levels for charged particles

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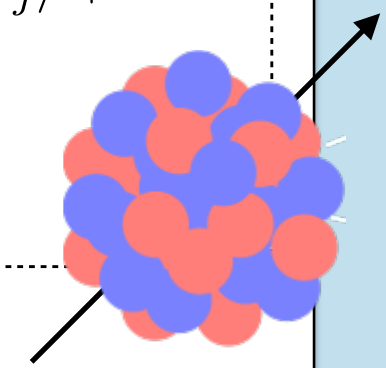
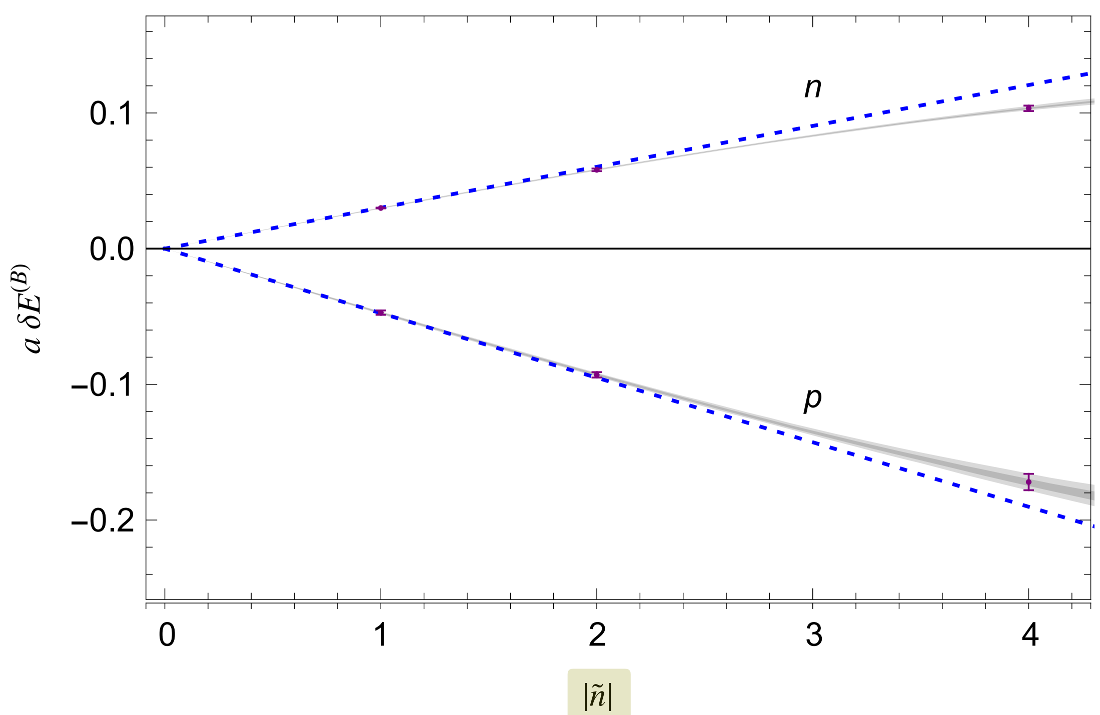
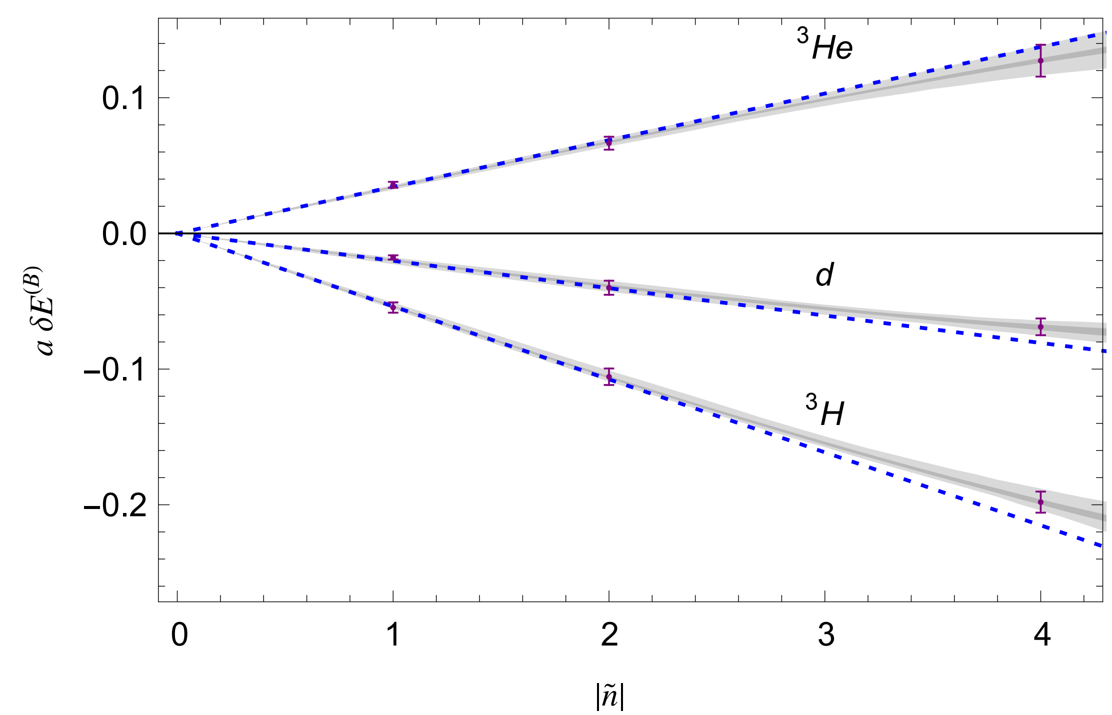


Nucleon



A quanta of magnetic field

Light nuclei



$N_f = 3,\ m_\pi = 0.806\ \text{GeV},\ a = 0.145(2)\ \text{fm}$

Beane et al.(NPLQCD), phys.rev.lett.113 (2014) 25, 252001.
Beane et al.(NPLQCD), phys.rev. D92 (2015) 11, 114502.

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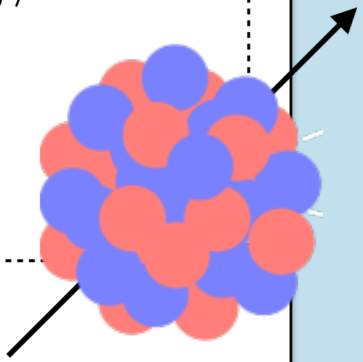
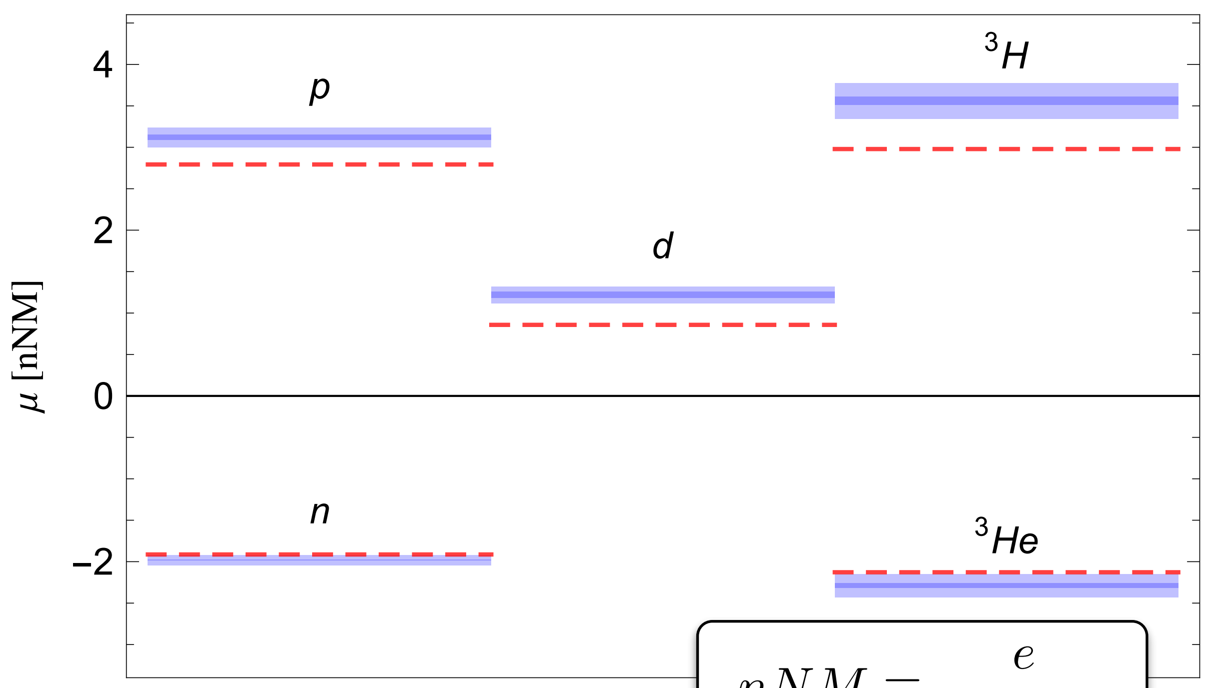
Landau levels for charged particles

Magnetic moment

Magnetic polarizabilities



Magnetic moment



$$nNM \equiv \frac{e}{2M_{@m_q}}$$

$$N_f = 3,\ m_\pi = 0.806\ \text{GeV},\ a = 0.145(2)\ \text{fm}$$

Beane et al.(NPLQCD), phys.rev.lett.113 (2014) 25, 252001.
Beane et al.(NPLQCD), phys.rev. D92 (2015) 11, 114502.

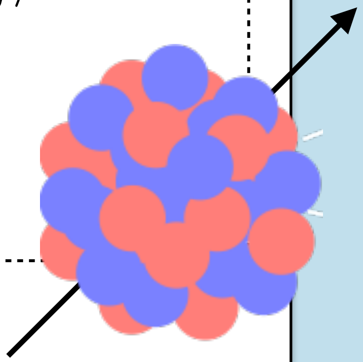
Here's an application of the background-field technique to obtain magnetic moment and polarizabilities of the nucleon:

$$E_{h;j_z}(\mathbf{B}) = \sqrt{M_h^2 + P_\parallel^2 + (2n_L+1)|Q_h e\mathbf{B}|} - \boldsymbol{\mu}_h \cdot \mathbf{B} - 2\pi\beta_h^{(M0)}|\mathbf{B}|^2 - 2\pi\beta_h^{(M2)}\langle \hat{T}_{ij} B_i B_j\rangle + \ldots$$

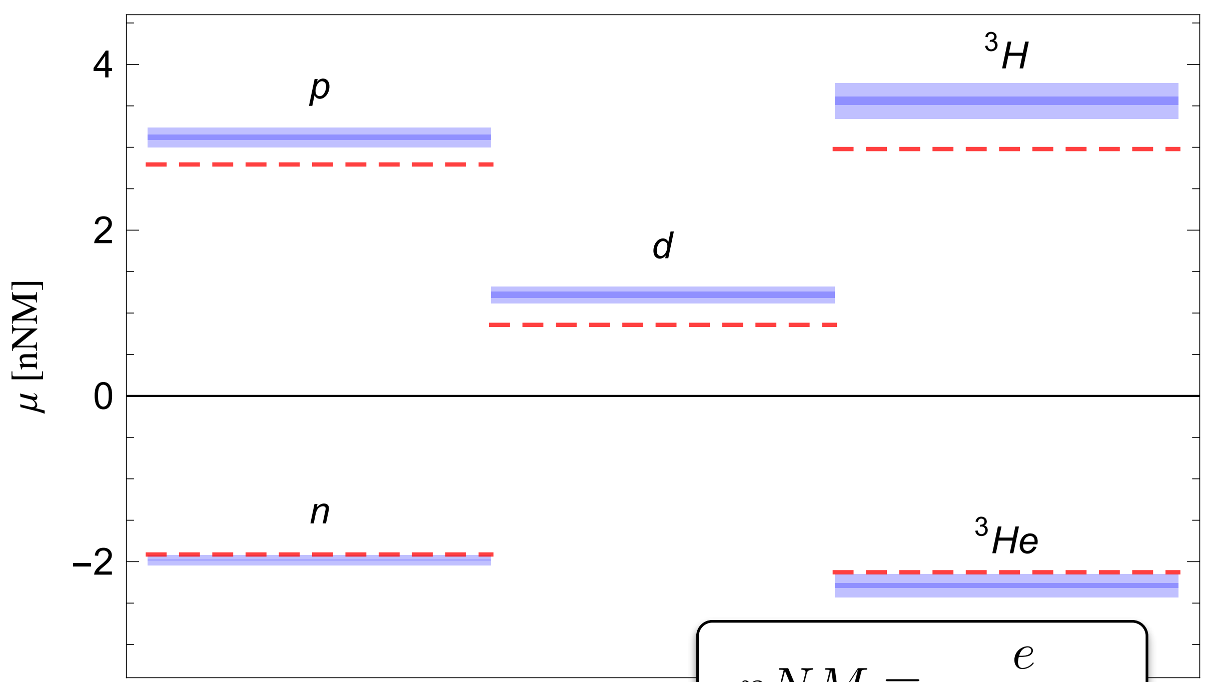
Landau levels for charged particles

Magnetic moment

Magnetic polarizabilities



Magnetic moment



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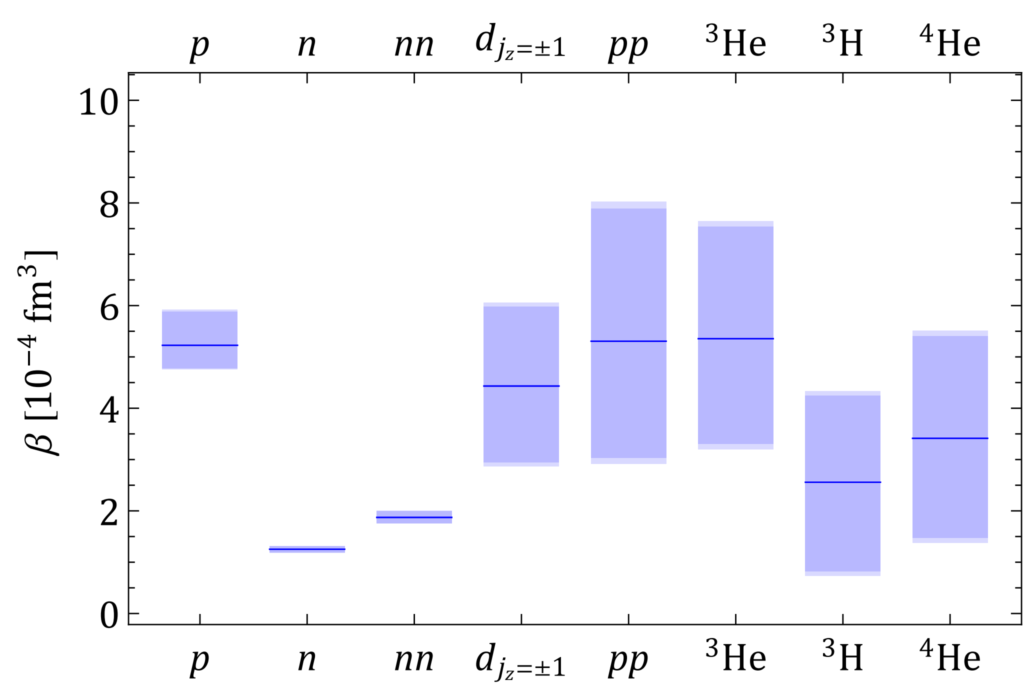
Magnetic polarizability



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Let’s enumerate a some of the methods that give access to structure quantities in general:

Three(four)-point functions

For e.g., form factors, moments of structure functions, Compton amplitude, transition amplitudes

Background-field methods

For e.g., EM moments and polarizabilities, charge radius, form factors and transition amplitudes.

Feynman-Hellmann inspired methods

Similar to background fields. For e.g., axial charge, form factors, EM moments, transition amplitudes

Hamiltonian as a function of a variable parameter

$$\hat{H}(\lambda) = \hat{H} + \lambda \hat{V}$$

Energy eigenvalue

$$\frac{\mathrm{d}E_n}{\mathrm{d}\lambda} = \frac{\langle \psi_n | \frac{\mathrm{d}\hat{H}}{\mathrm{d}\lambda} | \psi_n \rangle}{\langle \psi_n | \psi_n \rangle}$$

Energy eigenstate

Example: sigma term

$$m_q \frac{\partial m_N}{\partial m_q}\bigg|_{m_q = m_q^{\mathrm{phy}}} = \langle \mathcal{N} | m_q \bar{q} q | \mathcal{N} \rangle$$

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Generalization to correlation functions

$$C_\lambda(t) = \langle\lambda|\mathcal{O}(t)\mathcal{O}^\dagger(0)|\lambda\rangle = \frac{1}{\mathcal{Z}_\lambda}\int D\Phi e^{-S-S_\lambda}\mathcal{O}(t)\mathcal{O}^\dagger(0)$$

Just a 2pt function

$$S_\lambda = \lambda\int d^4x j(x)$$

Integrated matrix element

$$-\frac{\partial C_\lambda(t)}{\partial\lambda}\bigg|_{\lambda=0} = -C(t)\int dt'\langle\Omega|\mathcal{J}(t')|\Omega\rangle + \int dt'\langle\Omega|T\{\mathcal{O}(t)\mathcal{J}(t')\mathcal{O}^\dagger(0)\}|\Omega\rangle$$

$$\mathcal{J}(t) = \int d^3x j(t,\vec{x})$$

Buochard et al (CALLATT), Phys.Rev.D96,014504(2017).

Example: axial charge of the nucleon and triton!

Since the operator here is a quark bilinear, a clever to implement this is by modifying the quark propagator.

$$S^{(q)}_{\lambda_q;\Gamma}(x,y) = S^{(q)}(x,y) + \lambda_q \int dz\ S^{(q)}(x,z)\Gamma S^{(q)}(z,y)$$

Savage et al (NPLQCD), Phys.Rev.Lett.119,062002(2017).

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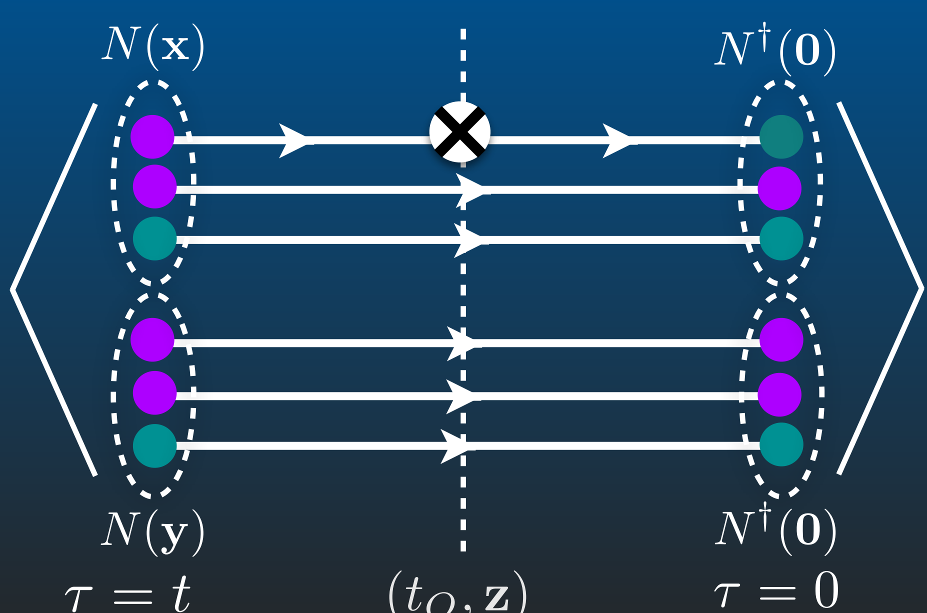
Buochard et al (CALLATT), Phys.Rev.D96,014504(2017).

e.g.,

$$C(\mathbf{P};t,t_O) = \sum_{\mathbf{p}_1+\mathbf{p}_2=\mathbf{P}} \sum_{\mathbf{x},\mathbf{y},\mathbf{z}} e^{i\mathbf{p}_1\cdot\mathbf{x}+i\mathbf{p}_2\cdot\mathbf{y}} \times$$



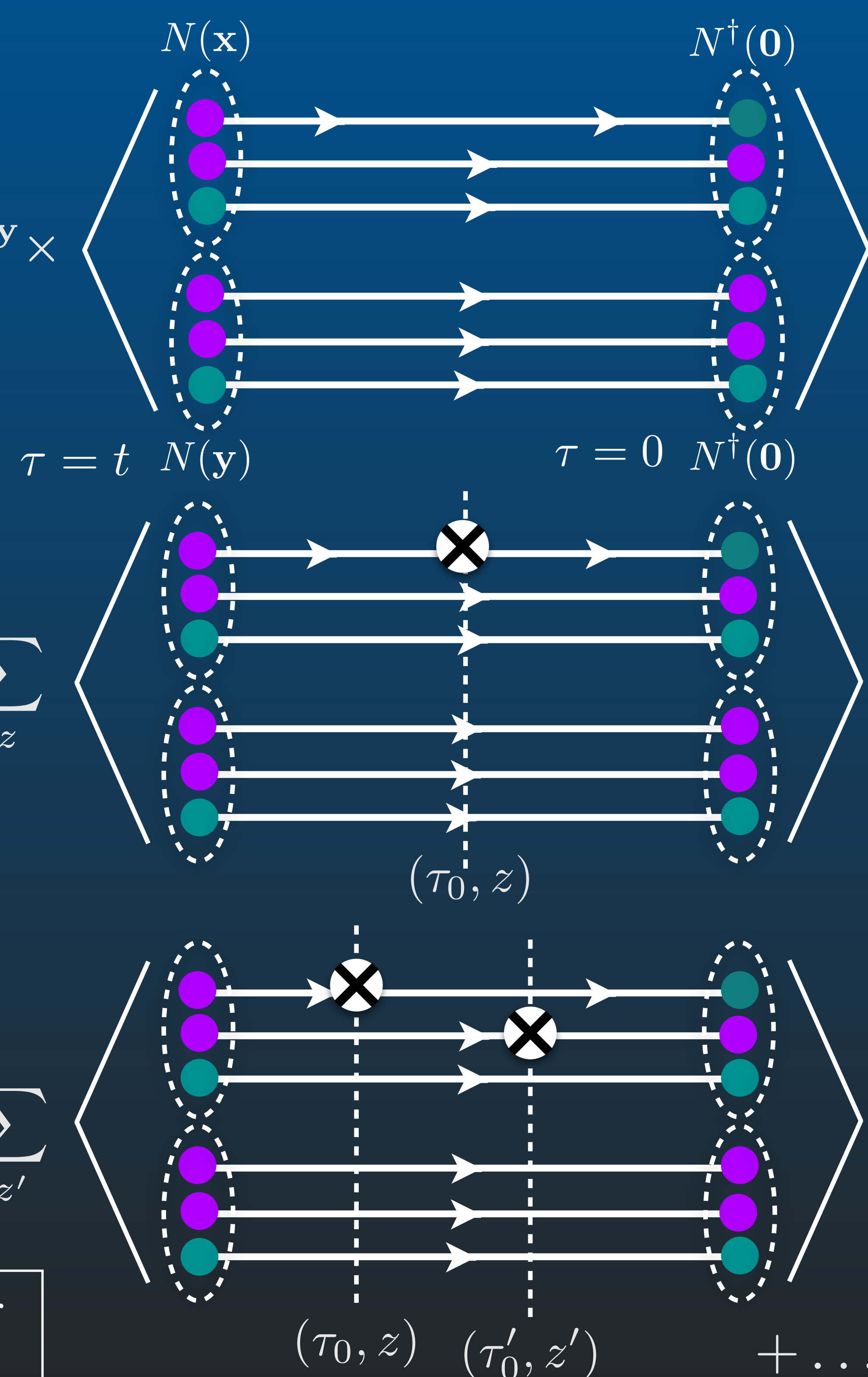
This gives more generally:

$$C_\lambda(\mathbf{P}; t, t_O) = \sum_{\mathbf{p}_1+\mathbf{p}_2=\mathbf{P}} \sum_{\mathbf{x},\mathbf{y},\mathbf{z}} e^{i\mathbf{p}_1\cdot\mathbf{x}+i\mathbf{p}_2\cdot\mathbf{y}} \times$$

All possibilities $\longrightarrow$ $+\lambda \sum_{\tau_0=0}^{T} \sum_{z}$

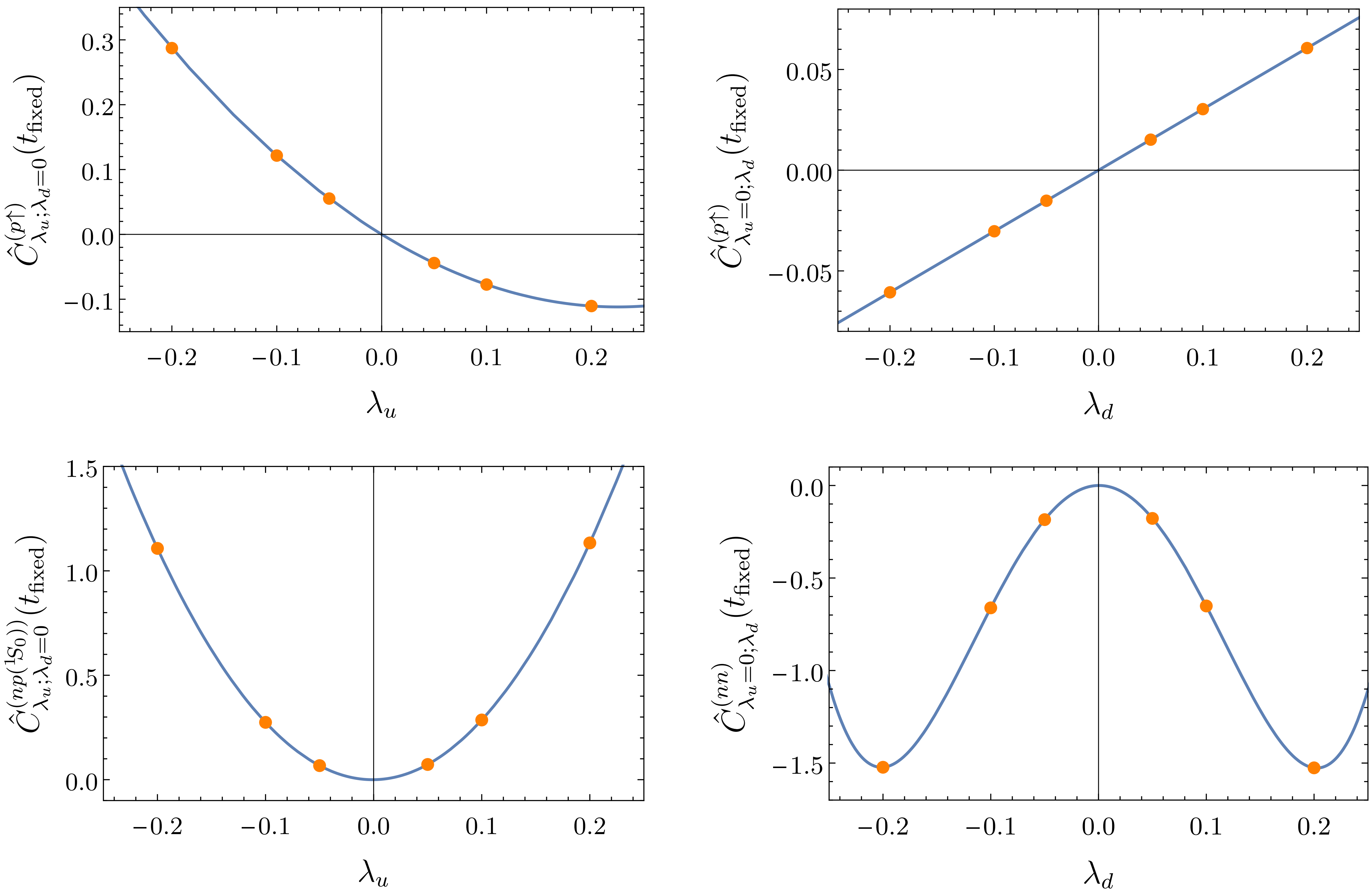
time-ordered producT

All possibilities $\longrightarrow$ $+\lambda^2 \sum_{\tau_0=0}^{T} \sum_{\tau_0'=0}^{T} \sum_{z} \sum_{z'}$



Double-current MEs are exact for isotensor quantities.

Matrix elements from a compound propagator/background field



Tiburzi et al (NPLQCD), Phys. Rev. D 96, 054505 (2017).

2pt functions we calculate is:

Has information about the matrix element we want

$$C^{(p\uparrow)}_{\lambda_u;\lambda_d=0}(t) = \sum_{\boldsymbol{x}} \left(\langle 0|\chi_{p\uparrow}(\boldsymbol{x},t)\chi^\dagger_{p\uparrow}(0)|0\rangle + \lambda_u \sum_{\boldsymbol{y}} \sum_{t_1=0}^{t} \langle 0|\chi_{p\uparrow}(\boldsymbol{x},t) J_3^{(u)}(\boldsymbol{y},t_1)\chi^\dagger_{p\uparrow}(0)|0\rangle \right) + d_2 \lambda_u^2,$$

$$C^{(p\uparrow)}_{\lambda_u=0;\lambda_d}(t) = \sum_{\boldsymbol{x}} \left(\langle 0|\chi_{p\uparrow}(\boldsymbol{x},t)\chi^\dagger_{p\uparrow}(0)|0\rangle + \lambda_d \sum_{\boldsymbol{y}} \sum_{t_1=0}^{t} \langle 0|\chi_{p\uparrow}(\boldsymbol{x},t) J_3^{(d)}(\boldsymbol{y},t_1)\chi^\dagger_{p\uparrow}(0)|0\rangle \right),$$

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Taking the linear term: $$C^{(p\uparrow)}_{\lambda_u;\lambda_d=0}(t)\Big|_{\mathcal{O}(\lambda_u)} = \sum_{\boldsymbol{x},\boldsymbol{y}} \sum_{t_1=0}^{t} \langle 0|\chi_{p\uparrow}(\boldsymbol{x},t) J_3^{(u)}(\boldsymbol{y},t_1)\chi^\dagger_{p\uparrow}(0)|0\rangle$$

$$= \sum_{\mathfrak{n},\mathfrak{m}} \sum_{\boldsymbol{x},\boldsymbol{y}} \sum_{t_1=0}^{t} \langle 0|\chi_{p\uparrow}(\boldsymbol{x},t)|\mathfrak{n}\rangle\langle\mathfrak{n}| J_3^{(u)}(\boldsymbol{y},t_1)|\mathfrak{m}\rangle\langle\mathfrak{m}|\chi^\dagger_{p\uparrow}(0)|0\rangle$$

Insert two complete set of states

Tiburzi et al (NPLQCD), Phys. Rev. D 96, 054505 (2017).

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Has information about the matrix element we want

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Insert two complete set of states

Isolate the ground state to ground state matrix element:

$$C^{(p\uparrow)}_{\lambda_u;\lambda_d=0}(t)\Big|_{\mathcal{O}(\lambda_u)} = \sum_{t_1=0}^{t} \sum_{\mathfrak{n},\mathfrak{m}} z_{\mathfrak{n}} z^\dagger_{\mathfrak{m}} e^{-E_{\mathfrak{n}}(t-t_1)} e^{-E_{\mathfrak{m}} t_1} \langle \mathfrak{n}|\tilde{J}_3^{(u)}|\mathfrak{m}\rangle$$

$$= \sum_{\mathfrak{n},\mathfrak{m}} z_{\mathfrak{n}} z^\dagger_{\mathfrak{m}} \frac{e^{-E_{\mathfrak{n}}t} - e^{-E_{\mathfrak{m}}t}}{aE_{\mathfrak{m}} - aE_{\mathfrak{n}}} \langle \mathfrak{n}|\tilde{J}_3^{(u)}|\mathfrak{m}\rangle$$

$$\xrightarrow{t\to\infty} |z_0|^2 e^{-E_0 t}\left[c + t\,\langle p\uparrow|\tilde{J}_3^{(u)}|p\uparrow\rangle + \mathcal{O}(e^{-\hat{\delta}t})\right]$$

Energy gap to excited states

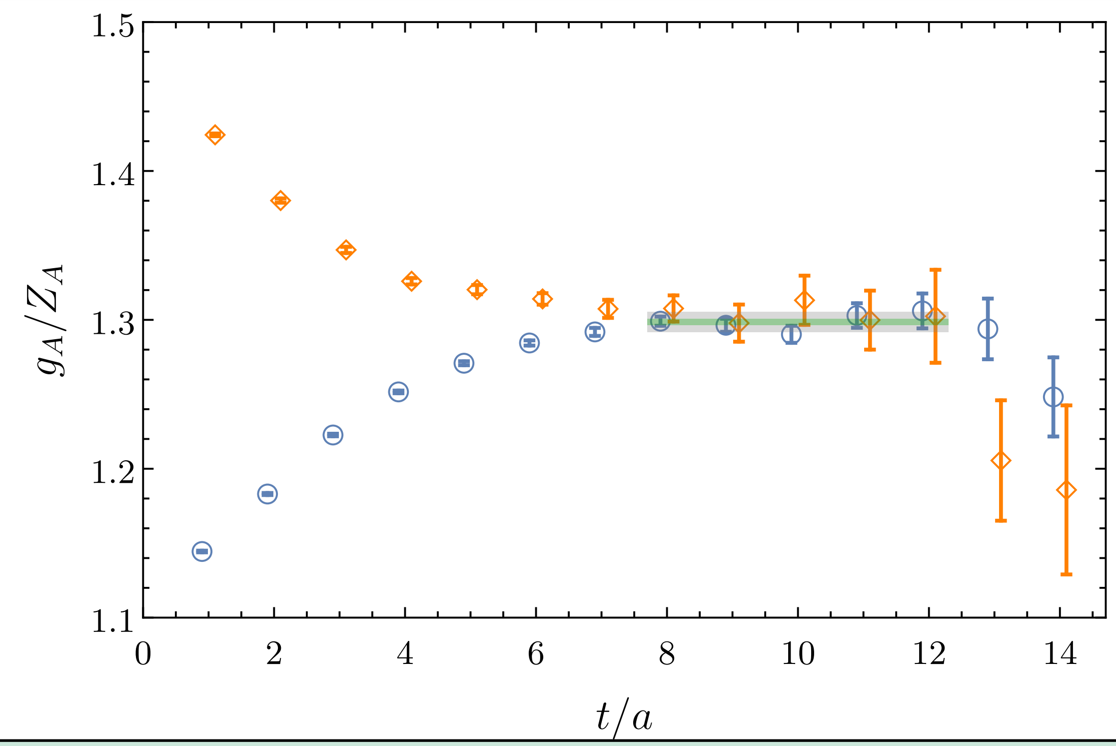
The desired matrix element

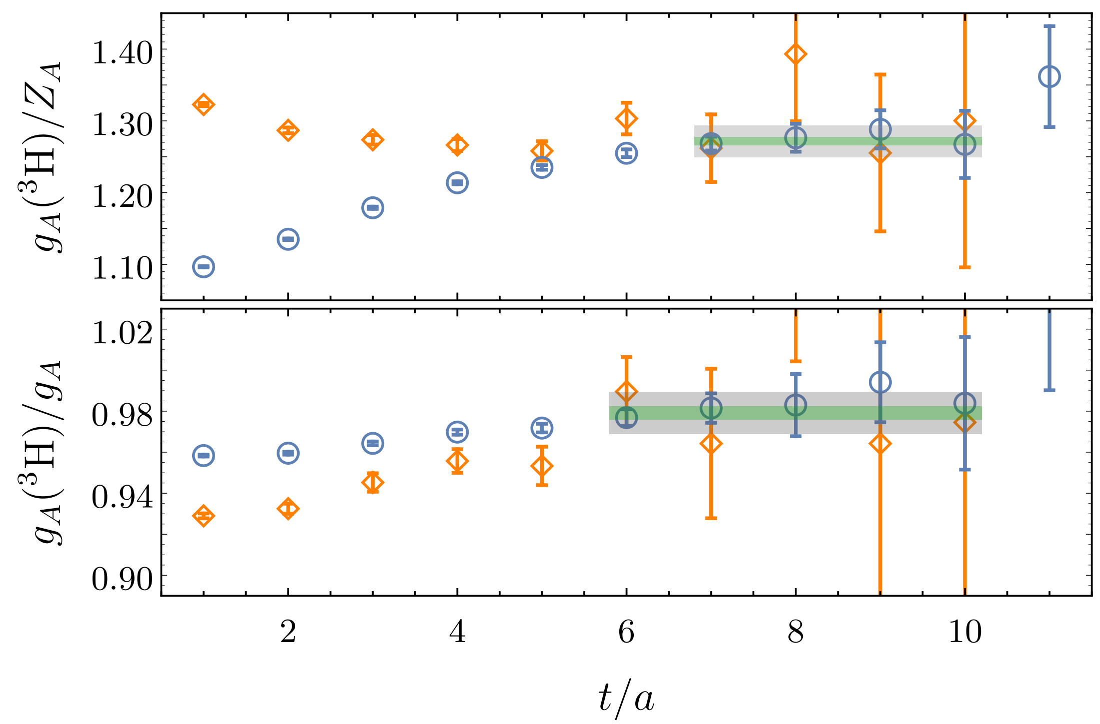
Tiburzi et al (NPLQCD), Phys. Rev. D 96, 054505 (2017).

Example of two works using the method:

$$\overline{R}_p(t) \equiv R_p(t+a) - R_p(t) \overset{t\to\infty}{\longrightarrow} \langle p|\tilde{J}_3^3|p\rangle = \frac{g_A}{2Z_A}$$

$N_f = 3,\ m_\pi = 0.806$ GeV, $a = 0.145(2)$ fm





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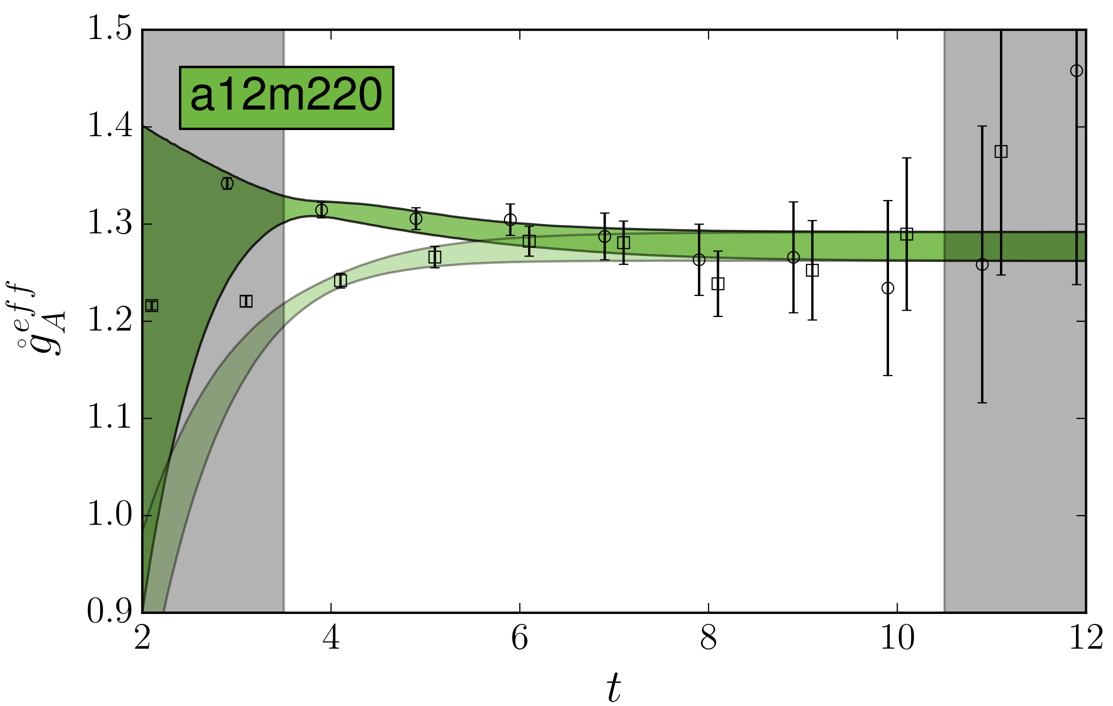

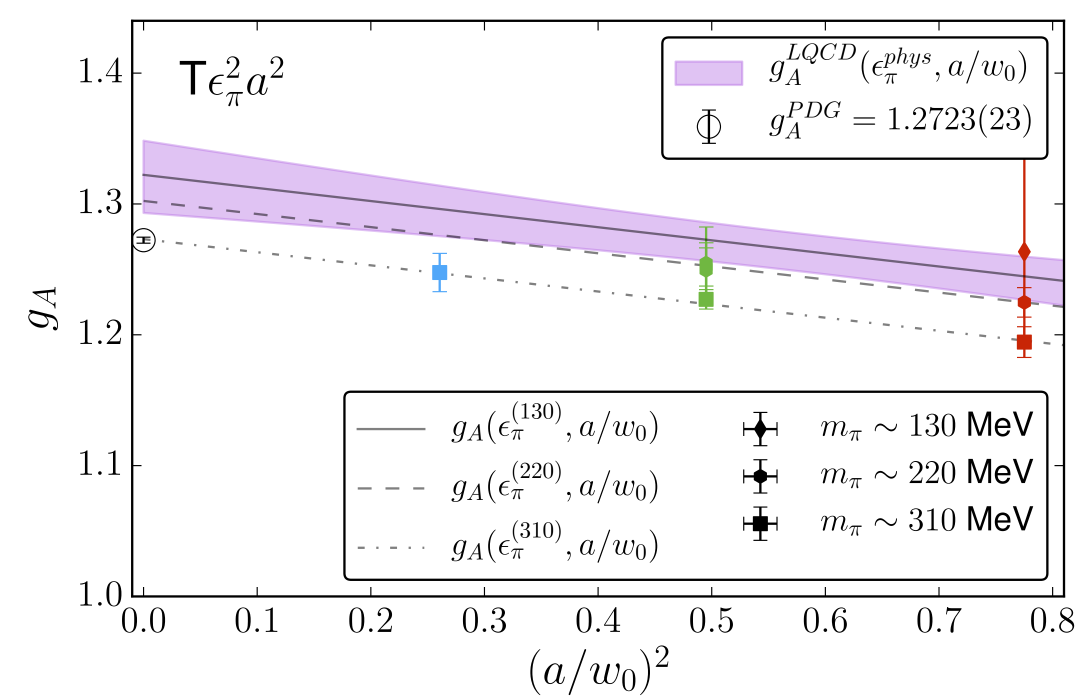

Chang at al (CALLATT), Nature volume 558, 91–94 (2018).

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Background-field methods

For e.g., EM moments and polarizabilities, charge radius, form factors and transition amplitudes.

Feynman-Hellmann inspired methods

Similar to background fields. For e.g., axial charge, form factors, EM moments, transition amplitudes

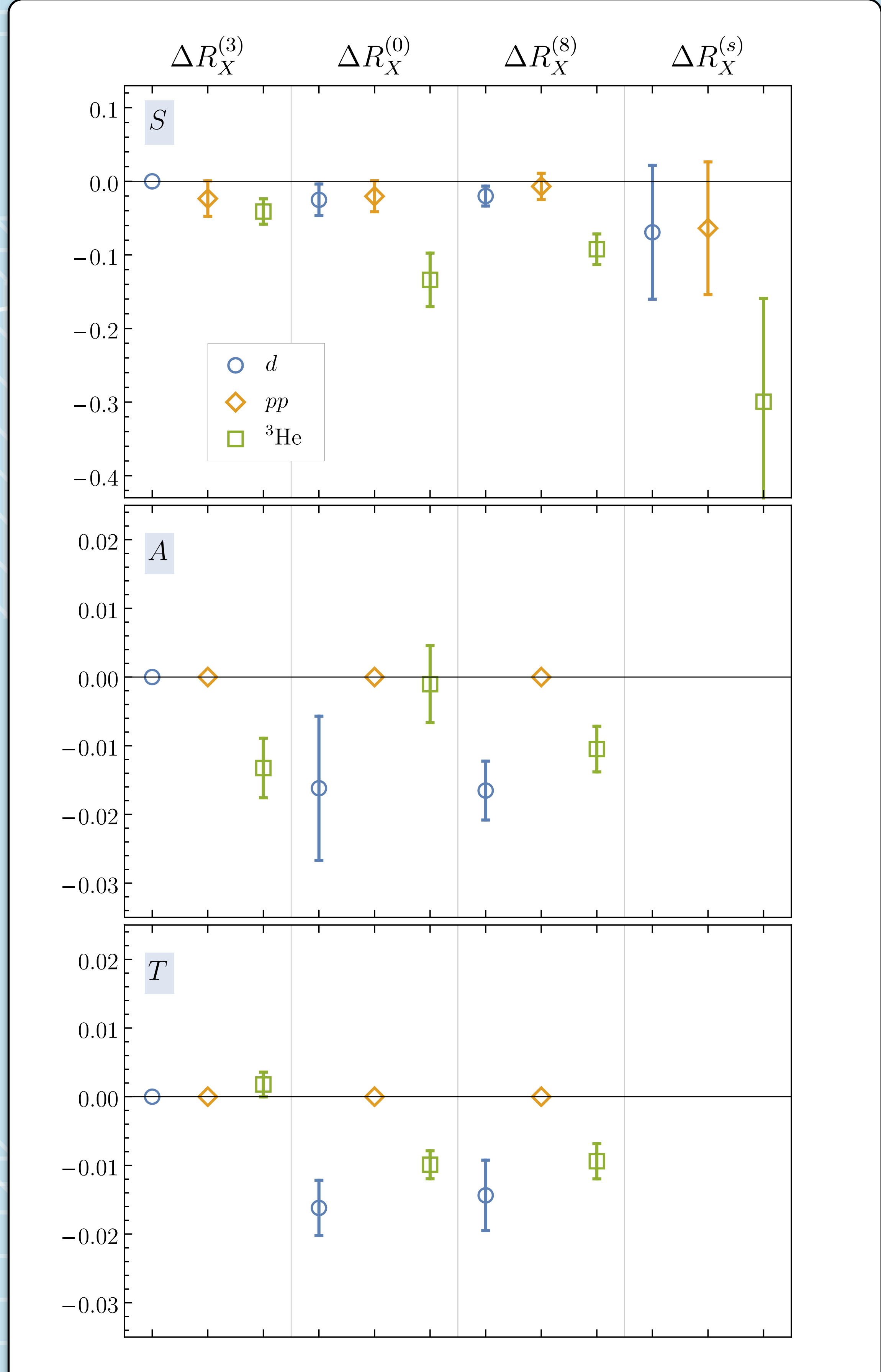
We did not discuss many other interesting directions in the field, e.g.,

- Moments of structure functions
- Quasi-PDFs and pseudo-PDFs
- Hadron tensor through inverse transform methods
- GPDs, TMDs, gluonic observables, etc.

LECTURE III: NUCLEAR STRUCTURE, CHALLENGES AND PROGRESS

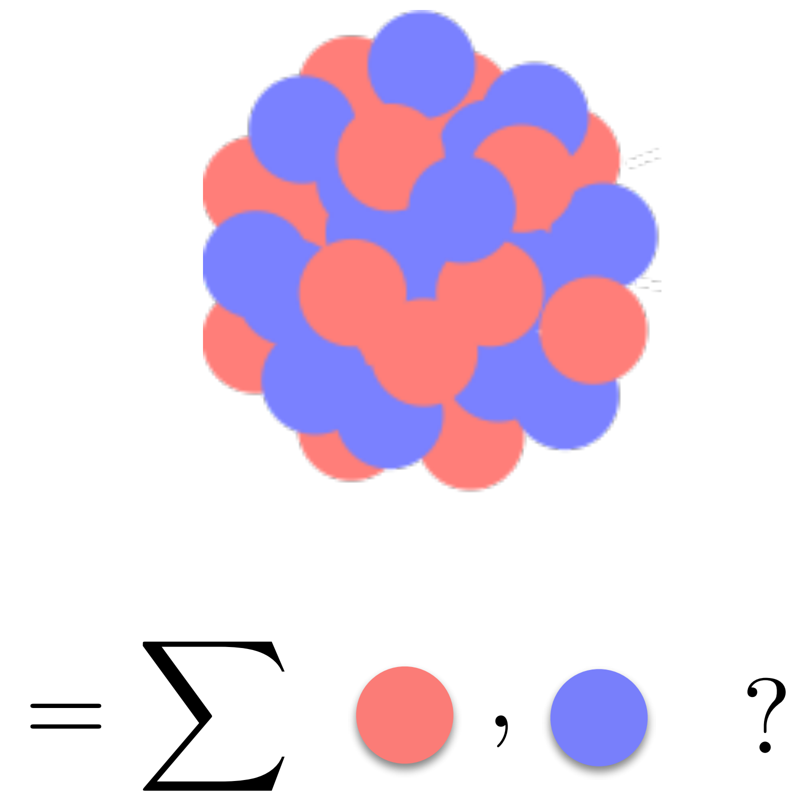
What about nuclear observables? Let’s see the application of these methods to two examples:

$N_f = 3,\ m_\pi = 0.806\ \text{GeV},\ a = 0.145(2)\ \text{fm}$



EMC effect from QCD?

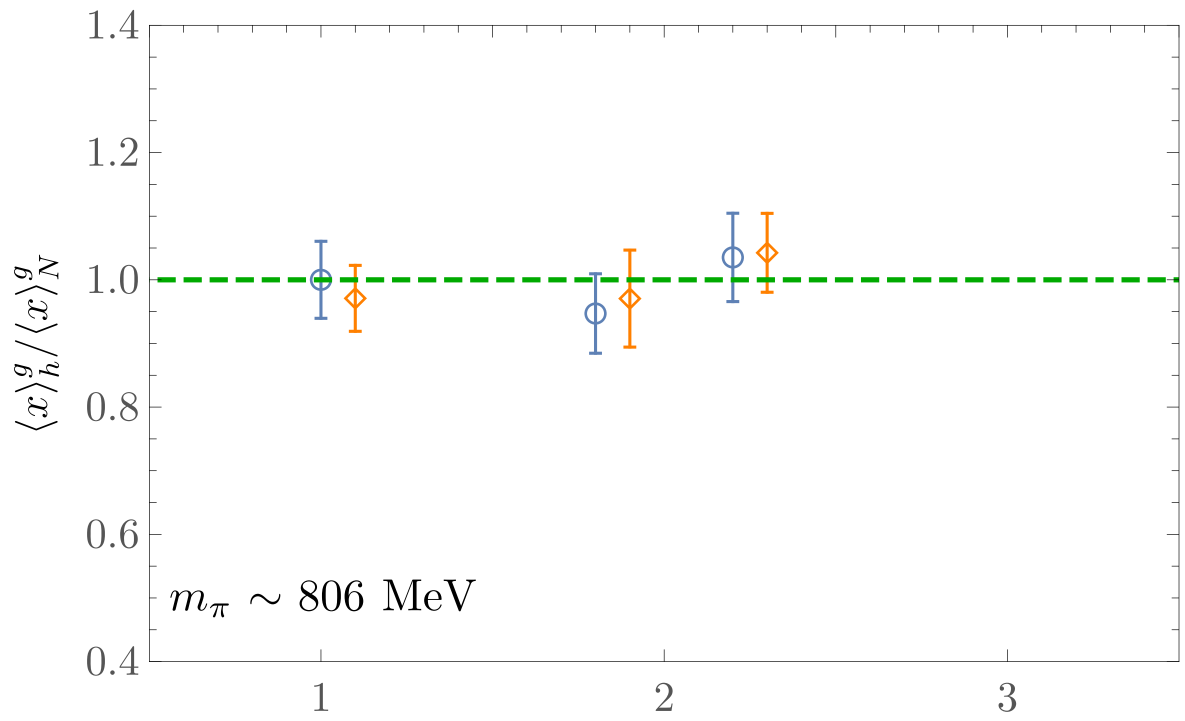
CHANG et al.(NPLQCD), Phys.Rev.Lett. 120 (2018) 15, 152002.

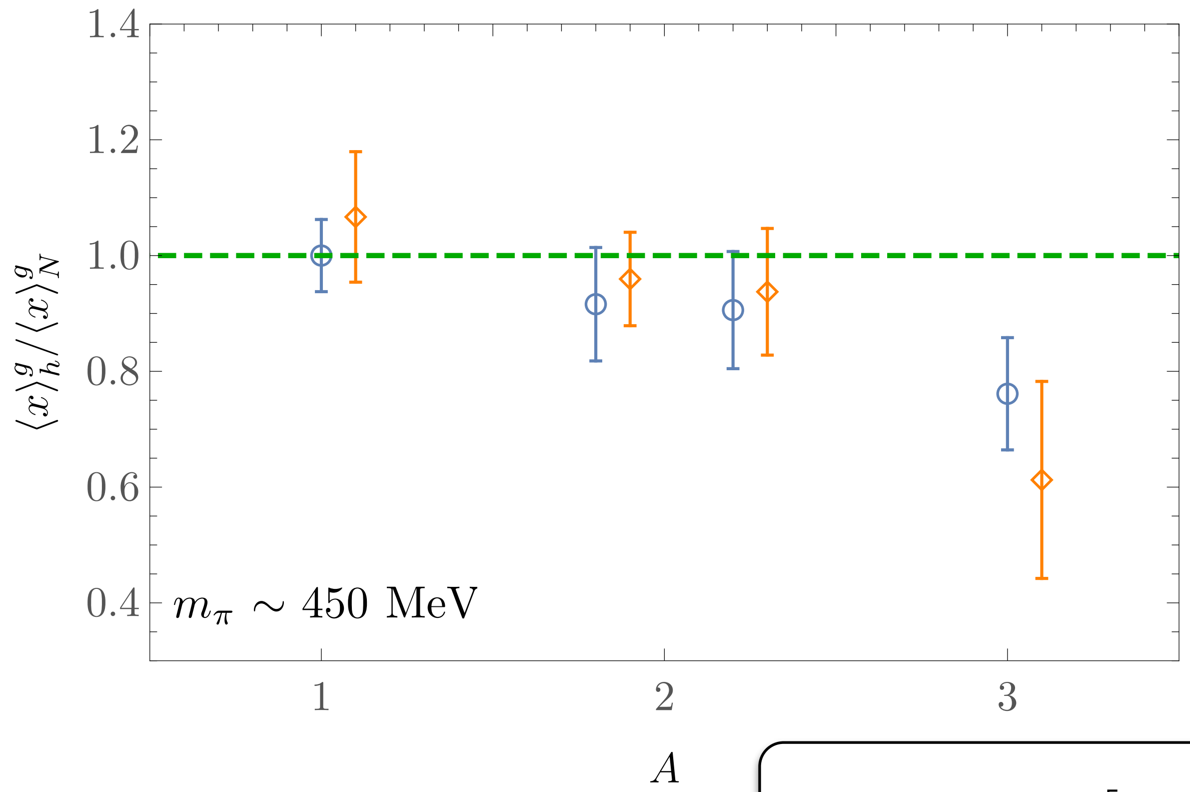


$$g_X^{(f)}(A) = \langle A|\bar{q}_f \Gamma_X q_f|A\rangle$$

$$R_X^{(f)}(A) = g_X^{(f)}(A)/g_X^{(f)}(p)$$

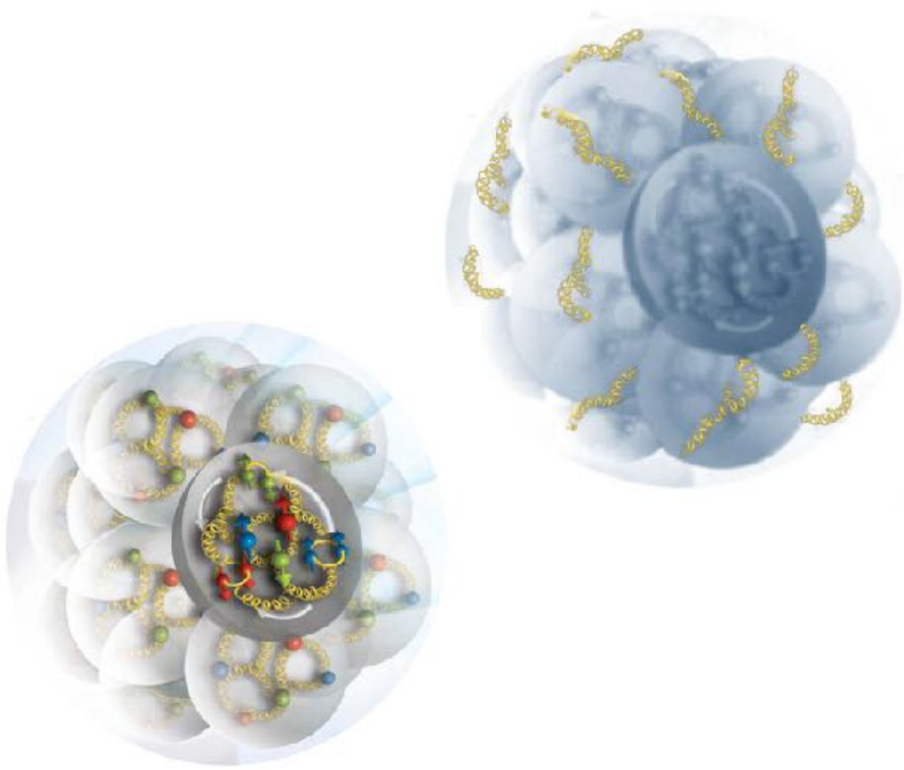
$$N_f = 2+1\ , m_\pi \approx 450\ \text{MeV}, a \approx 0.12\ \text{fm}$$





A gluonic EMC effect?

Winter et al.(NPLQCD), Phys.Rev. D96 (2017) 9, 094512



Graphic: EIC white paper (left), P. Shanahan (right)

$$\overline{\mathcal{O}}_{\mu_1 \ldots \mu_n}(\mu) = S\left[G_{\mu_1 \alpha} i \overleftrightarrow{D}_{\mu_3} \ldots i \overleftrightarrow{D}_{\mu_n} G_{\mu_2}^{\ \alpha}\right]$$

Three features make lattice QCD calculations of nuclei hard:

i) The complexity of systems grows rapidly with the number of quarks.

Detmold and Orginos, Phys. Rev. D 87, 114512 (2013).

See also: Detmold and Savage, Phys.Rev.D82 014511 (2010). Doi and Endres, Comput. Phys. Commun. 184 (2013) 117.

ii) Excitation energies of nuclei are much smaller than the QCD scale.

Beane at al (NPLQCD), Phys.Rev.D79 114502 (2009).
Beane, Detmold, Orginos, Savage, Prog. Part. Nucl. Phys. 66 (2011).
Junnakar and Walker-Loud, Phys.Rev. D87 (2013) 114510.
Briceno, Dudek and Young, Rev. Mod. Phys. 90 025001.

iii) There is a severe signal-to-noise degradation.

Paris (1984) and Lepage (1989).

Wagman and Savage, Phys. Rev. D 96, 114508 (2017).
Wagman and Savage, arXiv:1704.07356 [hep-lat].

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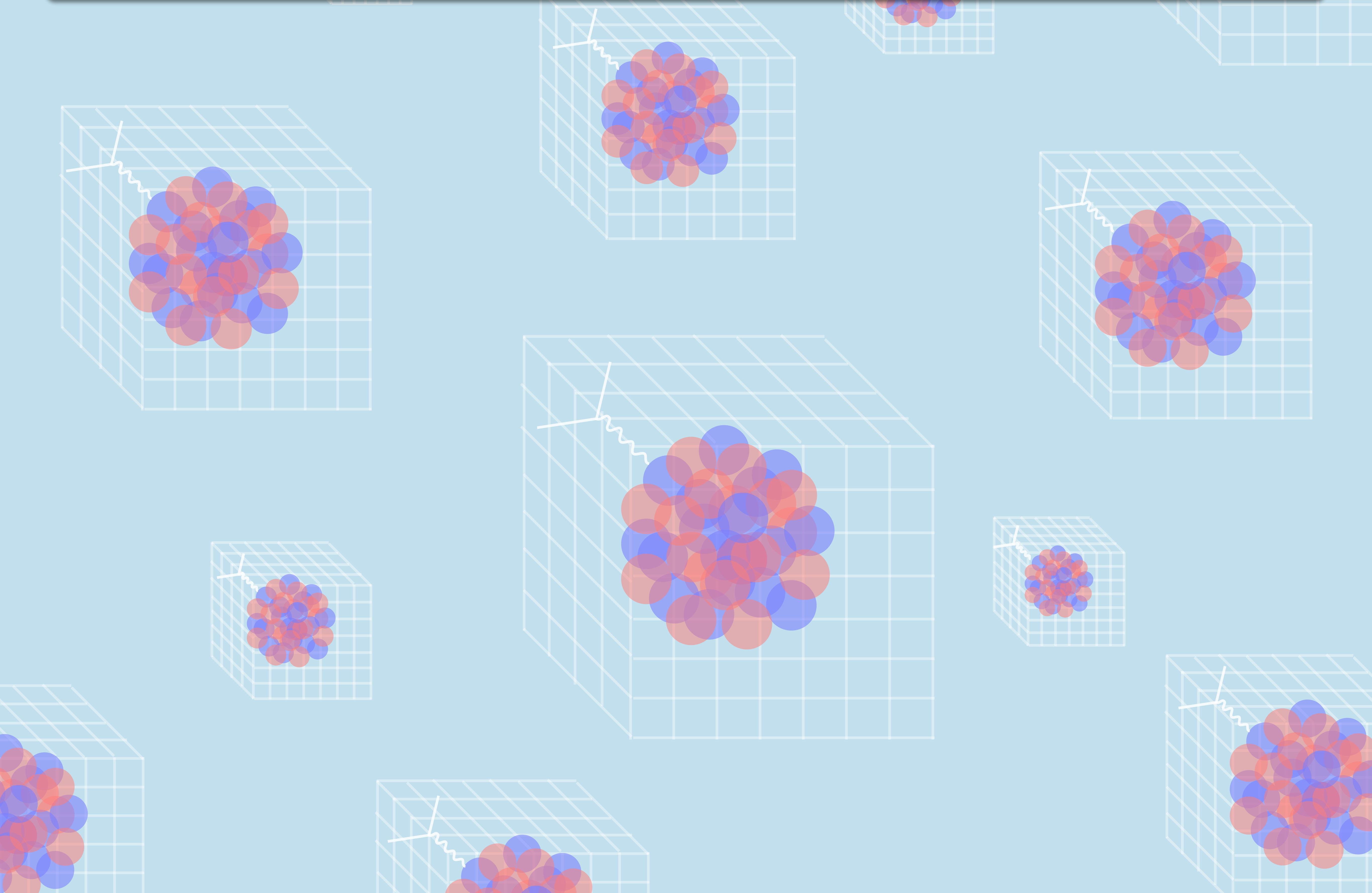
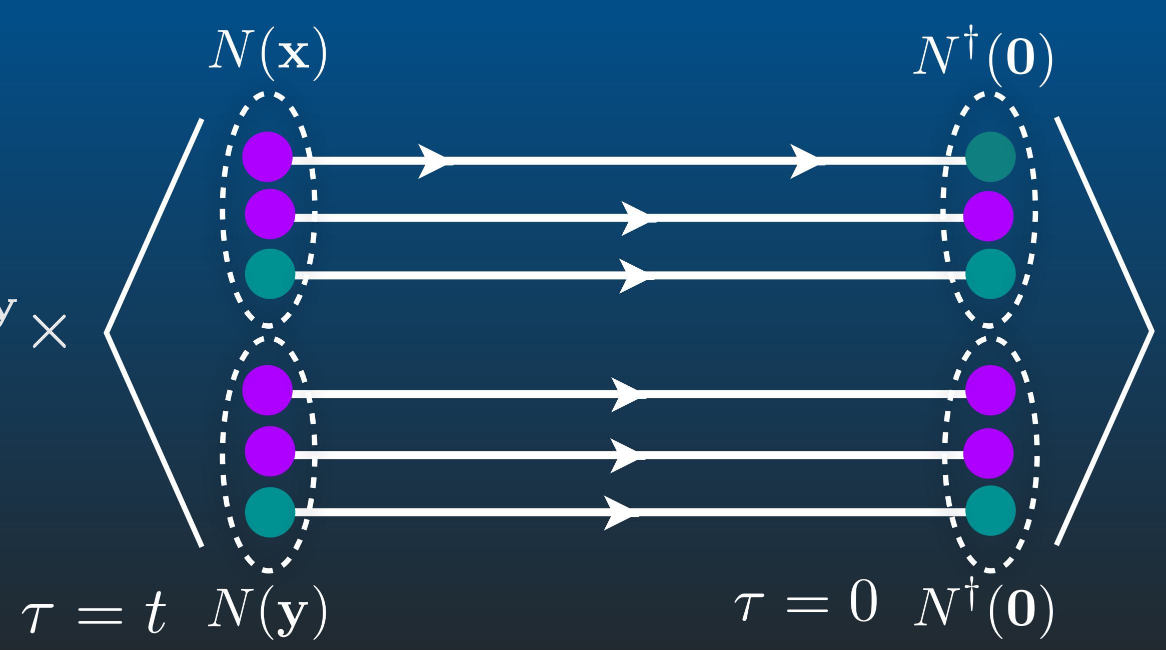


$$C(\mathbf{P}; t, t_O) = \sum_{\mathbf{p}_1+\mathbf{p}_2=\mathbf{P}} \sum_{\mathbf{x},\mathbf{y},\mathbf{z}} e^{i\mathbf{p}_1\cdot\mathbf{x}+i\mathbf{p}_2\cdot\mathbf{y}} \times$$



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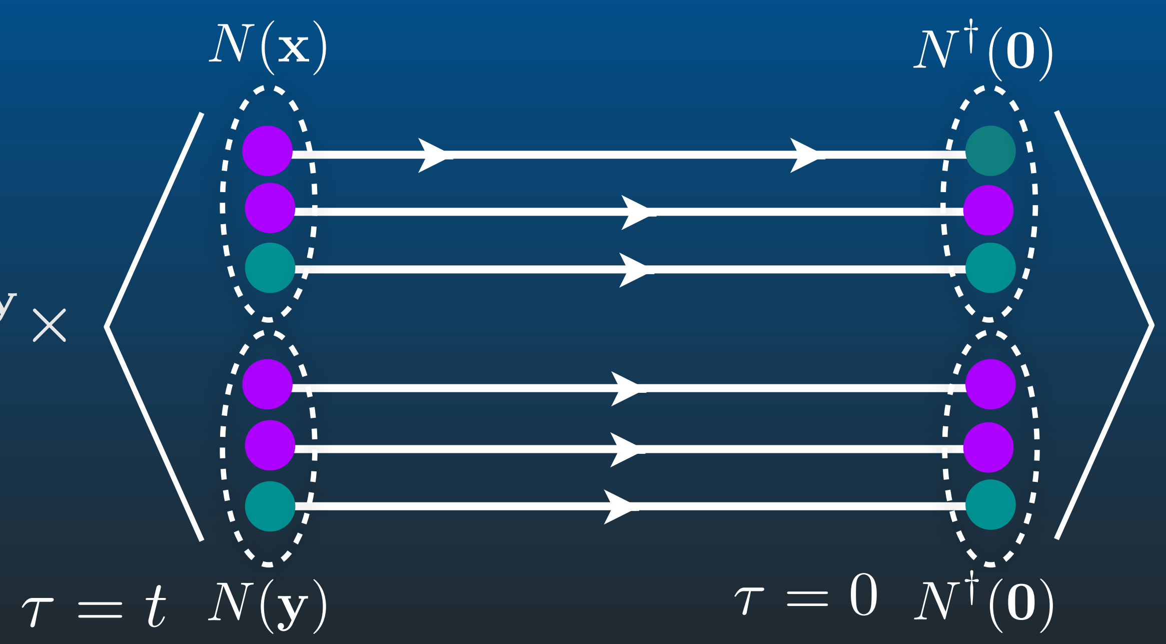

COMPLEXITIES OF
QUARK-LEVEL
INTERPOLATING FIELDS

COMPLEXITIES
OF QUARK
CONTRACTIONS

Detmold and Orginos (2013).

Naively the number of quark contractions for a nucleus goes as: $(2N_p + N_n)!\,(N_p + 2N_n)!$

How bad is this?
Example: Consider radium-226 isotope.
the number of contractions required is ~ 10^{1425}

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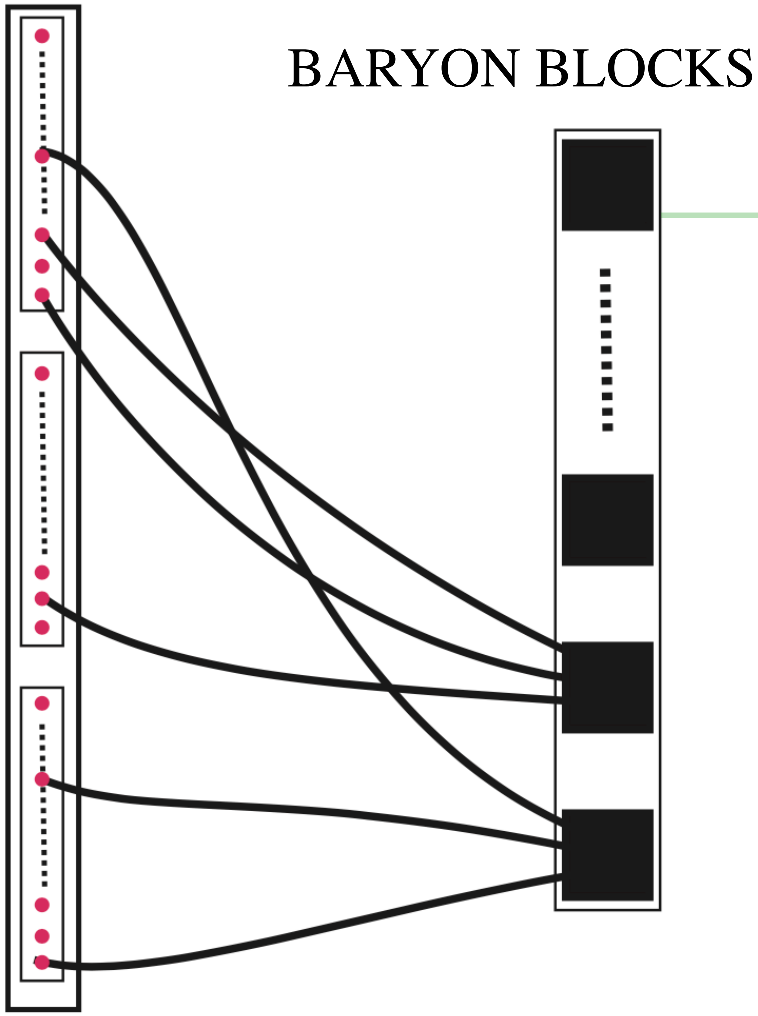
An example of a more efficient algorithm:



$$\mathcal{B}_b^{a_1,a_2,a_3}(\mathbf{p},t;x_0) = \sum_{\mathbf{x}} e^{i\mathbf{p}\cdot\mathbf{x}} \sum_{k=1}^{N_{B(b)}} \tilde{w}_b^{(c_1,c_2,c_3),k} \sum_{\mathbf{i}} \epsilon^{i_1,i_2,i_3} S(c_{i_1},x;a_1,x_0) S(c_{i_2},x;a_2,x_0) S(c_{i_3},x;a_3,x_0)$$

Can also start propagators at different locations.

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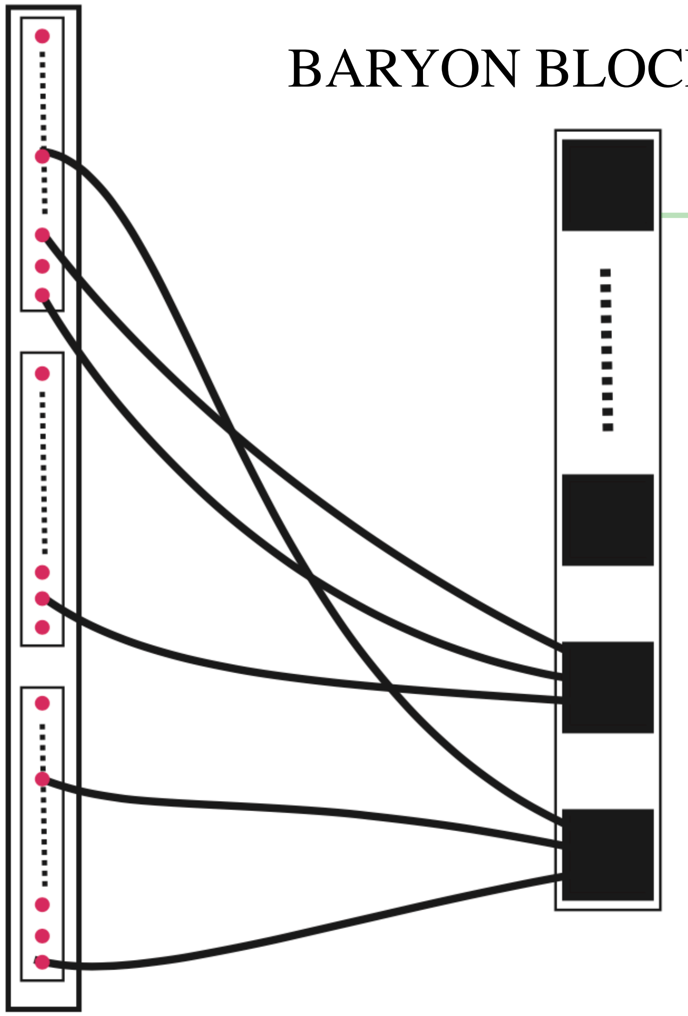
BARYON BLOCKS



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Can also start propagators at different locations.

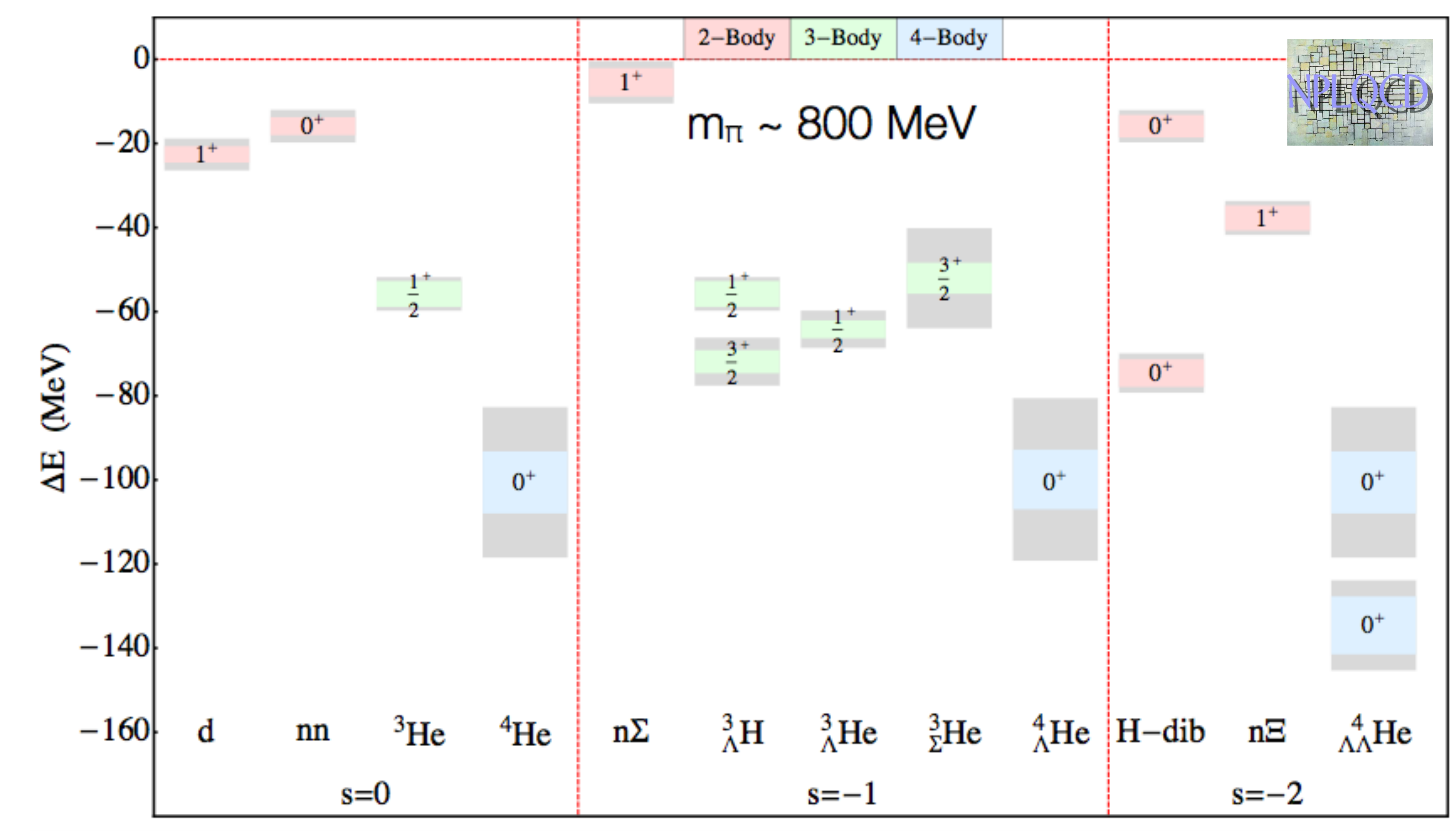
The new scaling is: $M_w \cdot N_w \cdot \dfrac{(3A)!}{(3!)^A}$

M_w: Number of terms in the sink

N_w: Number of terms in the source

Nuclei obtained from such an approach (at a heavier quark masses)

$N_f = 3,\ m_\pi = 0.806\ \text{GeV},\ a = 0.145(2)\ \text{fm}$

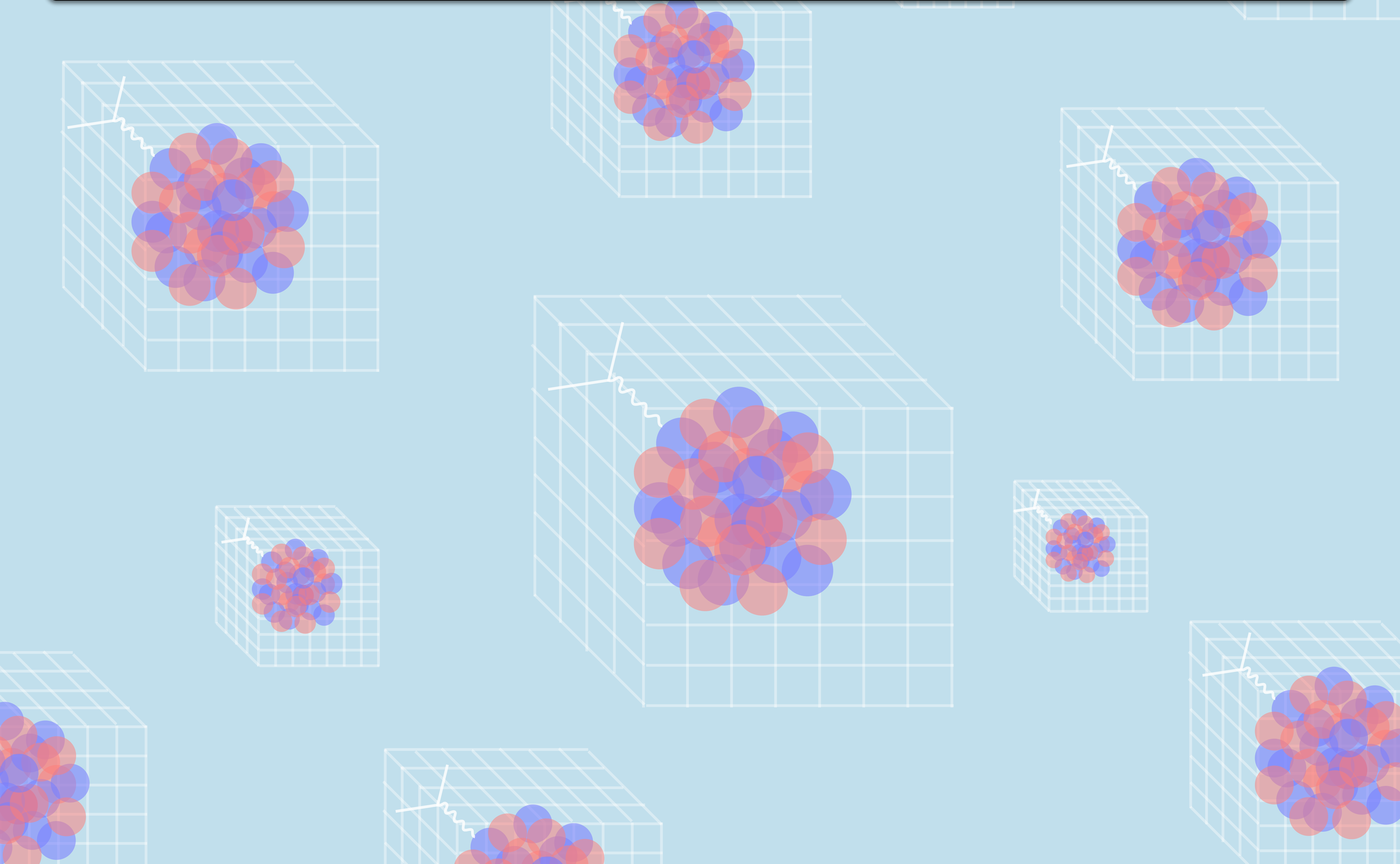


Beane, et al. (NPLQCD), Phys.Rev. D87 (2013) , Phys.Rev. C88 (2013)

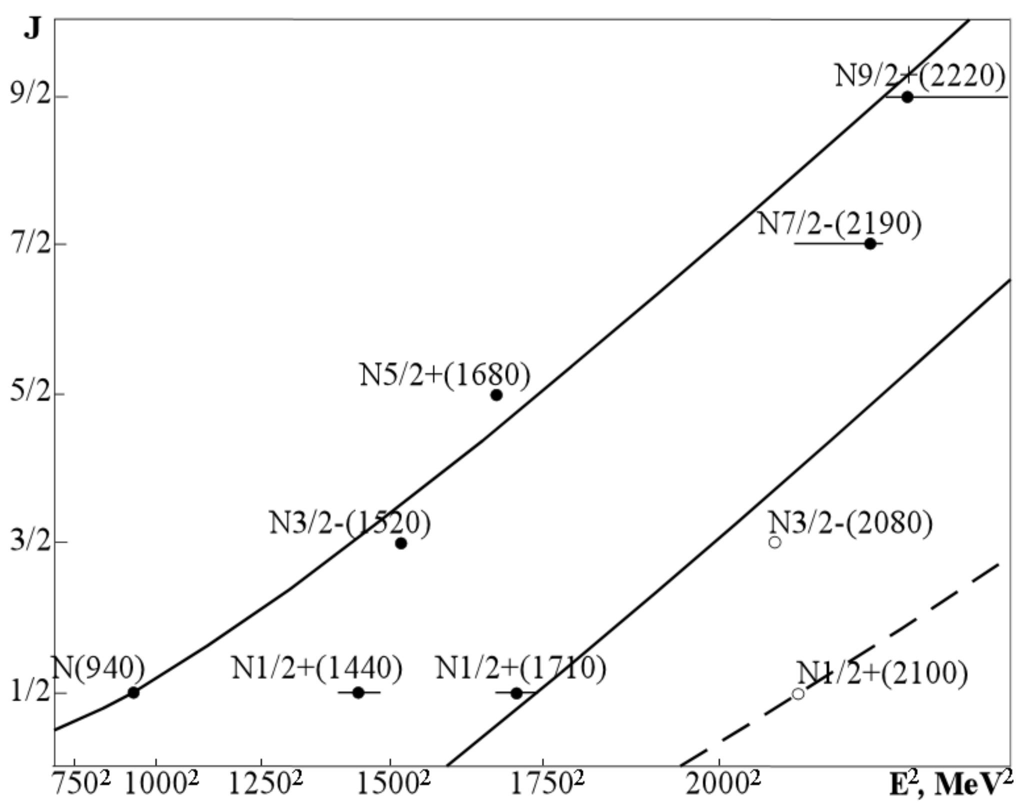
EXERCISE 8

According to the naive counting, how many contractions are required for a nucleus at the source and sink with atomic numbers A = 4, 8, 12, 16? How many contractions are there with the use of the efficient algorithm described? There are even more optimal algorithms that lead to a polynomial scaling with the number of the quarks.

ii) Excitation energies of nuclei are much smaller than the QCD scale.

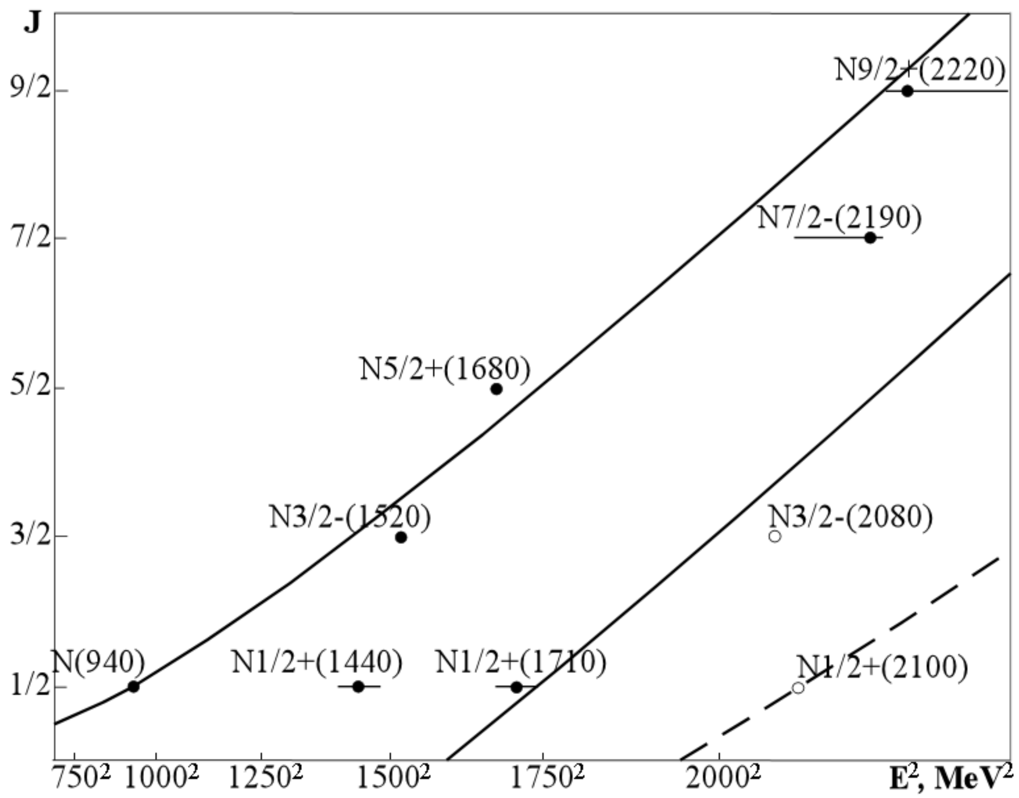


Nucleon excitations



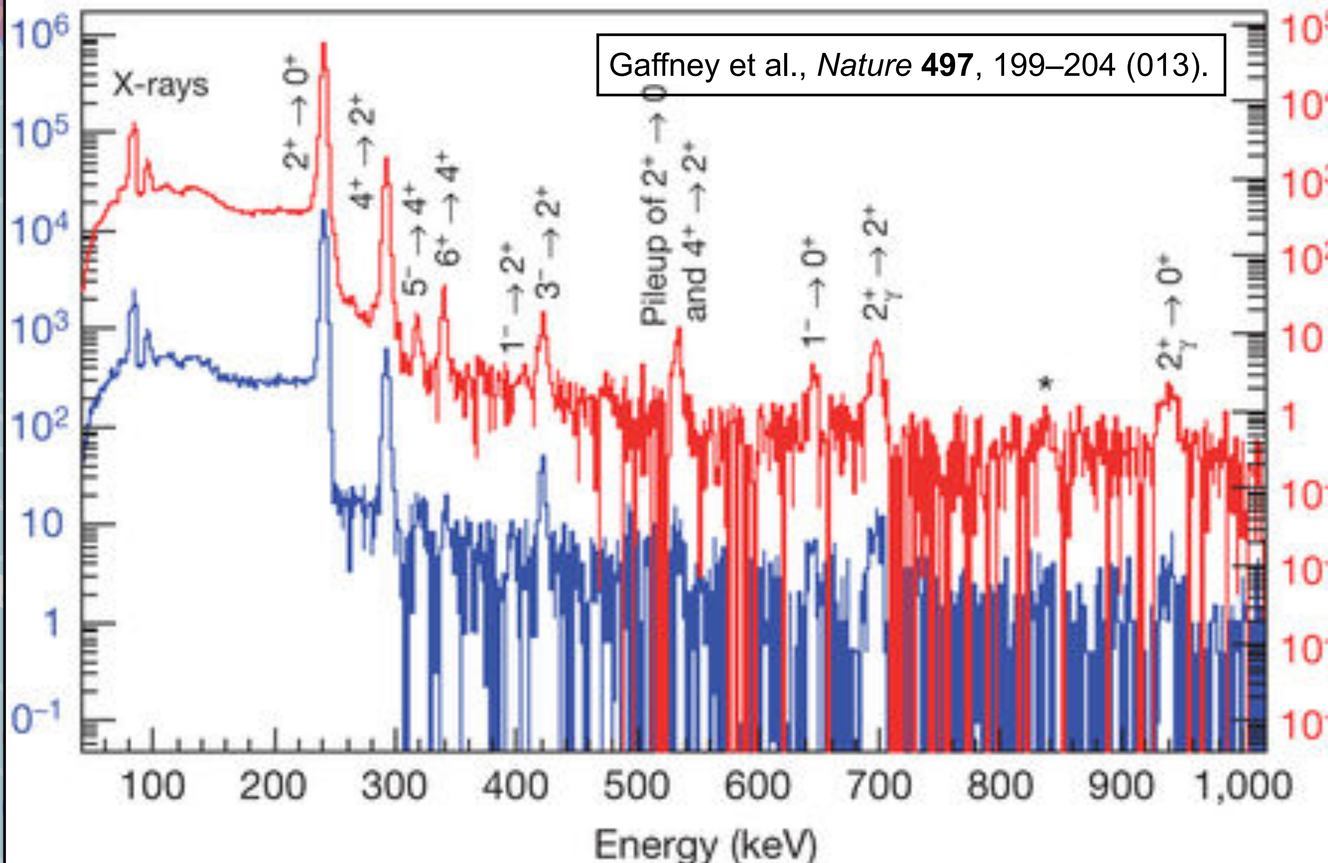
Kulikov, Dmitry A. et al., Central Eur.J.Phys. 11 (2013) .

Nucleon excitations



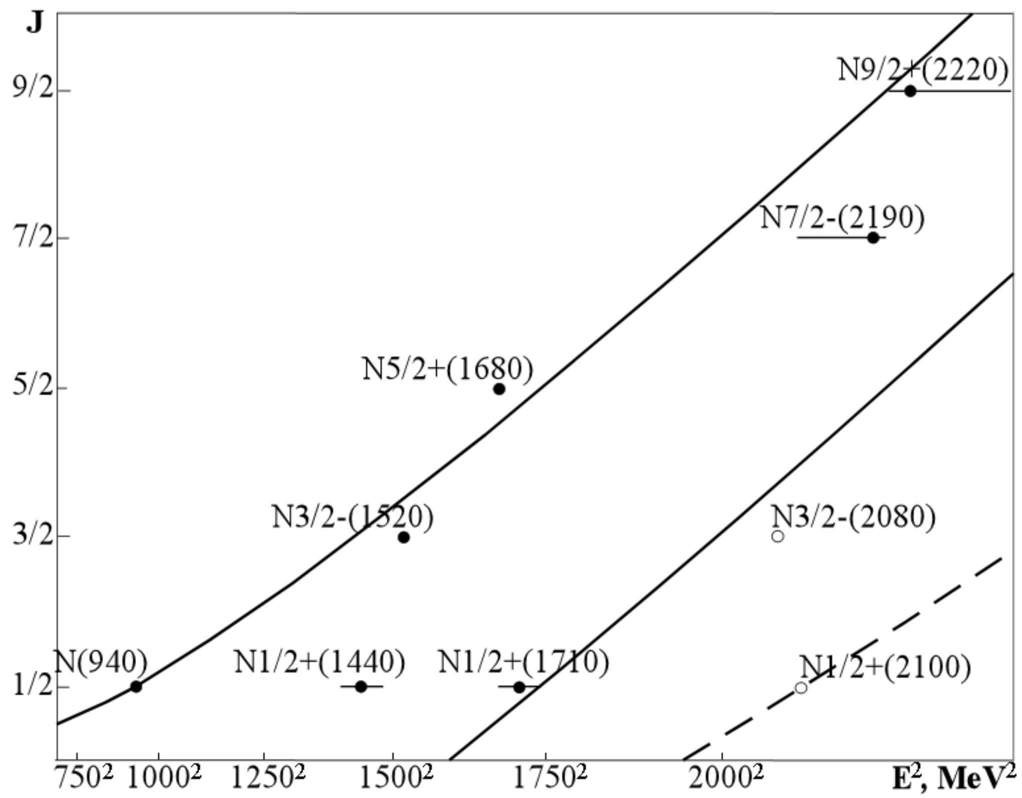
Kulikov, Dmitry A. et al., Central Eur.J.Phys. 11 (2013) .

Nuclear excitations of two pear-shaped nuclei (radium and radon)



Gaffney et al., *Nature* **497**, 199–204 (013).

Nucleon excitations



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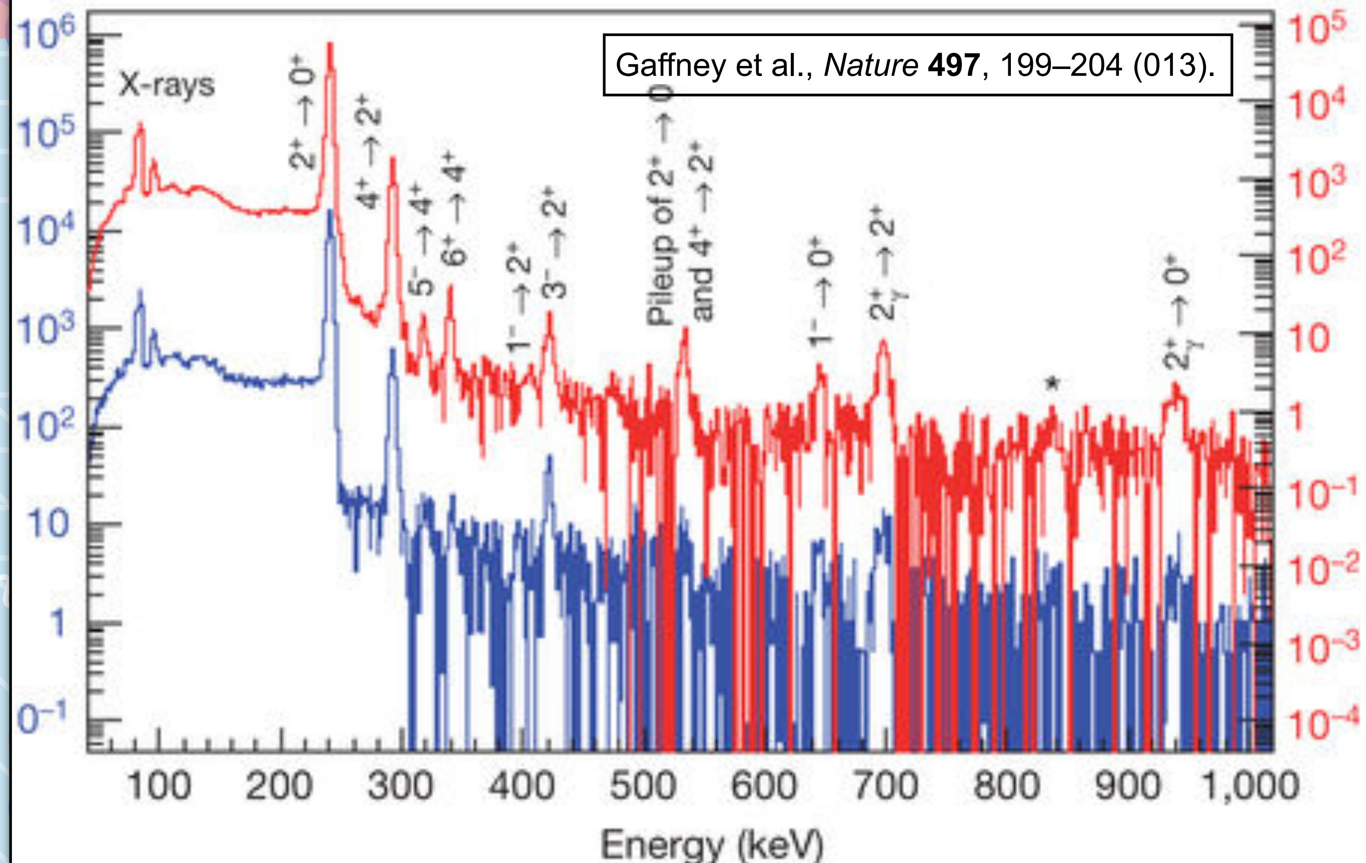


Gaffney et al., *Nature* **497**, 199–204 (013).

Getting radium directly from QCD will remain challenging for a long time! One should first compute A = 2, 3, 4 systems well. This is till not that easy: *B_d* = 2 MeV!

EXERCISE 9

With a given amount of computational resources, you have achieved a 1% statistical uncertainty on the extracted mass of the nucleon from your lattice QCD calculation. By what factor should you increase your computing resources (your statistics) to also achieve a 1% statistical uncertainty on the binding energy of the deuteron?

So what to do?

- With the most naive operators with similar overlaps to all states, unreasonably large times are needed to resolve nuclear energy gaps.

- The key to success of this program is in the use of good interpolating operators for nuclei. Since nucleons retain their identity in nuclei, forming baryon blocks at the sink turns out to be very advantageous.

- Ideally need to use a large set of operators for a variational analysis, but this has remained too costly in nuclear calculations.

Applications in mesonic sector: Briceno, Dudek and Young, Rev. Mod. Phys. 90 025001.

- Methods such as matrix Prony that eliminate the excited states in linear combinations of interpolators or correlations functions have shown to be useful.

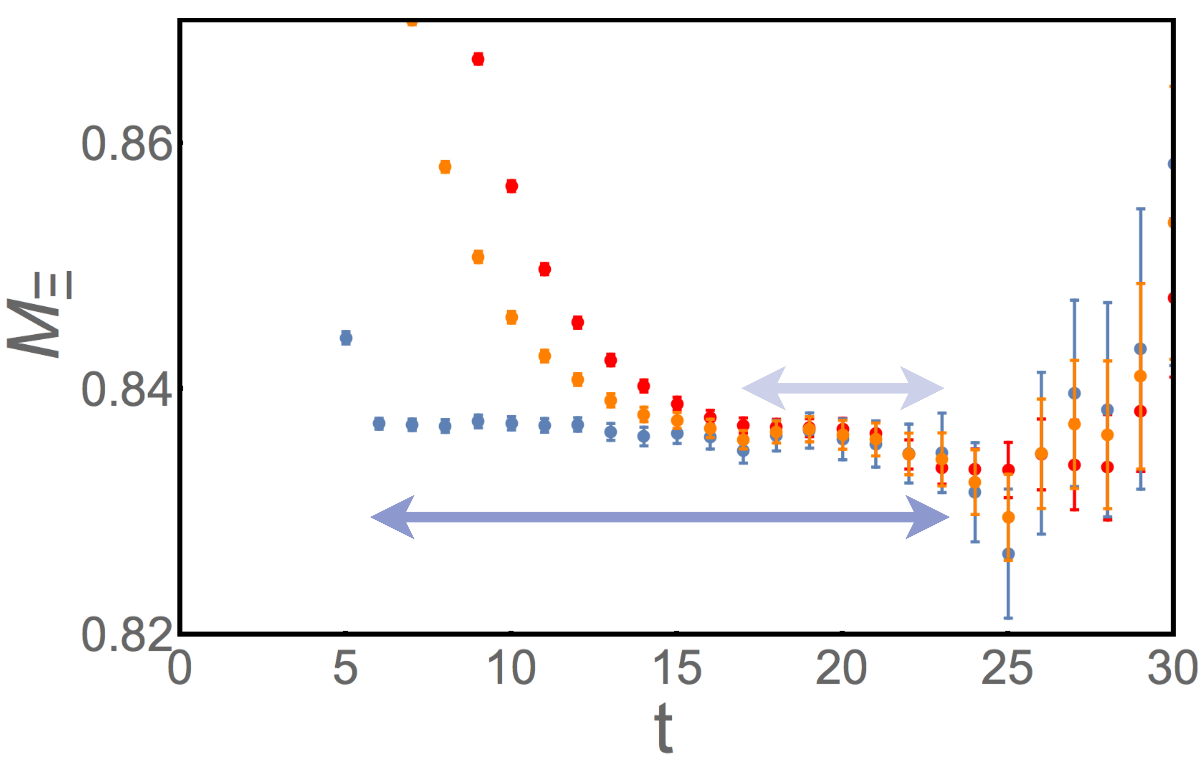
A good review: Beane, Detmold, Orginos, Savage, Prog. Part. Nucl. Phys. 66 (2011).

EXERCISE 10

Consider a simple two-state model in the spectral decomposition of an Euclidean two-point function. Demonstrate that the time scale to reach the ground state of the model with a finite statistical precision can depend highly on the corresponding overlap factor for the state. It is sufficient to show this numerically and for a set of chosen energies and overlap factors.

An example

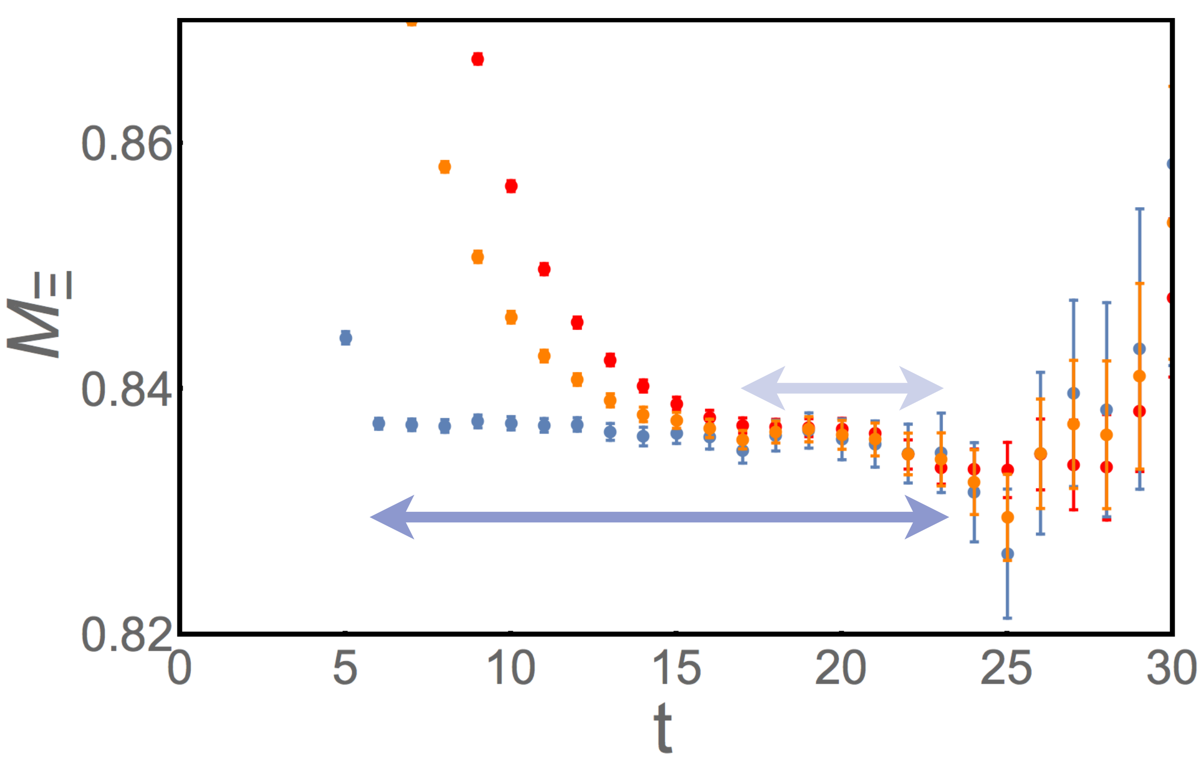
Linear combos. at the level of correlation functions



Beane et al (NPLQCD), Phys.Rev.D79:114502 (2009).

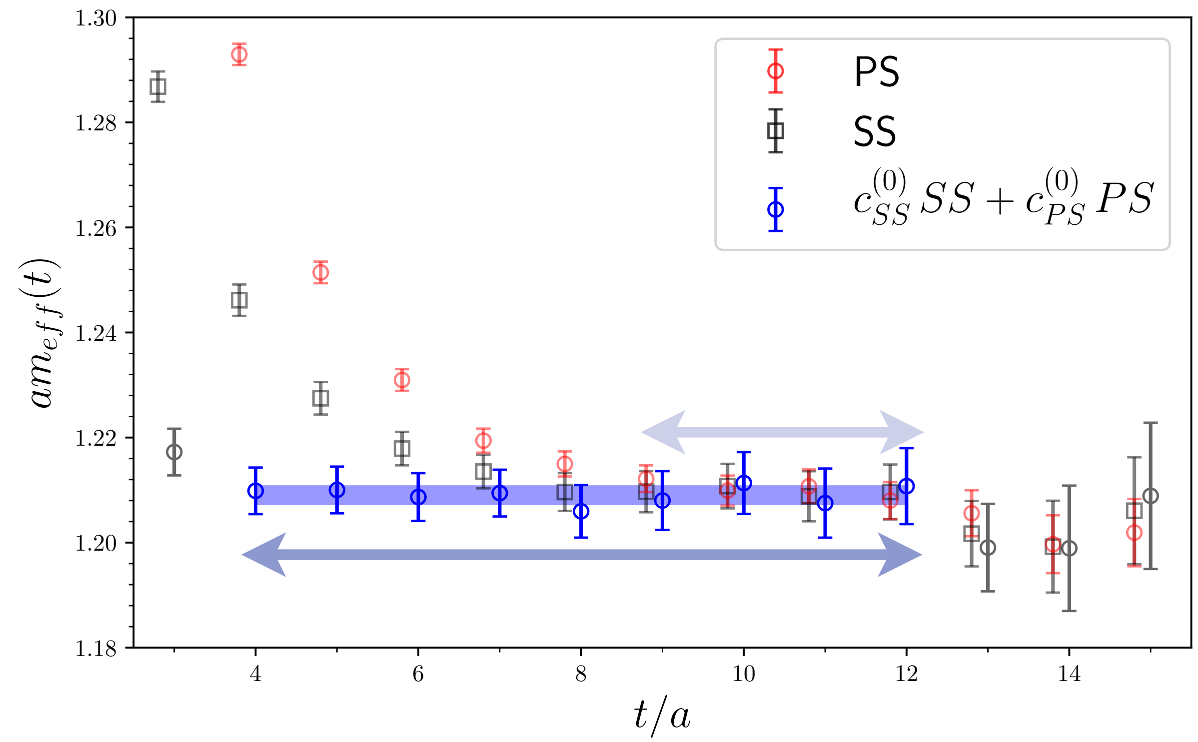
An example

Linear combos. at the level of correlation functions



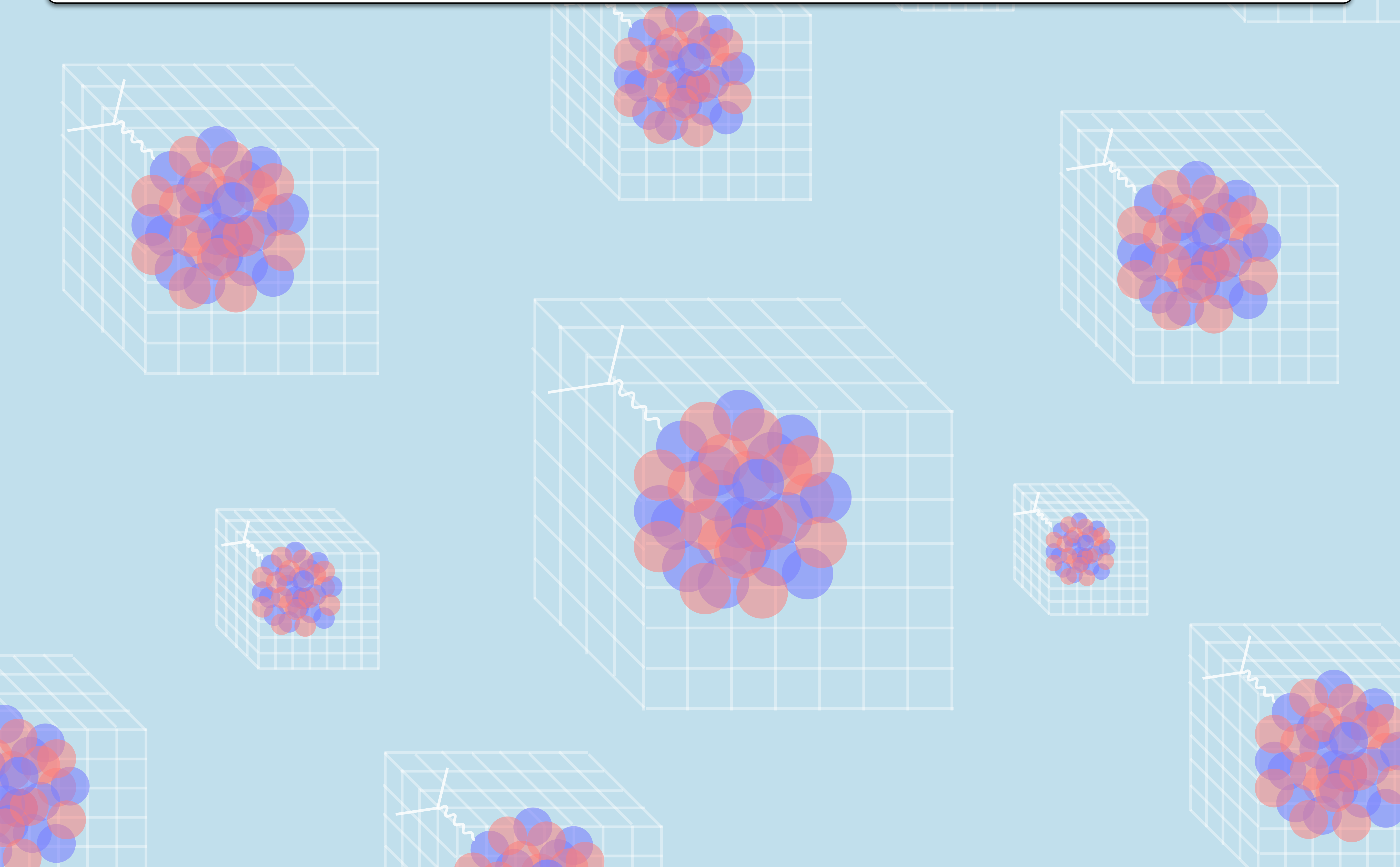
Beane et al (NPLQCD), Phys.Rev.D79:114502 (2009).

Berkowitz et al (CalLatt), arXiv:1710.05642(2017).



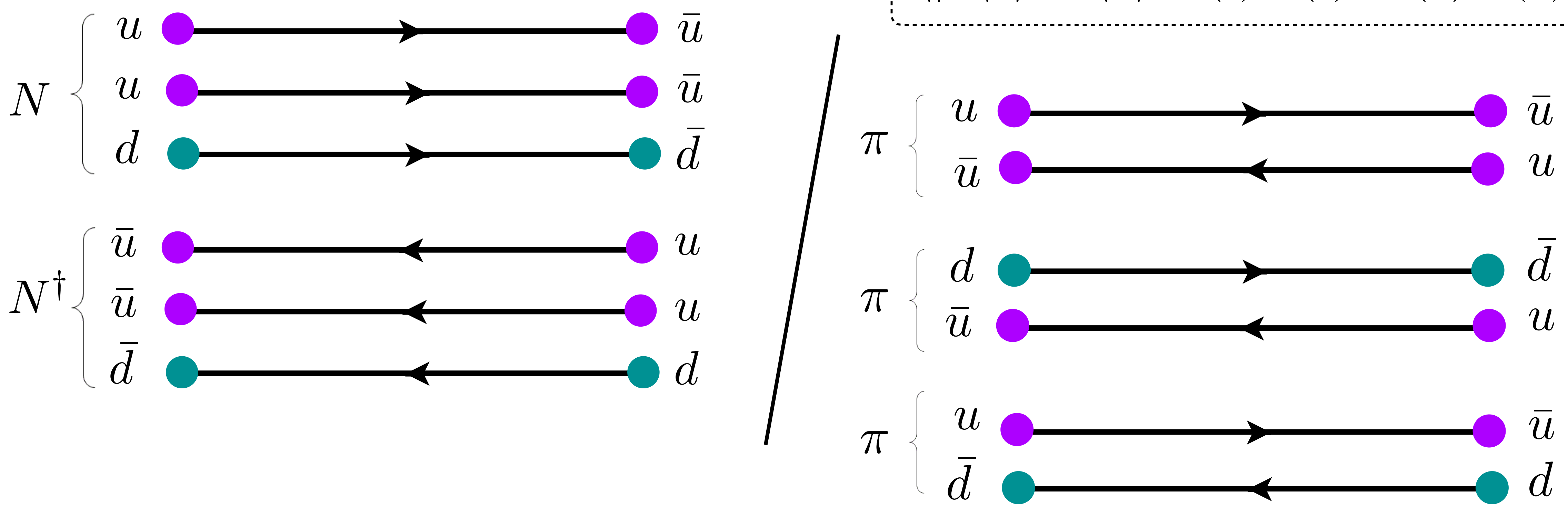
Linear combos. at the level of sink construction

iii) There is a severe signal-to-noise degradation.



The origin of noise

$$\langle |C|^2 \rangle = \langle 0 | N^\dagger(t) N(t) N^\dagger(0) N(0) | 0 \rangle$$



The origin of noise

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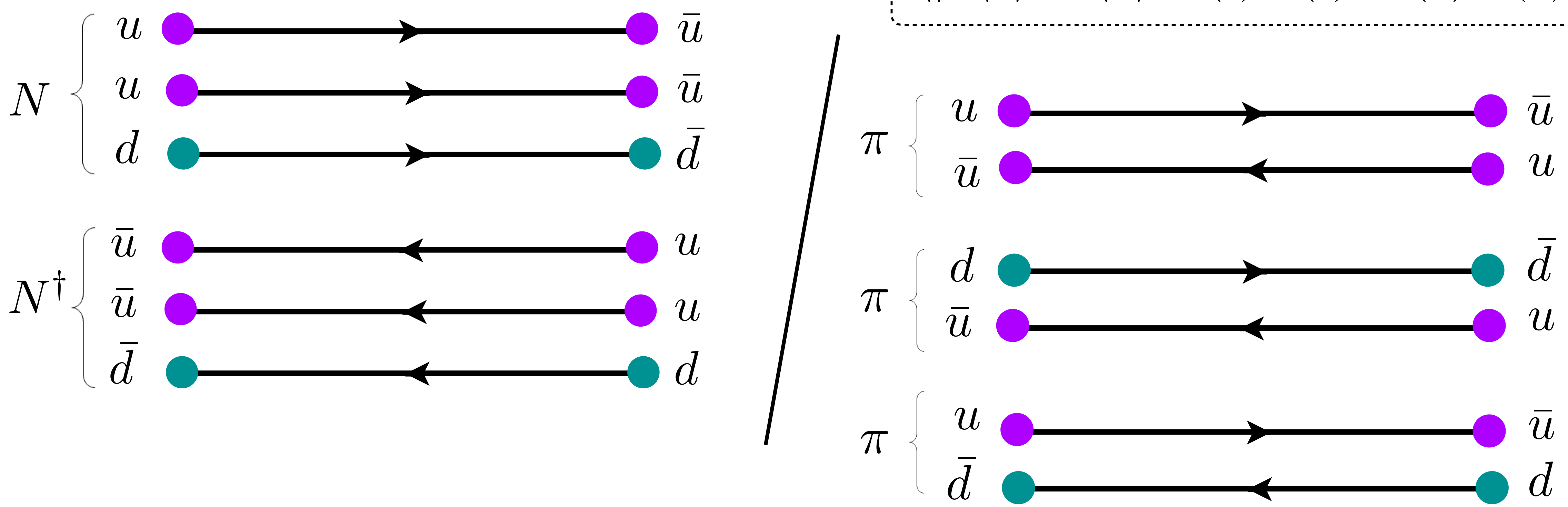
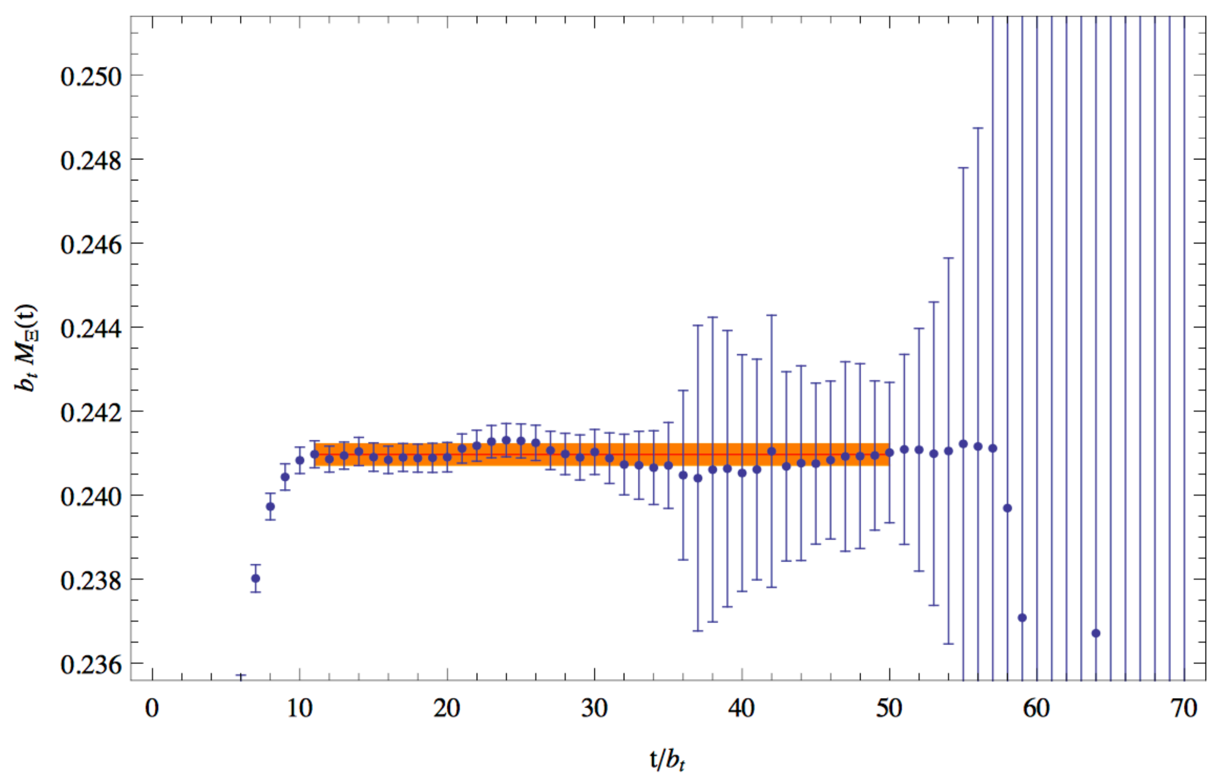


The ground-state of the variance correlator is three pions and not two nucleons:

$$\mathrm{StN}(C_i) \sim \frac{\langle C_i \rangle}{\sqrt{\langle |C_i|^2 \rangle}} \sim e^{-(M_N - \frac{3}{2} m_\pi) t}$$

Parisi (1984) and Lepage (1989).

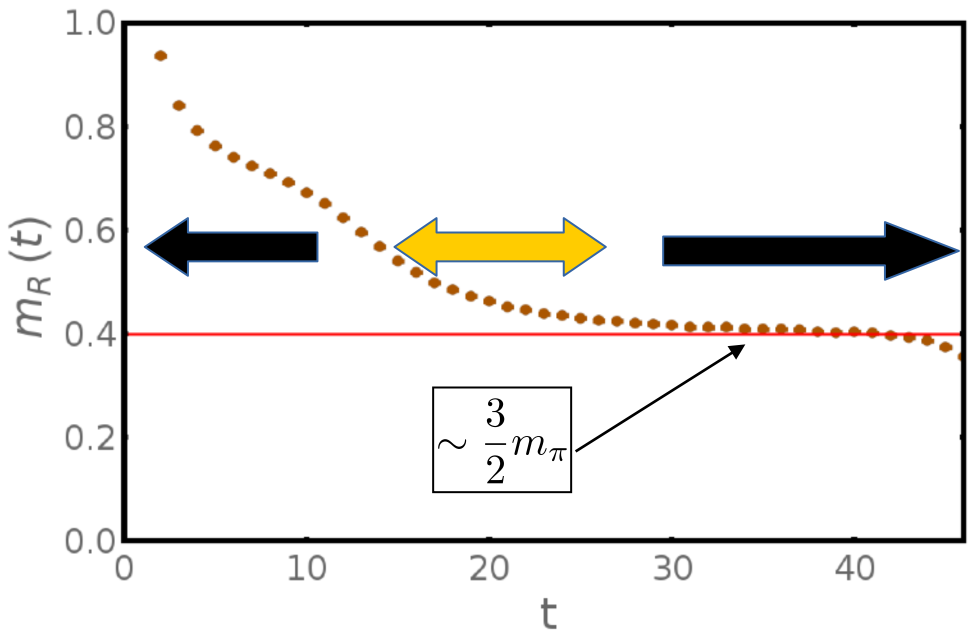


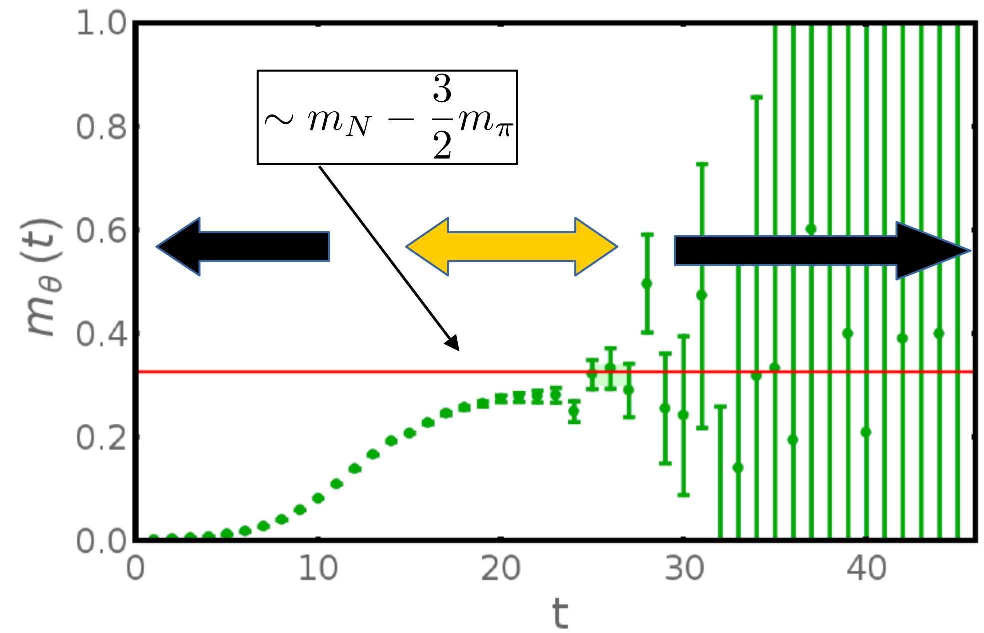
Can we understand better the noise in nuclear correlation function and control it?

$$\text{StN} \sim e^{-(m_N - \frac{3}{2}m_\pi)\Delta t}$$

Wagman and Savage (2016,2017).

Let's consider the magnitude and the phase of the correlation functions: $C_i(t) = e^{R_i(t)+i\theta_i(t)}$



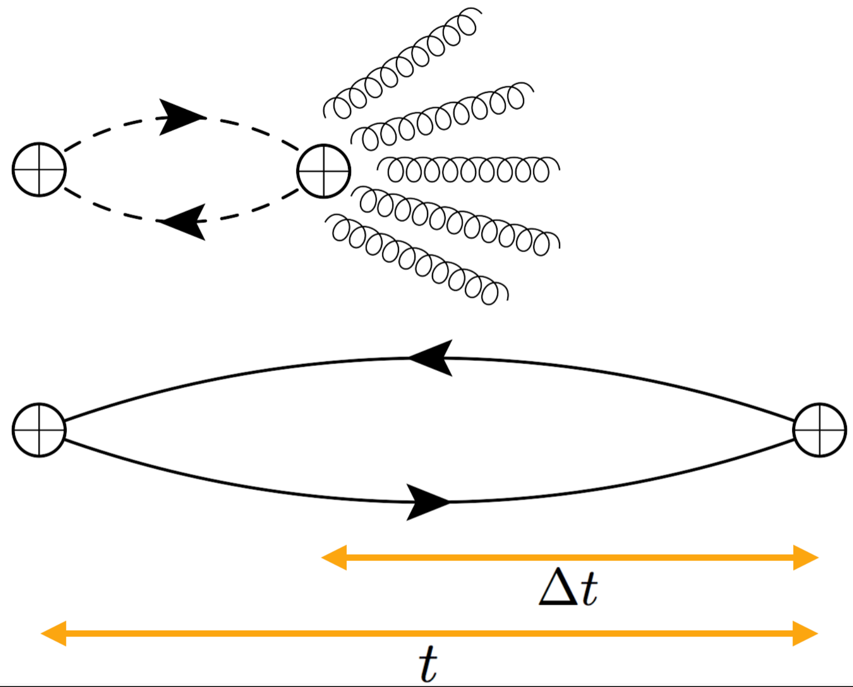


$$m_R(t) = \ln\left(\frac{\langle e^{R_i(t)}\rangle}{\langle e^{R_i(t+1)}\rangle}\right)$$

$$m_\theta(t) = \ln\left(\frac{\langle e^{i\theta_i(t)}\rangle}{\langle e^{i\theta_i(t+1)}\rangle}\right)$$

A phase reweighting method seems to work:

$$G^{\theta}(t, \Delta t) = \langle e^{-i\theta(t-\Delta t)} C(t) \rangle$$



$$M^{\theta}_{\rho}(t, \Delta t) \;=\; M_{\rho} \;+\; c\,\delta M_{\rho} e^{-\delta M_{\rho} \Delta t} \;+\; \ldots$$

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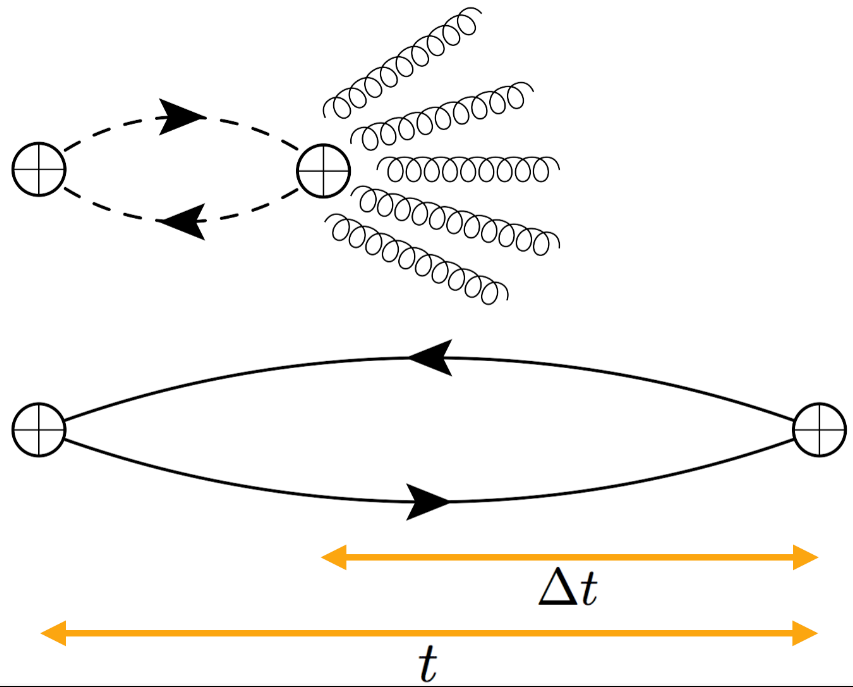
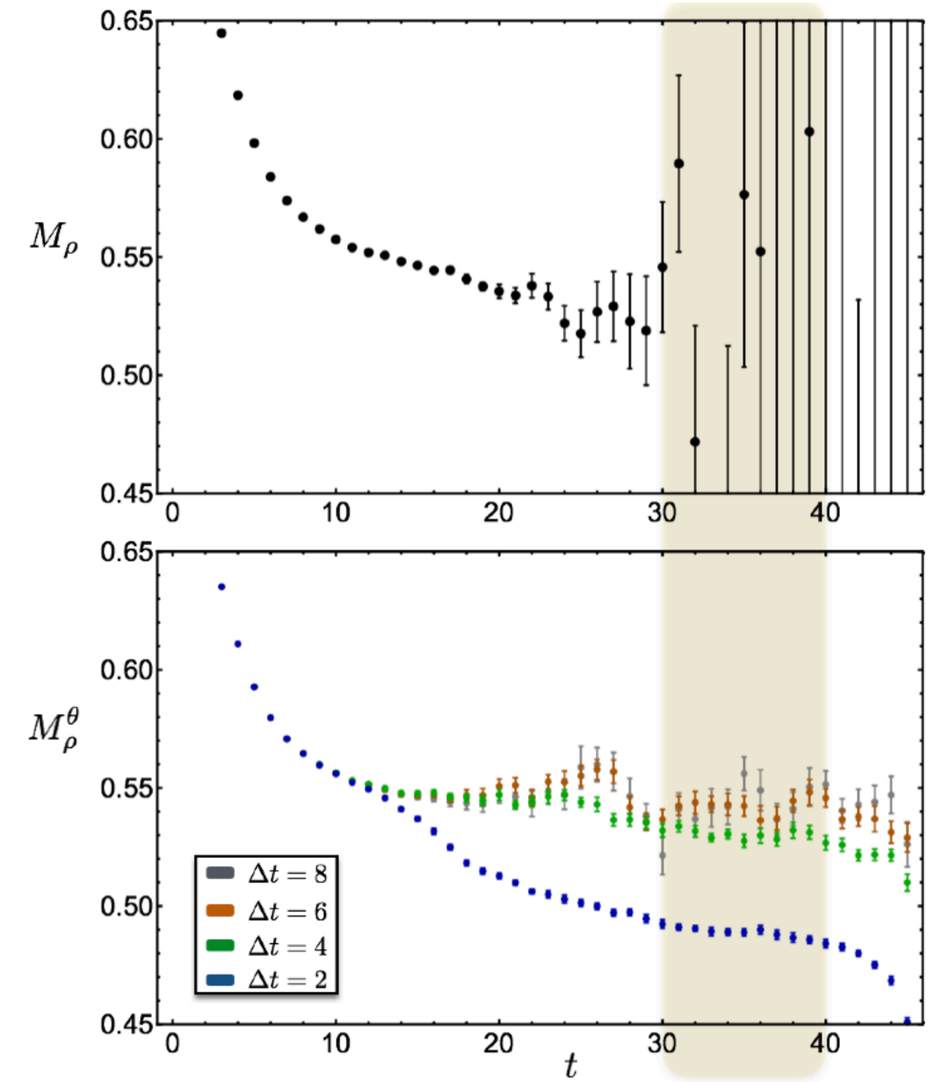
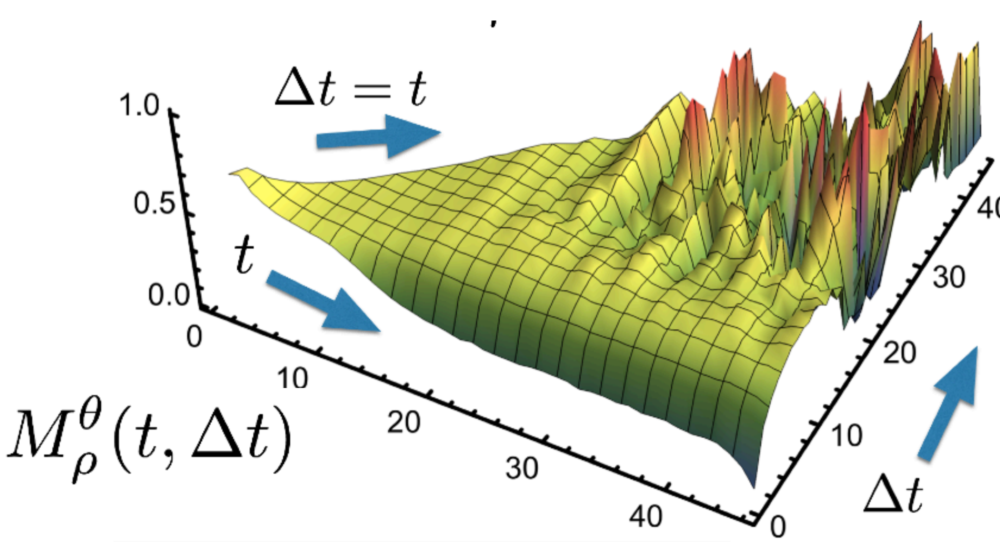
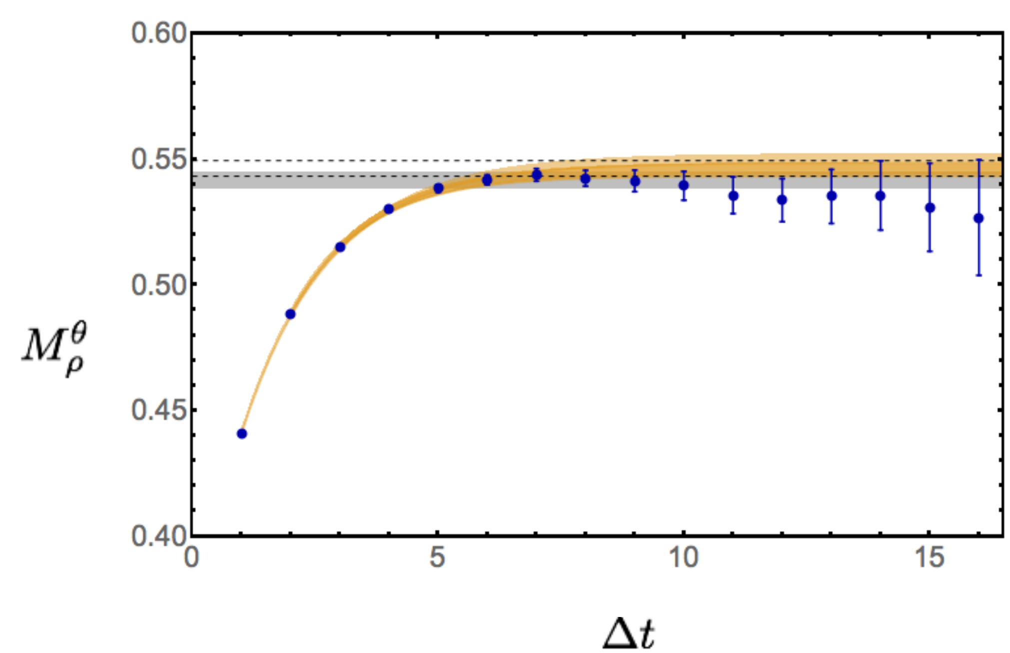


$$M_{\rho}^{\theta}(t, \Delta t) \;=\; M_{\rho} \;+\; c\, \delta M_{\rho} e^{-\delta M_{\rho} \Delta t} \;+\; \ldots$$







DESPITE CHALLENGES, GREAT PROGRESS HAS BEEN MADE. LQCD IS ON TRACK TO DELIVER RESULTS ON IMPORTANT QUANTITIES FOR THE EIC PHYSICS.

QUESTIONS?