Unifying the Background with Perturbations in Chaplygin-gas Cosmology II

Heba Abduhlraman
In collaboration with Amare Abebe
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North-West University
1. The background solutions

2. Cosmological perturbations

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4. Matter-Chaplygin gas System

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The aim of this project is to find a suitable model that could possibly solve the dark matter and dark energy problems, and make it possible to predict the future of the Universe.
The Friedmann Equation

\[
H^2 = \frac{8\pi G}{3} \rho - \frac{kc^2}{a^2}, \quad (1)
\]

The Fluid Equation

\[
\dot{\rho} + 3\frac{\dot{a}}{a} \left( \rho + \frac{p}{c^2} \right) = 0, \quad (2)
\]
The background solutions

Curvature and... stuff

Barotropic Equation of State for perfect fluids: relates pressure $p$ and energy density $\rho$ by

$$p = \omega \rho,$$

(3)

where $\omega$ is the Equation of State Parameter given by

$$\omega = \begin{cases} 
0 & \text{for dark and baryonic matter (dust),} \\
1/3 & \text{for radiation, and} \\
-1 & \text{for dark energy, etc...}
\end{cases}$$

(4)
The solution to the fluid equation (when substituting in different \( \omega \)) produces energy densities:

\[
\rho_d(a) = \rho_0 a^{-3} \quad (5)
\]
\[
\rho_r(a) = \rho_0 r a^{-4} \quad (6)
\]
\[
\rho_\Lambda(a) = \rho_0 \Lambda \quad (7)
\]

Figure: The contents of the Universe according to recent Planck data. Credit: Katie Mack.
The background solutions

The Chaplygin-Gas

- Chaplygin Gas: a hypothetical fluid that behaves like dark matter and dark energy combined.
- Equation of State for generalized CG:
  \[ p = -\frac{A}{\rho^\alpha} \] (8)
  
  Substitute (8) into fluid equation to produce energy density solution:
  \[ \rho_{ch}(a) = \left( A + \frac{B}{a^{3(1+\alpha)}} \right)^{\frac{1}{1+\alpha}} \] (9)

where \( A \) and \( B \) are positive constants, and \( 0 < \alpha \leq 1 \).
The background solutions

**CG behaves like DM and DE**

Let us take $\alpha = 1$ for simplicity (original CG):

$$\rho_{ch}(a) = \sqrt{A + \frac{B}{a^6}}$$  \hspace{1cm} (10)

- When the scale factor $a$ is very small (early Universe), then $\rho_{ch}(a) \approx \frac{\sqrt{B}}{a^3} \rightarrow$ similar to dust/dark matter ($\rho \sim a^{-3}$).
- When the scale factor $a$ is very large (late Universe), then $\rho_{ch}(a) \approx \sqrt{A} \rightarrow$ similar to dark energy ($\rho \sim constant$).
- A universe consisting of Chaplygin gas, radiation and baryonic matter:

$$\rho_{total}(a) = \left[ A + \frac{B}{a^3(1+\alpha)} \right] \frac{1}{(1+\alpha)} + \rho_0 r a^{-4} + \rho_0 b a^{-3}$$  \hspace{1cm} (11)
The background solutions

The Chaplygin gas solutions

(a) CG density versus scale factor for different values of $\alpha$ ($\alpha = 1$ is known as the original CG model).

(b) $\rho$ vs. $t$ for the CG system.

Figure
The deceleration parameter

\[ q \equiv -\frac{\ddot{a}}{a} \frac{1}{H^2} \]  

(12)

describes the rate of the expansion of the Universe. If \( q < 0 \), the expansion of the Universe is accelerating, and \( q > 0 \) means that the expansion of the Universe is decelerating.
The background solutions

Accelerated Expansion?

Figure: The deceleration parameter versus the redshift in a radiation-baryonic matter-CG system.
The metric approach.
Cosmological perturbations

- The metric approach.
  It is a non-local, linear and coordinate-dependent approach

- The covariant approach.
Cosmological perturbations

- The metric approach. It is a non-local, linear and coordinate-dependent approach.
- The covariant approach. It is local and valid in all coordinate systems.
The 1 + 3 covariant formalism

The space-time is split into space and time; it is divided into perpendicular 4-velocity field vector $u^a$

$$u^a = \frac{dx^a}{d\tau}, \quad u^a u_a = -1$$

In this approach the metric $g_{ab}$ is decomposed into the projected tensor $h_{ab}$ as follows;

$$g_{ab} = h_{ab} - u_a u_b$$  (13)

with

$$h_c^a h_b^c = h_b^a, \quad h_a^a = 3, \quad h_{ab} u^b = 0$$
Cosmological perturbations

Kinematic and dynamic quantities

\[ \nabla_a u_b = \tilde{\nabla}_a u_b - u_a \dot{u}_b = \frac{1}{3} \Theta h_{ab} + \sigma_{ab} + \omega_{ab} - u_a \dot{u}_b \]  (14)

The relativistic acceleration vector is given as

\[ \dot{u}^a = u^b \nabla_b u^a \]  (15)
The vector gradient variables are defined as:

\[ Z_a = a \tilde{\nabla}_a \theta \]  \hspace{1cm} (16)

\[ D^m_a = \frac{a}{\rho_m} \tilde{\nabla}_a \rho_m \]  \hspace{1cm} (17)

\[ D^{ch}_a = \frac{a \tilde{\nabla}_a \rho_{ch}}{\rho_{ch}} \]  \hspace{1cm} (18)
Conservation equations for non-interacting fluids

\[ \dot{\rho} = -\theta (\rho + p) + (\rho + p) \tilde{\nabla}^a \Psi_a \]  
\[ (\rho + p) \dot{u}_a + \tilde{\nabla}_a p + \dot{\Psi}_a - (3c_s^2 - 1) \frac{\theta}{3} \Psi_a + \Pi_a = 0 \]

where \( \Psi_a = \frac{q_a}{\rho + p} \) is the heat flux of the fluids and \( \Pi_a = \tilde{\nabla}^a \pi_{ab} \) is anisotropic stress.
For totally non-interacting fluids, the time derivative of the four-velocity vector is

$$\dot{u}_a = \frac{\alpha \omega_{ch} \rho_{ch} D_{a}^{ch}}{a(\rho_t + p_t)} - \frac{\omega_m \rho_m D_{a}^{m}}{a(\rho_t + p_t)}$$  \hspace{1cm} (21)$$

The another key equation for general fluid so called Raychaudhuri equation and it can be expressed as

$$\dot{\Theta} = -\frac{1}{3} \Theta^2 - \frac{1}{2} (\rho_t + 3p_t) - \tilde{\nabla}^a \dot{u}_a .$$  \hspace{1cm} (22)$$
Perturbations

Linear evolution equations

\[
\dot{Z}_a = -\frac{2\theta}{3} Z_a - \left( \frac{1}{2} + \frac{3\omega_m}{2} + \frac{\theta^2 \omega_m}{3(\rho_t + p_t)} + \frac{\omega_m(\rho_t + 3p_t)}{2(\rho_t + p_t)} \right) \rho_m \ D_a^m (23)
\]

\[
- \left( \frac{1}{2} - \frac{3\alpha w_{ch}}{2} - \frac{\alpha \theta^2 w_{ch}}{3(\rho_t + p_t)} - \frac{\alpha w_{ch}(\rho_t + 3p_t)}{2(\rho_t + p_t)} \right) \rho_{ch} \ D_a^{ch} - a \nabla^2 \dot{u}_a
\]

\[
\dot{D}_a^m = -(1 + \omega_m) Z_a + \omega_m \theta D_a^m
\]

\[
\dot{D}_a^{ch} = -(1 + \omega_{ch}) Z_a + \omega_{ch} \theta D_a^{ch}
\]

\[
\dot{D}_a^t = \theta(1 + \omega_t) D_a^t - (1 + \omega_t) Z_a - \frac{\theta \rho_m}{\rho_t} D_a^m - \frac{\theta \rho_{ch}}{\rho_t} D_a^{ch}
\] (26)
The scalar decomposition

\[ a \tilde{\nabla} aX_b = X_{ab} = \frac{1}{3} h_{ab}X + \Sigma^{X}_{ab} + X_{[ab]} \]  

(27)

where \( \Sigma^{X}_{ab} = X_{(ab)} - 3h_{ab}X \)

We extract the scalar parts of the gradient variables as:

\[ \Delta_m = a \tilde{\nabla}^a D^m_a \]  

(28)

\[ Z = a \tilde{\nabla}^a Z_a \]  

(29)

\[ \Delta_{ch} = a \tilde{\nabla}^a D^c^h_a \]  

(30)
Perturbations

Harmonic decomposition

\[ \ddot{X} + A\dot{X} + BX = C(Y, \dot{Y}), \quad (31) \]

\[ X = \sum_k X^k Q_k(\vec{x}), \quad Y = \sum_k Y^k(t) Q_k(\vec{x}), \quad (32) \]

where \( Q_k(x) \) are the eigenfunctions of the covariantly defined spatial Laplace-Beltrami operator such that

\[ \tilde{\nabla}^2 Q = -\frac{k^2}{a^2} Q. \quad (33) \]

The order of the harmonic (wave number) is given by

\[ k = \frac{2\pi a}{\lambda}, \quad (34) \]

where \( \lambda \) is the physical wavelength of the mode.
Perturbations

Second-order evolution equations

The harmonic decomposition equations become

\[
\Delta'' + \frac{1}{1+z}\left(\frac{1}{2} - 4\omega_m\right)\Delta' + \left\{ \frac{(1 + \omega_m)}{(1 + z)^2} \left[ \frac{1}{2} (1 + 3\omega_m) \right.ight.
\]
\[
+ \frac{\omega_m}{(\Omega_{ch} + \Omega_m)(1 + \omega_t)} + \frac{\omega_m(1 + 3\omega_t)}{2(1 + \omega_t)} + \frac{9\pi^2 (1 + \omega_m)^2 \omega_m}{3\lambda^2 (1 + z)^3 (1 + \omega_m)(\Omega_{ch} + \Omega_m)} \right\} 3\Omega_m
\]
\[
+ \left[ 3\omega_m - \frac{3}{2} \omega_m(\Omega_{ch} + \Omega_m)(1 + \omega_t) \right] \Delta^k_m
\]
\[
+ \left( \frac{1 + \omega_m}{(1 + z)^2} \left[ \frac{1}{2} - \frac{3\alpha\omega_{ch}}{2} \right. \right.
\]
\[
- \frac{\alpha\omega_{ch}}{(\Omega_{ch} + \Omega_m)(1 + \omega_t)} - \frac{\alpha\omega_{ch}(1 + 3\omega_t)}{2(1 + \omega_t)} +
\]
\[
\frac{9\pi^2 (1 + \omega_m)^2 \alpha\omega_{ch}}{3\lambda^2 (1 + z)^3 (1 + \omega_m)(\Omega_{ch} + \Omega_m)} \right\} 3\Omega_{ch} \Delta^k_{ch},
\]
(35)
\[ \Delta''_h = \frac{3\sqrt{4} - 6}{\sqrt{4}(1 + z)} \left( \omega_ch(1 + \alpha) - \omega_ch + \frac{2}{3} \right) \Delta'_ch \]
\[ + \frac{(1 + \omega_ch)}{(1 + z)^2} \left[ \frac{1}{2}(1 + 3\omega_m) + \frac{\omega_m}{3(\Omega_m + \Omega_ch)(1 + \omega_t)} + \frac{\omega_m(1 + 3\omega_t)}{2(1 + \omega_t)} \right] \]
\[ + \frac{9\pi^2(1 + \omega_m)^2\omega_m}{3\lambda^2(1 + z)^3(1+\omega_m)(\Omega_ch + \Omega_m)} \left[ 3\Omega_m\Delta^k_m \right] \]
\[ + \frac{1}{(1 + z)^2} \left\{ (1 + \omega_ch) \left[ \frac{1}{2} - \frac{3\alpha\omega_ch}{2} - \frac{\alpha\omega_ch}{(\Omega_m + \Omega_ch)(1 + \omega_t)} \right] - \frac{\alpha\omega_ch(1 + 3\omega_t)}{2(1 + \omega_t)} + \frac{9\pi^2(1 + \omega_m)^2\alpha\omega_ch}{\lambda^2(A + B(1 + z)^3(1+\alpha))\frac{1}{1+\alpha}(\Omega_ch + \Omega_m)} \right\} 3\Omega_ch \]
\[ - \omega_ch \left[ 3 + \frac{3}{2}(1 + 3\omega_t)(\Omega_m + \Omega_ch) \right] + 9\omega_ch \left[ (1 + \alpha)(1 + 2\omega_ch)3\Omega_ch + \frac{2}{3} \right] \right\} \Delta^k_ch , \quad (36) \]
Here we consider two non-interacting fluids namely radiation-Chaplygin gas and dust-Chaplygin system for further analysis.
We consider the growth of the radiation fluctuations as a background of CG fluid, and $\Delta_r \gg \Delta_{ch}$, i.e., $\Delta_{ch} \approx 0$. Then solutions of our leading equation Eq. 35 or short-wavelength range becomes

$$
\Delta(z) = C_1 (1 + z)^{\frac{1}{12}} \text{BesselJ}\left(\frac{\Sigma}{24(\Omega_{ch} + \Omega_r)}, \frac{4\pi \sqrt{\Omega_r}}{3\lambda \sqrt{\Omega_{ch} + \Omega_r}(1 + z)^2}\right)
$$

$$
+ C_2 (1 + z)^{\frac{1}{12}} \text{BesselY}\left(\frac{\Sigma}{24(\Omega_{ch} + \Omega_r)}, \frac{4\pi \sqrt{\Omega_r}}{3\lambda \sqrt{\Omega_{ch} + \Omega_r}(1 + z)^2}\right), \quad (37)
$$

for long-wavelength range is given as

$$
\Delta(z) = C_1 (1 + z)^{\Omega_{ch} + \Omega_r + \psi} + C_2 (1 + z)^{-(\Omega_{ch} + \Omega_r - \psi)}, \quad (38)
$$
Matter-Chaplygin gas System

Radiation-Chaplygin dominated Universe

(a) $\delta(z)$ with $z$ for short-wavelength, for $\Omega_r = 1 - \Omega_{ch}$ and for different wavelengths.

(b) $\delta(z)$ with $z$ for long-wavelength and for $\Omega_r = 1 - \Omega_{ch}$. We use $\Omega_r = 4.48 \times 10^{-5}$

Figure
The solutions of our leading evolution equations for the density perturbations Eq.(35) dust-CG system becomes

\[ \Delta(z) = C_1(1 + z)^{\frac{3}{4} + \frac{1}{4} \sqrt{9 + 24 \Omega_d}} + C_2(1 + z)^{\frac{3}{4} - \frac{1}{4} \sqrt{9 + 24 \Omega_d}}. \]  

(39)
Dust-Chaplygin dominated Universe

Figure: $\delta(z)$ with $z$ for dust and for $\Omega_d = 1 - \Omega_{ch}$. We use $\Omega_d = 0.32$. 
In this case $\Delta_{ch} \gg \Delta_m$ and $\Delta_m \approx 0$, then the solution of our leading equation (36) read as

$$
\Delta(z) = C_1 \log(1+z) \sin(\Omega_r + \Omega_d + \Omega_{ch} + 5) + C_2 \log(1+z) \cos(\Omega_r + \Omega_d + \Omega_{ch} + 5) .
$$

(40)
Figure: $\delta(z)$ with $z$ for CG and for $1 - \Omega_r + \Omega_d = \Omega_{ch}$. 
Conclusion

The CG complies to the behaviour of dark energy and dark matter.
It explains well the background expansion history.
It has a significant role in the formation of the large-scale structure.
A detailed exploration of cosmological perturbations of growth density with red-shift by unified the dark energy and dark matter in Chaplygin gas.

The Future

More needs to be done to constrain the CG model with the cosmological observations.