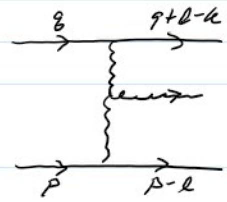
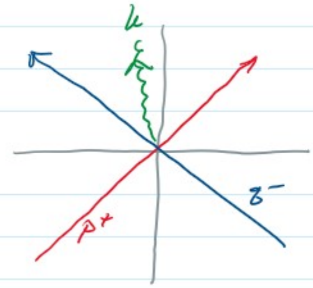
Solutions

Wednesday, July 31, 2019 6:44 PM

Exercise #1: Eikonal Soft Gluon Radiation

Start with the kinematics.
Consider all particles massless





Collinear / CMS frame:

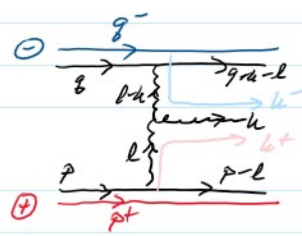
$$\begin{cases} p^\mu = (p^+, 0^-, \vec{0}_T) \\ q^\mu = (0^+, q^-, \vec{0}_T) \\ k^\mu = (k^+, \frac{k_T^2}{2k^+}, \vec{k}_T) \end{cases} \qquad k^+ \ll p^+$$

$$l^- = \cancel{p^-}_{0} - (p-l)^- = 0^- - \frac{l_T^2}{2(p^+ - \cancel{l^+})} \approx -\frac{l_T^2}{2p^+} \approx 0$$

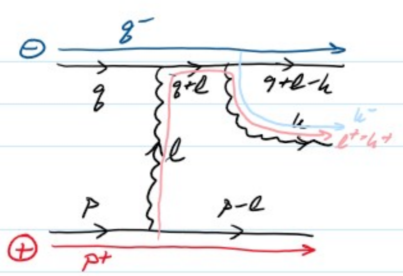
$$l^+ = (q+l-k)^+ - \cancel{q^+}_{0} + k^+ = \frac{\cancel{(l-k)_T^2}}{2(q^- + \cancel{l^-} - \cancel{k^-})} + \frac{k_T^2}{2k^-}$$

$q^- \gg k^-$

$$\boxed{\begin{cases} p^\mu = (p^+, 0^-, \vec{0}_T) \\ q^\mu = (0^+, q^-, \vec{0}_T) \\ k^\mu = (k^+, \frac{k_T^2}{2k^+}, \vec{k}_T) \\ l^\mu \approx (k^+, 0^-, l_T) \end{cases}}$$



A:



First start with the virtualities of the propagators.

$$\left[\begin{aligned} (q+l)^2 &= 2(\cancel{q^+} + l^+)(q^- + \cancel{l^-}) - (q+l)_T^2 \\ &= 2q^- k^+ - (q+l)_T^2 \\ &= \left(\frac{q^-}{k^-}\right) k_T^2 - \cancel{(q+l)_T^2} \\ (q+l)^2 &\approx 2q^- k^+ \end{aligned}\right. \qquad \left[\begin{aligned} l^2 &= 2l^+ l^- - l_T^2 \\ &= 2(k^+)\left(-\frac{l_T^2}{2p^+}\right) - l_T^2 \\ &= -\cancel{\left(\frac{k^+}{p^+}\right) l_T^2} - l_T^2 \\ l^2 &\approx -l_T^2 \end{aligned}\right.$$

$$A = \left[\bar{u}(q+l-k)\, \epsilon^*(k)\, (\not{q} + \not{l})\, \gamma_\mu u(q)\right]\left[\bar{u}(p-l)\, \gamma^\mu u(p)\right] \times \left(\frac{1}{l^2}\right)\left(\frac{1}{(q+l)^2}\right) \times \text{coeff}$$

$$\times \left(\frac{1}{\ell^2}\right)\left(\frac{1}{(q+\ell)^2}\right) \times \text{coeff}$$

$$= \left[\bar{u}(q+\ell-k)\, \not{\epsilon}^*(k)\, (\cancel{q^-}\gamma^+)\, \gamma_\mu\, u(q)\right]\left[\bar{u}(p-\ell)\,\gamma^\mu\, u(p)\right]$$
$$\times \left(-\frac{1}{\ell_\perp^2}\right)\left(+\frac{1}{2k^+\cancel{q^-}}\right) \times \text{coeff}$$

Gordon Identity: $\bar{u}(p)\gamma^\mu u(p) = 2p^\mu$

so $\left[\bar{u}(p-\ell)\gamma^\mu u(p)\right] \approx \left[\bar{u}(p^+)\gamma^\mu u(p^+)\right]$
$$= 2p^+ \underbrace{g^{\mu -}}_{\text{note: } = g^{\mu}{}_{+}}$$

$$A = \left(-\frac{1}{2k^+}\,\frac{1}{\ell_\perp^2}\right)\left[\bar{u}(q+\ell-k)\,\not{\epsilon}^*(k)\,\gamma^+\gamma_\mu\, u(q)\right]$$
$$\times \left(2p^+ g^{\mu -}\right) \times \text{coeff}$$

$$= -\left(\frac{p^+}{k^+}\right)\frac{1}{\ell_\perp^2}\left[\bar{u}(q+\ell-k)\,\not{\epsilon}^*(k)\,\gamma^+\gamma^-\, u(q)\right] \times \text{coeff}$$

<u>Dirac Eqn:</u>
$$0 = \not{q}\, u(q)$$
$$\approx (q^-\gamma^+)\, u(q^-)$$
$$\Leftrightarrow \quad 0 = \gamma^+ u(q^-)$$

so $$\gamma^+\gamma^- u(q) = \Big(\underbrace{\{\gamma^+,\gamma^-\}}_{2g^{+-}=2} - \gamma^-\underbrace{\cancel{\gamma^+}}_{0}\Big)\, u(q)$$

$$\gamma^+\gamma^- u(q) = 2\, u(q)$$

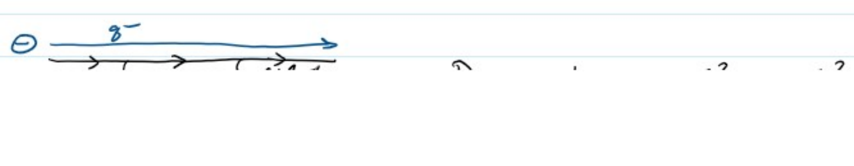
$$A = -2\left(\frac{p^+}{k^+}\right)\frac{1}{\ell_\perp^2}\left[\bar{u}(q+\ell-k)\,\not{\epsilon}^*(k)\, u(q)\right] \times \text{coeff}$$

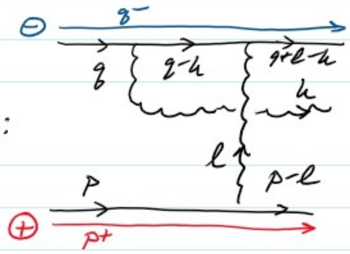
$$\left[\bar{u}(q+\ell-k)\,\not{\epsilon}^*(k)\, u(q)\right] \approx \left(\epsilon^*(k)\right)_\mu \left[\bar{u}(q^-)\gamma^\mu u(q^-)\right]$$
$$= \left(\epsilon^*(k)\right)_\mu \left(2q^- g^{\mu +}\right)$$
$$= 2q^-\left(\epsilon^*(k)\right)^+$$

$$A = -\frac{4p^+q^-}{\ell_\perp^2} \times \text{coeff} \times \left[\frac{\epsilon^{*+}(k)}{k^+}\right]$$

$$s = 2p^+q^-$$
$$k^+ = k_\perp^2/2k^-$$

$$\boxed{A = -\left(\frac{4s}{\ell_\perp^2}\right)\left(\frac{k^-\epsilon^{*+}(k)}{k_\perp^2}\right) \times \text{coeff}}$$



B: 

Propagators: $\ell^2 \approx -\ell_T^2$
$(q-k)^2 \approx -2k^+q^-$

$$B = \left[\bar{u}(q+\ell-k)\gamma_\mu(\not q-\not k)\,\not\epsilon^*(k)\,u(q)\right] \times \left[\bar{u}(p-\ell)\gamma^\mu u(p)\right] \times \frac{1}{(q-k)^2} \times \frac{1}{\ell^2} \times \text{coeff}$$

$$= \left(+\frac{1}{2k^+q^-}\frac{1}{\ell_T^2}\right)\left[\bar{u}(q+\ell-k)\gamma_\mu(\not q-\not k)\,\not\epsilon^*(k)\,u(q)\right]\left[\bar{u}(p-\ell)\gamma^\mu u(p)\right] \times \text{coeff}$$

$$= \left(\frac{1}{\cancel{2}k^+\cancel{q^-}}\frac{1}{\ell_T^2}\right)\left[\bar{u}(q+\ell-k)\gamma^-(\cancel{q^-}\gamma^+)\,\not\epsilon^*(k)\,u(q)\right](\cancel{2}p^+) \times \text{coeff}$$

$$= \left(\frac{p^+}{k^+}\right)\frac{1}{\ell_T^2}\left[\bar{u}(q^-)\gamma^-\gamma^+\,\not\epsilon^*(k)\,u(q)\right] \times \text{coeff}$$

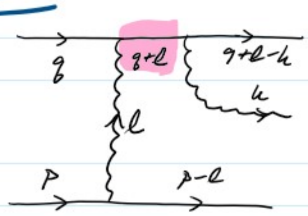
$$= 2\left(\frac{p^+}{k^+}\right)\frac{1}{\ell_T^2}\left[\bar{u}(q^-)\,\not\epsilon^*(k)\,u(q)\right] \times \text{coeff}$$

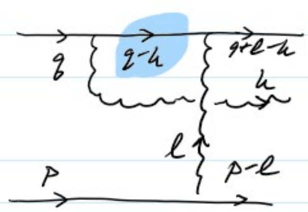
$$= 4\left(\frac{p^+}{k^+}\right)\frac{1}{\ell_T^2}\, q^-(\epsilon^*(k))^+ \times \text{coeff}$$

$$= \frac{2s}{\ell_T^2}\,\frac{(\epsilon^*(k))^+}{k^+} \times \text{coeff}$$

$$\boxed{B = \frac{4s}{\ell_T^2 k_T^2}\, k^-(\epsilon^*(k))^+ \times \text{coeff}}$$

Compare:

A: 

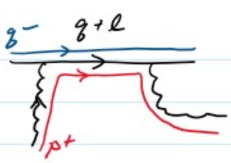
B: 

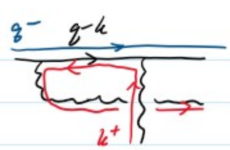
$$A = -\left(\frac{4s}{\ell_T^2}\right)\left(\frac{k^-\epsilon^{*+}(k)}{k_T^2}\right) \times \text{coeff}$$

$$B = +\frac{4s}{\ell_T^2 k_T^2}\, k^-(\epsilon^*(k))^+ \times \text{coeff}$$

$(q+\ell)^2 \approx +2k^+q^-$

$(q-k)^2 \approx -2k^+q^-$

Timelike: 

Spacelike: 

Relative sign due to virtualities: forward vs backward propagation of k^+

<u>For QED:</u>

In both A & B

$$\text{coeff} = (-ie)^3(i)(-i) = ie^3$$

$$A + B = 0$$

<u>For QCD</u>:

In A, $\text{coeff} = (ig)^3(i)(-i)[t^a t^b]\otimes[t^b] = -ig^3[t^a t^b]\otimes[t^b]$

In B, $\text{coeff} = (ig)^3(i)(-i)[t^b t^a]\otimes[t^b] = -ig^3[t^b t^a]\otimes[t^b]$

$$\text{In } A+B: \; (-ig^3)\,[\underbrace{t^a t^b - t^b t^a}_{[t^a, t^b] = if^{abc}t^c}]\otimes[t^b]$$

$$\hookrightarrow \; + g^3 f^{abc}[t^c]\otimes[t^b]$$

$$\boxed{A+B = -4g^3 f^{abc}[t^c]\otimes[t^b]\,\frac{s}{\ell_T^2 k_T^2}\left[k^- \varepsilon^{+*}(k)\right]}$$

By analogy: swap top/bottom
$(+ \leftrightarrow -)$ and $(b \leftrightarrow c)$

$$\boxed{C+D = +4g^3 f^{abc}[t^c]\otimes[t^b]\,\frac{s}{\ell_T^2 k_T^2}\left[k^+ \varepsilon^{-*}(k)\right]}$$

→ Commutator puts A–D in a form compatible with the 3G diagram E.

The total amplitude, including E, is:

$$\mathcal{M}_{tot} = 4ig^3 f^{abc}[t^b]\otimes[t^c]\left(\frac{s}{\ell_T^2}\right)\left[-\frac{(k-\ell)_T^{\mu}}{(k-\ell)_T^2} + \frac{k_T^{\mu}}{k_T^2}\right]\varepsilon_\mu^*(k)$$

Then the amplitude squared is:

$$\langle|\mathcal{M}|^2\rangle = 16g^3 f^{abc} f^{ab'c'} \cdot \frac{1}{N_c}\text{tr}[t^b t^{b'}]\,\frac{1}{N_c}\text{tr}[t^c t^{c'}]\left(\frac{s}{\ell_T^2}\right)^2\left[\frac{k_T^{\mu}}{k_T^2} - \frac{(k-\ell)_T^{\mu}}{(k-\ell)_T^2}\right]\left[\frac{k_T^{\nu}}{k_T^2} - \frac{(k-\ell)_T^{\nu}}{(k-\ell)_T^2}\right](-g_{\mu\nu})$$

<u>Color Algebra</u>: $\text{tr}[t^a t^b] = \frac{1}{2}\delta^{ab}$

and $f^{abc} f^{abd} = N_c \delta^{cd}$

so $f^{abc} f^{ab'c'}\,\frac{1}{N_c}\text{tr}[t^b t^{b'}]\,\frac{1}{N_c}\text{tr}[t^c t^{c'}] = \frac{1}{4N_c^2} f^{abc} f^{abc}$

$$= \frac{1}{4N_c^2}\left(N_c \sum_{c=1}^{N_c^2-1} \delta^{cc}\right)$$

$$= \frac{1}{4N_c^2}\left(N_c \sum_{c=1}^{N_c^2-1} \delta^{cc}\right)$$

$$= \frac{N_c^2-1}{4N_c} = \frac{1}{2} C_F$$

$$\langle |\mathcal{M}|^2 \rangle = 8 g^3 C_F \left(\frac{s}{l_T^2}\right)^2 \left[\frac{k_T^\mu}{k_T^2} - \frac{(k-l)_T^\mu}{(k-l)_T^2}\right]\left[\frac{k_T^\nu}{k_T^2} - \frac{(k-l)_T^\nu}{(k-l)_T^2}\right](-g_{\mu\nu})$$

$$= 8 g^3 C_F \left(\frac{s}{l_T^2}\right)^2 \left[\frac{\vec{k}_T}{k_T^2} - \frac{\vec{(k-l)}_T}{(k-l)_T^2}\right]^2$$

Vectors: $$\left[\frac{\vec{k}_T}{k_T^2} - \frac{\vec{(k-l)}_T}{(k-l)_T^2}\right]^2 = \frac{1}{k_T^2} - \frac{2\vec{k}_T \cdot \vec{(k-l)}_T}{k_T^2 (k-l)_T^2} + \frac{1}{(k-l)_T^2}$$

$$= \frac{1}{k_T^2 (k-l)_T^2}\left[(k-l)_T^2 - 2\vec{k}_T \cdot \vec{(k-l)}_T + \vec{k}_T^{\,2}\right]$$

$$= \frac{1}{k_T^2 (k-l)_T^2}\left[(\cancel{k}-l)_T - \cancel{k}_T\right]^2$$

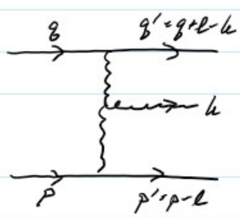
$$\left[\frac{\vec{k}_T}{k_T^2} - \frac{\vec{(k-l)}_T}{(k-l)_T^2}\right]^2 = \frac{l_T^2}{k_T^2 (k-l)_T^2}$$

$$\langle |\mathcal{M}|^2 \rangle = 8 g^3 C_F \frac{s^2}{l_T^2 k_T^2 (k-l)_T^2}$$

or

$$\boxed{\left\langle \left|\frac{\mathcal{M}}{2s}\right|^2 \right\rangle = 2 (4\pi\alpha_s)^3 C_F \frac{1}{l_T^2 k_T^2 (k-l)_T^2}}$$

Then the cross section is given by



$$d\sigma = \frac{1}{2s}\left[\frac{1}{2(2\pi)^3}\right]^3 \frac{d^3q'}{E_q'} \frac{d^3p'}{E_p'} \frac{d^3k}{E_k}$$
$$\times \langle |\mathcal{M}|^2 \rangle \cdot (2\pi)^4 \delta^4(p+q-p'-q')$$

$$= \frac{1}{2s} \frac{1}{8(2\pi)^5}\left(d^2q' \frac{dq'^-}{q'^-}\right)\left(d^2p' \frac{dp'^+}{p'^+}\right)\left(d^2k \frac{dk^+}{k^+}\right)$$
$$\times \langle |\mathcal{M}|^2 \rangle \, \delta(p^+ - p'^+)\, \delta(q^- - q'^-)\, \delta^2(\vec{p}_T' + \vec{q}_T')$$

and change variables $q' = q + l - k$, $d^2q' \to d^2l$

$$d\sigma = \frac{1}{2s} \frac{1}{8(2\pi)^5}\left(\frac{1}{p^+ q^-}\right) d^2l\, d^2k \frac{dk^+}{k^+} \langle |\mathcal{M}|^2 \rangle$$

$$= \frac{d^2l}{(2\pi)^2} \frac{d^2k}{(2\pi)^2} \frac{dk^+}{4\pi k^+} \left\langle \left|\frac{\mathcal{M}}{2s}\right|^2 \right\rangle$$

$$y = \frac{1}{2}\ln\frac{k^+}{k^-} = \frac{1}{2}\ln\left(\frac{2(k^+)^2}{k_T^2}\right)$$
$$y = \ln\left(\sqrt{2}\frac{k^+}{k_T}\right)$$
$$\hookrightarrow d^2k\, dy = d^2k \frac{dk^+}{k^+} \quad \left(\text{also} = d^2k \frac{dk^-}{k^-}\right)$$

$\hookrightarrow \; d^2k \, dy = d^2k \frac{dk^+}{k^+} \quad \left(\text{also} = d^2k \frac{dk^-}{k^-}\right)$

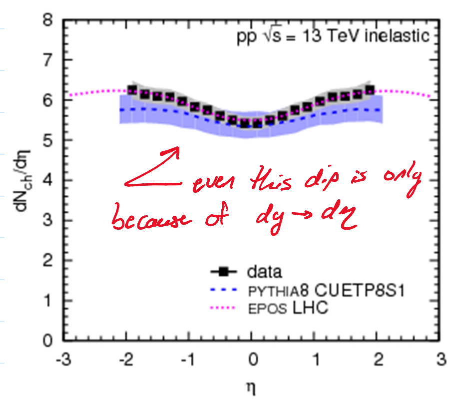
$$\boxed{\frac{d\sigma}{d^2l \, d^2k \, dy} = \frac{1}{2(2\pi)^5} \left\langle \left|\frac{\mathcal{M}}{2s}\right|^2 \right\rangle}$$

$$= \frac{1}{2(2\pi)^5} \left[2 (4\pi\alpha_s)^3 C_F \frac{1}{l_T^2 \, k_T^2 \, (k-l)_T^2} \right]$$

$$\boxed{\frac{d\sigma}{d^2k \, d^2l \, dy} = \frac{2\alpha_s^3 C_F}{\pi^2} \frac{1}{l_T^2 \, k_T^2 \, (k-l)_T^2}}$$

* Independent of rapidity (so long as the eikonal approximation is satisfied)

* At leading order, high-energy QCD is boost-invariant



even this dip is only because of $dy \to d\eta$

Exercise #2: Evolution of the Gluon Density

$$\frac{d\rho}{dY} = \alpha_s \cdot a \cdot \rho - \frac{\alpha_s^2 b}{Q^2} \rho^2$$

At weak coupling: $\alpha_s \ll 1$ and $\rho \sim \mathcal{O}(1)$

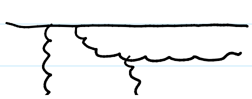
$$\frac{d\rho}{dY} \approx \alpha_s \cdot a \cdot \rho$$

$$\int_{Y=0}^{Y} \frac{d\rho}{\rho} = \int_{Y=0}^{Y} \alpha_s a \, dY$$

$$\ln \frac{\rho(Y)}{\rho(Y=0)} = \alpha_s a Y$$

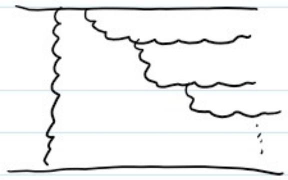
$$\boxed{\rho(Y) = \rho(Y=0) \cdot e^{\alpha_s \cdot a \cdot Y}}$$

$\rightarrow$ exponential growth in density with rapidity



$\rightarrow$ exponential growth in density with rapidity

$$\rho \sim e^{\alpha_s a Y} \sim \left[e^{\ln 1/x}\right]^{\alpha_s a} \sim \left(\frac{1}{x}\right)^{\alpha_s a}$$



In the full nonlinear case

$$\frac{d\rho}{dY} = \alpha_s \cdot a \cdot \rho - \frac{\alpha_s^2 b}{Q^2}\rho^2$$

$$\text{Let } \begin{cases} A \equiv \alpha_s a \\ B \equiv \alpha_s^2 b / Q^2 \end{cases}$$

Then

$$\frac{d\rho}{dY} = A\rho - B\rho^2$$

$$\frac{d\rho}{\rho(A-B\rho)} = dY$$

Partial fractions: try to find C, D such that:

$$\frac{1}{\rho(A-B\rho)} = C\left(\frac{1}{\rho}\right) + D\left(\frac{1}{A-B\rho}\right) \quad \text{for any } \rho.$$

- Multiply by ρ, then set $\rho = 0$: $C = 1/A$
- Multiply by $(A-B\rho)$, then set $\rho = A/B$: $D = B/A$

$$\hookrightarrow \boxed{\frac{1}{\rho(A-B\rho)} = \frac{1}{A\rho} + \frac{B}{A(A-B\rho)}}$$

Then

$$\frac{d\rho}{\rho(A-B\rho)} = \frac{1}{A}\frac{d\rho}{\rho} + \frac{B}{A}\frac{d\rho}{A-B\rho} = dY$$

$$\frac{d\rho}{\rho} + \frac{d\rho}{A/B - \rho} = A\,dY$$

$$\int_{Y=0}^{Y}\frac{d\rho}{\rho} - \int_{Y=0}^{Y}\frac{d(\rho - A/B)}{\rho - A/B} = A\int_{Y=0}^{Y} dY$$

$$\ln\frac{\rho}{\rho_0} - \ln\frac{\rho - A/B}{\rho_0 - A/B} = AY$$

$$\ln\left(\frac{\rho}{\rho - A/B}\cdot\frac{\rho_0 - A/B}{\rho_0}\right) = AY$$

$$\rho = \frac{\rho_0}{\rho_0 - A/B}\,(\rho - A/B)\, e^{AY}$$

$$\rho\left[1 - \frac{\rho_0}{\rho_0 - A/B}\, e^{AY}\right] = -\frac{A}{B}\,\frac{\rho_0}{\rho_0 - A/B}\, e^{AY}$$

$$\rho(Y) = \frac{-A/B\ \frac{\rho_0}{\rho_0 - A/B}\, e^{AY}}{1 - \frac{\rho_0}{\rho_0 - A/B}\, e^{AY}} \cdot \frac{-(\rho_0 - A/B)\, e^{-AY}}{-(\rho_0 - A/B)\, e^{-AY}}$$

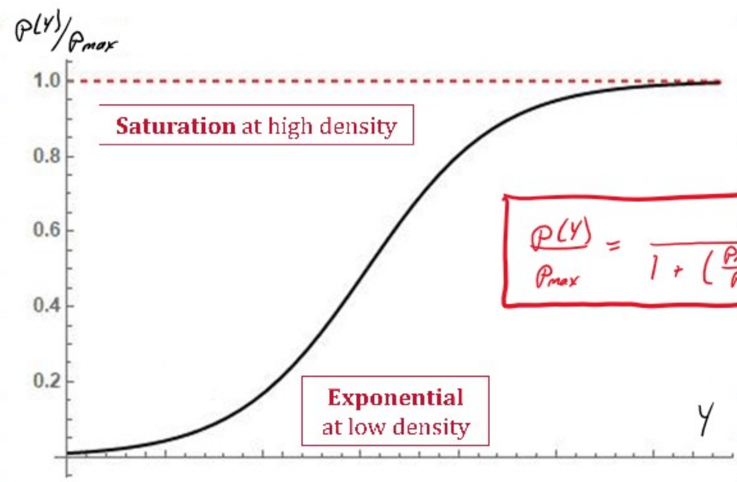
$$\rho(Y) = \frac{\left(\frac{A}{B}\right)\rho_0}{\rho_0 - (\rho_0 - A/B)\, e^{-AY}}$$

Then for $\begin{cases} A \equiv \alpha_s a \\ B \equiv \alpha_s^2 b / Q^2 \end{cases}$, $\quad \frac{A}{B} = \frac{a}{b}\frac{Q^2}{\alpha_s}$

for $\alpha_s \ll 1$ this is large

$$\rho_0 < \frac{A}{B} = \frac{a}{b}\frac{Q^2}{\alpha_s}$$

$$\boxed{\rho(Y) = \frac{\left(\frac{a}{b}\frac{Q^2}{\alpha_s}\right)\rho_0}{\rho_0 + \left(\frac{a}{b}\frac{Q^2}{\alpha_s} - \rho_0\right) e^{-a\alpha_s Y}}}$$



$$\frac{\rho(Y)}{\rho_{max}} = \frac{1}{1 + \left(\frac{\rho_{max}}{\rho_0} - 1\right) e^{-a\alpha_s Y}}$$

For $a\alpha_s Y \ll 1$, drop the constant ρ_0 compared to $\frac{a}{b}\frac{Q^2}{\alpha_s}$

↳ Recover $\rho(Y) \sim \rho_0\, e^{+a\alpha_s Y}$

For $a\alpha_s Y \gg 1$, use binomial expansion:

$$\rho(Y) = \left(\frac{a}{b}\frac{Q^2}{\alpha_s}\right)\left[1 + \left(\frac{1}{\rho_0}\frac{a}{b}\frac{Q^2}{\alpha_s} - 1\right) e^{-a\alpha_s Y}\right]^{-1}$$

$$\approx \left(\frac{a}{b}\frac{Q^2}{\alpha_s}\right)\left[1 - \left(\frac{1}{\rho_0}\frac{a}{b}\frac{Q^2}{\alpha_s} - 1\right) e^{-a\alpha_s Y}\right]$$

↳ Asymptotically saturates to $\rho_{max} = \frac{a}{b}\frac{Q^2}{\alpha_s}$

Since ρ = # gluons and $Q^2 \sim 1$

$$\text{Since} \quad \rho = \frac{\#\text{ gluons}}{\text{area}} \quad \text{and} \quad Q^2 \sim \frac{1}{\Delta r_T^2} \leftarrow \text{transverse area}$$

$$\rho_{max} \sim \frac{Q^2}{\alpha_s} \sim \frac{(1/\alpha_s)\text{ gluons}}{\Delta r_T^2 \quad \text{area}}$$

The number of gluons is <u>large</u>!

So large that $(\alpha_s \cdot \rho_{max} \cdot Q^2) \sim \mathcal{O}(1)$ is <u>not</u> a higher order correction!