

Theory of the Oscillating-Wire Measurements

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Overview on Wire Measurements

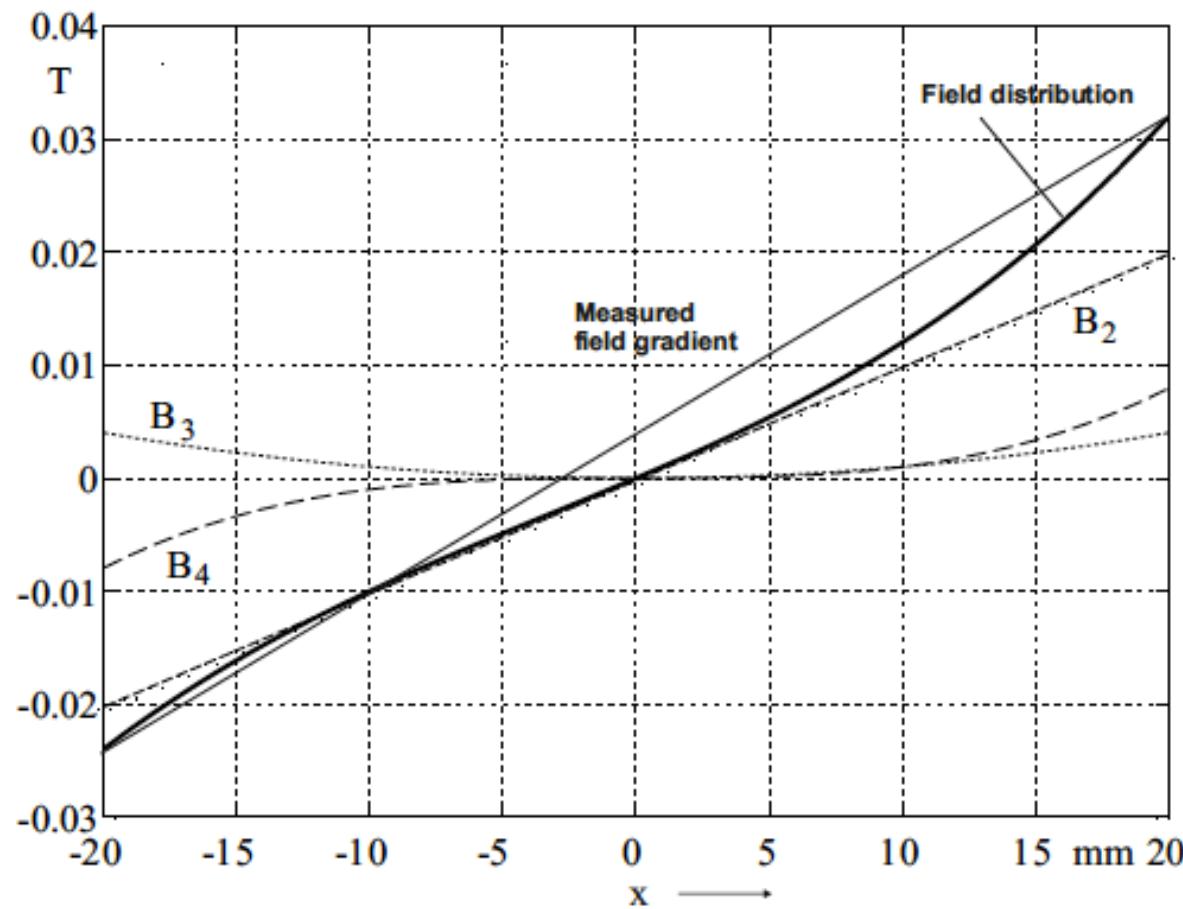
Method	Frequency	Applications	Measurand	Principle
Floating	Static	Spectrometers	Equilibrium orbit	$I/T \equiv -e/p$
SSW	Static	Quadrupoles Sextupoles	Magnetic axis and gradient Strength	Circular / horizontal scan
Vibrating	First and higher resonances	Solenoids, quadrupoles, Multipole correctors	Magnetic centers Axis alignment Longitudinal profiles	Coordinate search
Oscillating	20 - 40 Hz	Quadrupoles Multipole correctors	Circular harm. Elliptic harm.	$\mathcal{F}\{d_x\}$
Pulsed	Transient	Strings of magnets	Longitudinal mag. centers	

DiMarco 2007, Le Bec 2012, Panofski 1965, Themnykh 1997, Vogel 1964, Wolf 2005

Irvin and Caughey 1974, Lee and Perkins 1992



Measuring the Gradient (the Dilemma)



Proposal: Measure gradient with SSW method and correct the result with harmonics obtained with the **same experimental setup** by using the system in the oscillation mode (20 – 40 Hz).

Correcting the Gradient Measurements

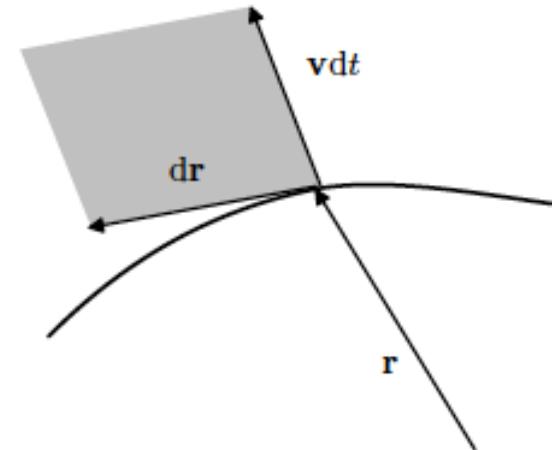
$$\int_{\partial \mathcal{A}} (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{r} = - \int_{\partial \mathcal{A}} \frac{d}{dt} (\mathbf{B} \cdot d\mathbf{a}) = - \frac{d\Phi}{dt},$$

Flux linkage when the wire is moved from z_1 to z_2 in the complex plane

$$\begin{aligned}\Phi &= L \operatorname{Re} \left\{ \int_{z_1}^{z_2} (\bar{B}_y + i\bar{B}_x) dz \right\} = L \operatorname{Re} \left\{ \int_{z_1}^{z_2} \sum_{n=1}^{\infty} (B_n + iA_n) \left(\frac{z}{r_0} \right)^{n-1} dz \right\} \\ &= L \operatorname{Re} \left\{ \sum_{n=1}^{\infty} \left[\frac{B_n + iA_n}{nr_0^{n-1}} z^n \right]_{z_1}^{z_2} \right\},\end{aligned}$$

Easy result for movement on the horizontal plane

$$\Phi = L \sum_{n=1}^{\infty} \frac{B_n}{nr_0^{n-1}} x^n.$$



Correction of the Gradient Measurements

Repeat the measurement in both directions and average

$$\Phi = L \sum_{n=1}^{\infty} \frac{B_n}{2nr_0^{n-1}} (x^n + (-x)^n).$$

Introduce the definition of the gradient

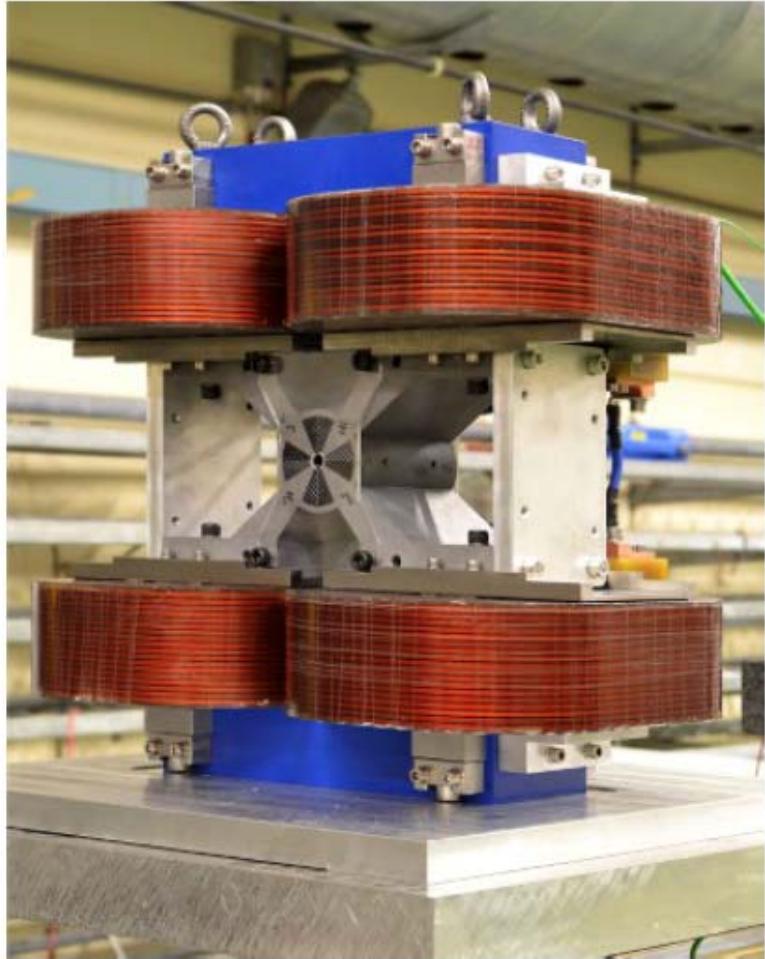
$$\Phi = L\bar{g} \frac{r_0}{B_2} \sum_{n=1}^{\infty} \frac{B_n}{2nr_0^{n-1}} (x^n + (-x)^n) = L\bar{g} \left(\frac{1}{2}x^2 + \sum_{n=4,6,\dots} \frac{b_n}{n} \frac{x^n}{r_0^{n-2}} \right).$$

Rearrange to obtain correction factor

$$\begin{aligned} \bar{g}^c &= \bar{g}^m \left(1 + \sum_{n=4,6,\dots} \frac{2b_n}{n} \left(\frac{x}{r_0} \right)^{n-2} \right)^{-1} \\ &= \bar{g}^m \left(1 + \frac{1}{2}b_4 \left(\frac{x}{r_0} \right)^2 + \frac{1}{3}b_6 \left(\frac{x}{r_0} \right)^4 + \frac{1}{4}b_8 \left(\frac{x}{r_0} \right)^6 + \dots \right)^{-1}, \end{aligned}$$



Correction of the Gradient Measurements



$$R^c = \frac{|\bar{g}^c(-x, x) + \bar{g}^c(-y, y)|}{|\bar{g}^c(-x, x)|}$$

$$R^m = \frac{|\bar{g}^m(-x, x) + \bar{g}^m(-y, y)|}{|\bar{g}^m(-x, x)|}$$

I A	$L\bar{g}^m(-x, x)$ T	$L\bar{g}^m(-y, y)$ T	R^m	$L\bar{g}^c(-x, x)$ T	$L\bar{g}^c(-y, y)$ T	R^c
0.0	9.245	-9.327	0.009	8.886	-8.907	0.002
6.2	32.686	-32.786	0.003	32.557	-32.618	0.002
7.6	39.174	-39.274	0.003	39.144	-39.177	0.001
15.5	51.358	-51.462	0.002	51.504	-51.430	0.001
7.6	39.523	-39.618	0.002	39.498	-39.530	0.001
6.2	33.077	-33.164	0.003	32.959	-33.002	0.001
0.0	9.238	-9.307	0.007	8.880	-8.883	0.000

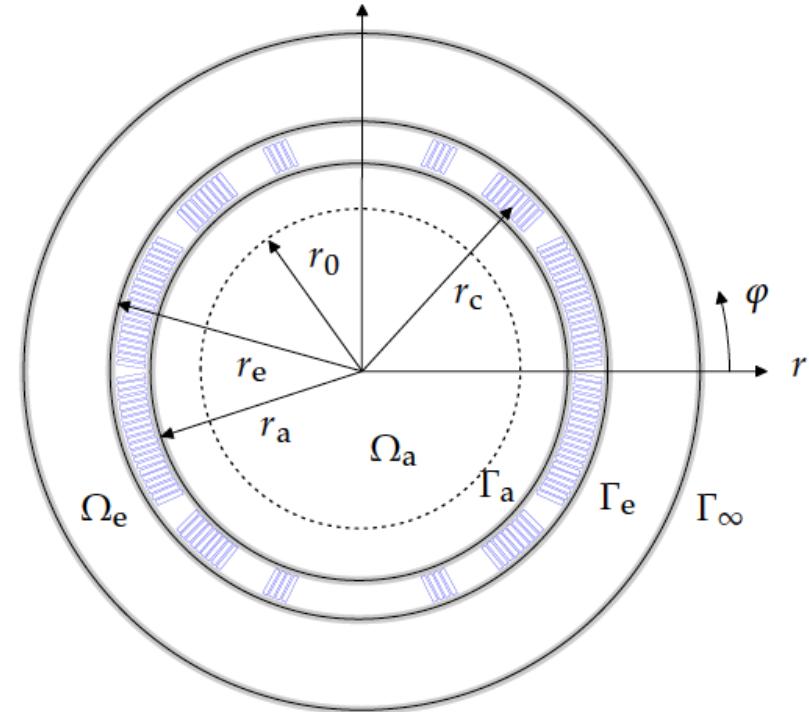
Solving of Boundary Value Problems

1. Governing equation in the air domain

$$\nabla^2 A_z = 0,$$

2. Choose a suitable coordinate system

$$r^2 \frac{\partial^2 A_z}{\partial r^2} + r \frac{\partial A_z}{\partial r} + \frac{\partial^2 A_z}{\partial \varphi^2} = 0,$$



3. Make a guess, look it up in a book, use the method of separation, that is, find **eigenfunctions**. Coefficients are not known at this stage

$$A_z(r, \varphi) = \sum_{n=1}^{\infty} (\mathcal{E}_n r^n + \mathcal{F}_n r^{-n})(\mathcal{G}_n \sin n\varphi + \mathcal{H}_n \cos n\varphi).$$



Solving of Boundary Value Problems

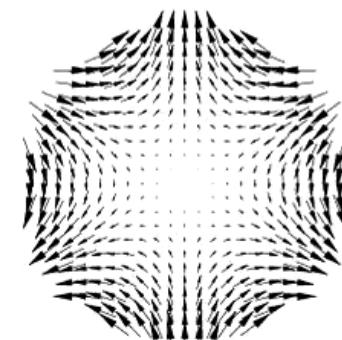
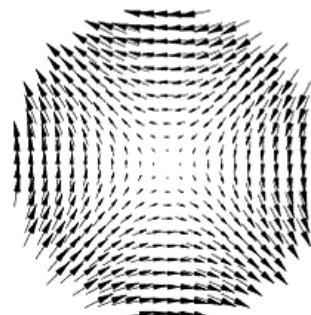
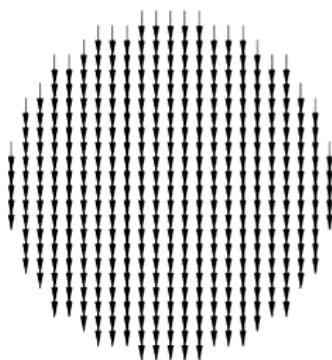
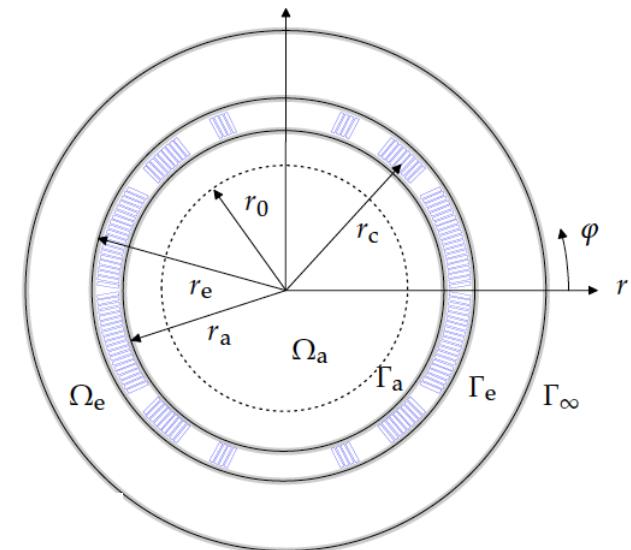
4. Incorporate a bit of knowledge and rename

$$A_z(r, \varphi) = \sum_{n=1}^{\infty} r^n (\mathcal{A}_n \sin n\varphi + \mathcal{B}_n \cos n\varphi).$$

2. Calculate a field components

$$B_r(r, \varphi) = \frac{1}{r} \frac{\partial A_z}{\partial \varphi} = \sum_{n=1}^{\infty} nr^{n-1} (\mathcal{A}_n \cos n\varphi - \mathcal{B}_n \sin n\varphi),$$

$$B_\varphi(r, \varphi) = -\frac{\partial A_z}{\partial r} = -\sum_{n=1}^{\infty} nr^{n-1} (\mathcal{A}_n \sin n\varphi + \mathcal{B}_n \cos n\varphi),$$



Solving of Boundary Value Problems

$$B_r(r, \varphi) = \frac{1}{r} \frac{\partial A_z}{\partial \varphi} = \sum_{n=1}^{\infty} n r^{n-1} (\mathcal{A}_n \cos n\varphi - \mathcal{B}_n \sin n\varphi),$$

3. Measure or calculate the field on a reference radius and perform a discrete Fourier analysis (develop into the eigenfunctions).
Coefficients are known here.

$$B_r(r_0, \varphi) = \sum_{n=1}^{\infty} (B_n(r_0) \sin n\varphi + A_n(r_0) \cos n\varphi),$$

$$A_n(r_0) \approx \frac{2}{N} \sum_{k=0}^{N-1} B_r(r_0, \varphi_k) \cos n\varphi_k, \quad \varphi_k = \frac{2\pi k}{N}, \quad k = 0, 1, 2, \dots, N-1.$$
$$B_n(r_0) \approx \frac{2}{N} \sum_{k=0}^{N-1} B_r(r_0, \varphi_k) \sin n\varphi_k.$$



Solving the Boundary Value Problem

4: Compare the known and unknown coefficients

$$B_r(r, \varphi) = \frac{1}{r} \frac{\partial A_z}{\partial \varphi} = \sum_{n=1}^{\infty} n r^{n-1} (\mathcal{A}_n \cos n\varphi - \mathcal{B}_n \sin n\varphi),$$

$$B_r(r_0, \varphi) = \sum_{n=1}^{\infty} (B_n(r_0) \sin n\varphi + A_n(r_0) \cos n\varphi),$$

$$\mathcal{A}_n = \frac{1}{n r_0^{n-1}} A_n(r_0), \quad \mathcal{B}_n = \frac{-1}{n r_0^{n-1}} B_n(r_0).$$

5. Put this into the original solution for the entire air domain

$$A_z(r, \varphi) = - \sum_{n=1}^{\infty} \frac{r_0}{n} \left(\frac{r}{r_0} \right)^n (B_n(r_0) \cos n\varphi - A_n(r_0) \sin n\varphi).$$



Solving the Boundary Value Problem

6: Calculate fields and potential in the entire air domain

$$B_r(r, \varphi) = \sum_{n=1}^{\infty} \left(\frac{r}{r_0} \right)^{n-1} (B_n(r_0) \sin n\varphi + A_n(r_0) \cos n\varphi)$$

$$B_\varphi(r, \varphi) = \sum_{n=1}^{\infty} \left(\frac{r}{r_0} \right)^{n-1} (B_n(r_0) \cos n\varphi - A_n(r_0) \sin n\varphi)$$

7: Remember de Moivre

$$B_x(r, \varphi) = \sum_{n=1}^{\infty} \left(\frac{r}{r_0} \right)^{n-1} (B_n(r_0) \sin((n-1)\varphi) + A_n(r_0) \cos((n-1)\varphi))$$

$$B_y(r, \varphi) = \sum_{n=1}^{\infty} \left(\frac{r}{r_0} \right)^{n-1} (B_n(r_0) \cos((n-1)\varphi) - A_n(r_0) \sin((n-1)\varphi))$$



Solving the Boundary Value Problem

Finally: Take any 2π periodic function and develop according to

$$\frac{C_0}{2} + \sum_{n=1}^{\infty} (C_n(r_0) \sin n\varphi + D_n(r_0) \cos n\varphi).$$

	B_r	B_φ	B_x	B_y	A_z	ϕ_m
$B_n =$	C_n	D_n	C_{n-1}	D_{n-1}	$\frac{-nD_n}{r_0}$	$\frac{-n\mu_0 C_n}{r_0}$
$A_n =$	D_n	$-C_n$	D_{n-1}	$-C_{n-1}$	$\frac{nC_n}{r_0}$	$\frac{-n\mu_0 D_n}{r_0}$

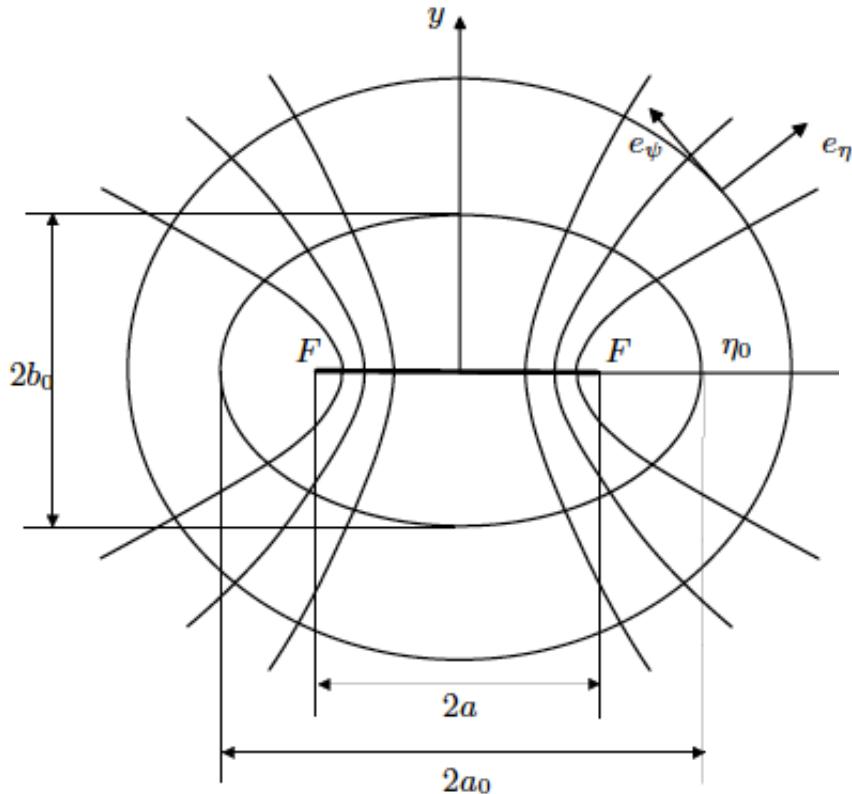
The operators div, grad, curl of vector **analysis** are isomorphic to **algebraic** operators in the L2 space of our field harmonics

For the oscillating wire technique: Use the **oscillating amplitudes** measured at **one position** longitudinally (to be questioned later) as we move the wire such that they become the generators of a cylinder inside the magnet aperture



In Principle also for the Elliptic Coordinate System

$$B_\eta(\eta, \psi) = \frac{1}{h_2} \sum_{n=1}^{\infty} (n A_n \sinh n\eta \cos n\psi - n B_n \cosh n\eta \sin n\psi) .$$



$$h_1 = h_2 = a \sqrt{\cosh^2 \eta - \cos^2 \psi}.$$

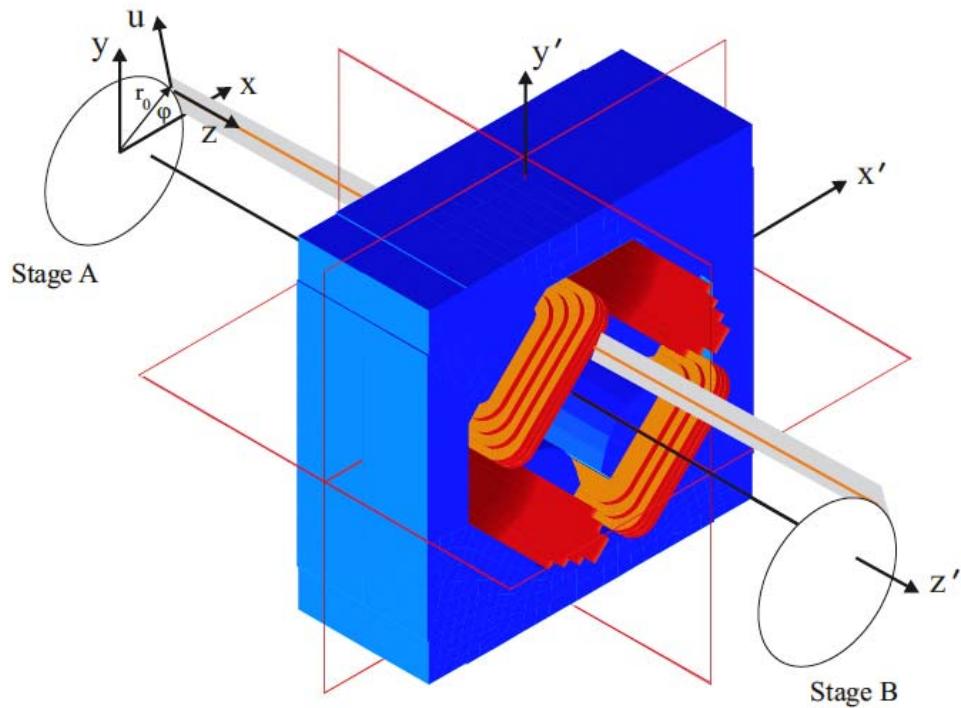
$$B_\eta(\eta_0, \psi) = \sum_{n=1}^{\infty} (B_n(\eta_0) \sin n\psi + A_n(\eta_0) \cos n\psi),$$

$$B_\eta = \frac{1}{h_1} (a \sinh \eta \cos \psi B_x + a \cosh \eta \sin \psi B_y) .$$

Solution: Develop the metric coefficient numerically (Schnizer 2009) or work with the covariant derivative, i.e, differential forms (Auchmann, Kurz, Russenschuck 2011)



Use Wire Displacements



Proposition: We are done if:

$$d_y^k(r_0) = \lambda_y \int_0^L B_x(r_0, \varphi_k) dz$$

$$d_x^k(r_0) = \lambda_x \int_0^L B_y(r_0, \varphi_k) dz.$$

$$\tilde{A}_n(r_0) = \frac{2}{K} \sum_{k=0}^{K-1} d_y^k(r_0) \cos n\varphi_k,$$

$$\tilde{B}_n(r_0) = \frac{2}{K} \sum_{k=0}^{K-1} d_y^k(r_0) \sin n\varphi_k.$$

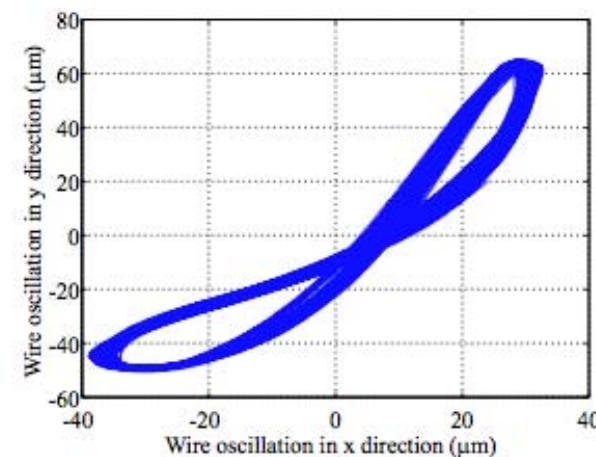
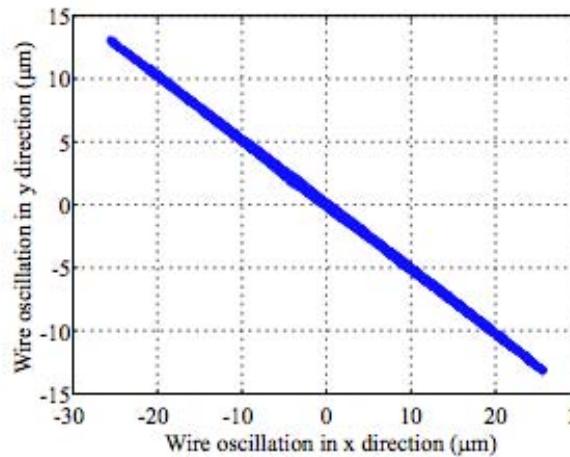
$$a_{n+1}(r_0) = \frac{\tilde{A}_n(r_0)}{\tilde{B}_N(r_0)},$$

$$b_{n+1}(r_0) = \frac{\tilde{B}_n(r_0)}{\tilde{B}_N(r_0)}.$$

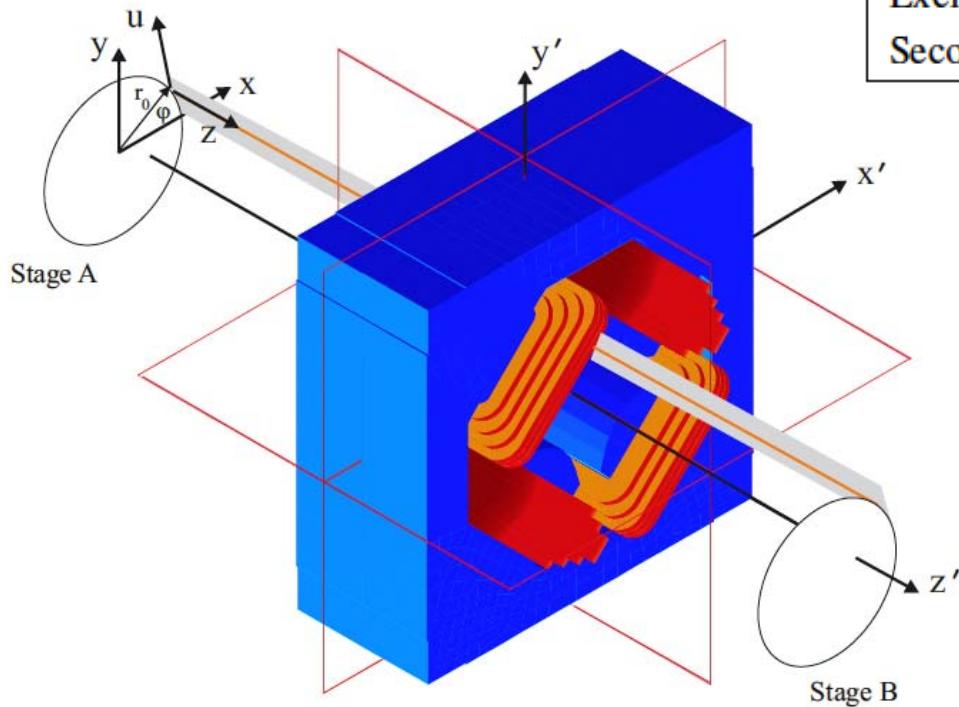


Solution of the Wave Equation (Assumptions)

- Uniformity: The string has a constant mass density ρ .
- Planar oscillations: The string is suspended at the origin $z = 0$ and $z = L$. The string deflection $u(z, t)$ is caused by the distributed force, which is proportional to the normal field to this plane: $f(z, t) = I(t)B_n(z)$.
- Uniform tension: Each segment of the string pulls on its neighboring segments with the same magnitude of force T .
- The only force in the wire is its tension and the Lorentz force. No gravitational, frictional, or other external forces (wind) are considered.
- Small vibrations: The slope $du(z, t)/dz$ remains small in the interval $[0, L]$.
- Steady state oscillations: After an initial setting time, the string oscillates in the form of a standing wave. Then there will be no energy flow along the string and no energy loss in the fixed suspensions.



Use Wire Displacements



Mass density of wire	ρ	$7 \cdot 10^{-5} \text{ kg m}^{-1}$
Damping coefficient	α	$6.44 \cdot 10^{-4} \text{ kg m}^{-1} \text{ s}^{-1}$
Wire tension	T	9 N
Excitation current	I_0	1 mA
Second resonance frequency for $L = 1 \text{ m}$	f_2	179.3 Hz

$$\rho \frac{\partial^2 u}{\partial t^2} + \alpha \frac{\partial u}{\partial t} - T \frac{\partial^2 u}{\partial z^2} = -I(t)B_n(z),$$

$$T \left(\frac{\partial u}{\partial z} \right) \Big|_{z+dz} - T \left(\frac{\partial u}{\partial z} \right) \Big|_z - \rho g \mathbf{e}_y \cdot \mathbf{e}_u dz + I(t)B_n(z)dz - \alpha \frac{\partial u}{\partial t} dz - \rho \frac{\partial^2 u}{\partial t^2} dz = 0.$$



Solution of the Wave Equation (Steady State)

$$u(z, t) = \frac{2I_0}{L} \sum_m \frac{\int_0^L B_n(z) \sin\left(\frac{m\pi}{L}z\right) dz}{\sqrt{\left[T\left(\frac{m\pi}{L}\right)^2 - \rho\omega^2\right]^2 + (\alpha\omega)^2}} \sin\left(\frac{m\pi}{L}z\right) \sin(\omega t - \varphi_m),$$

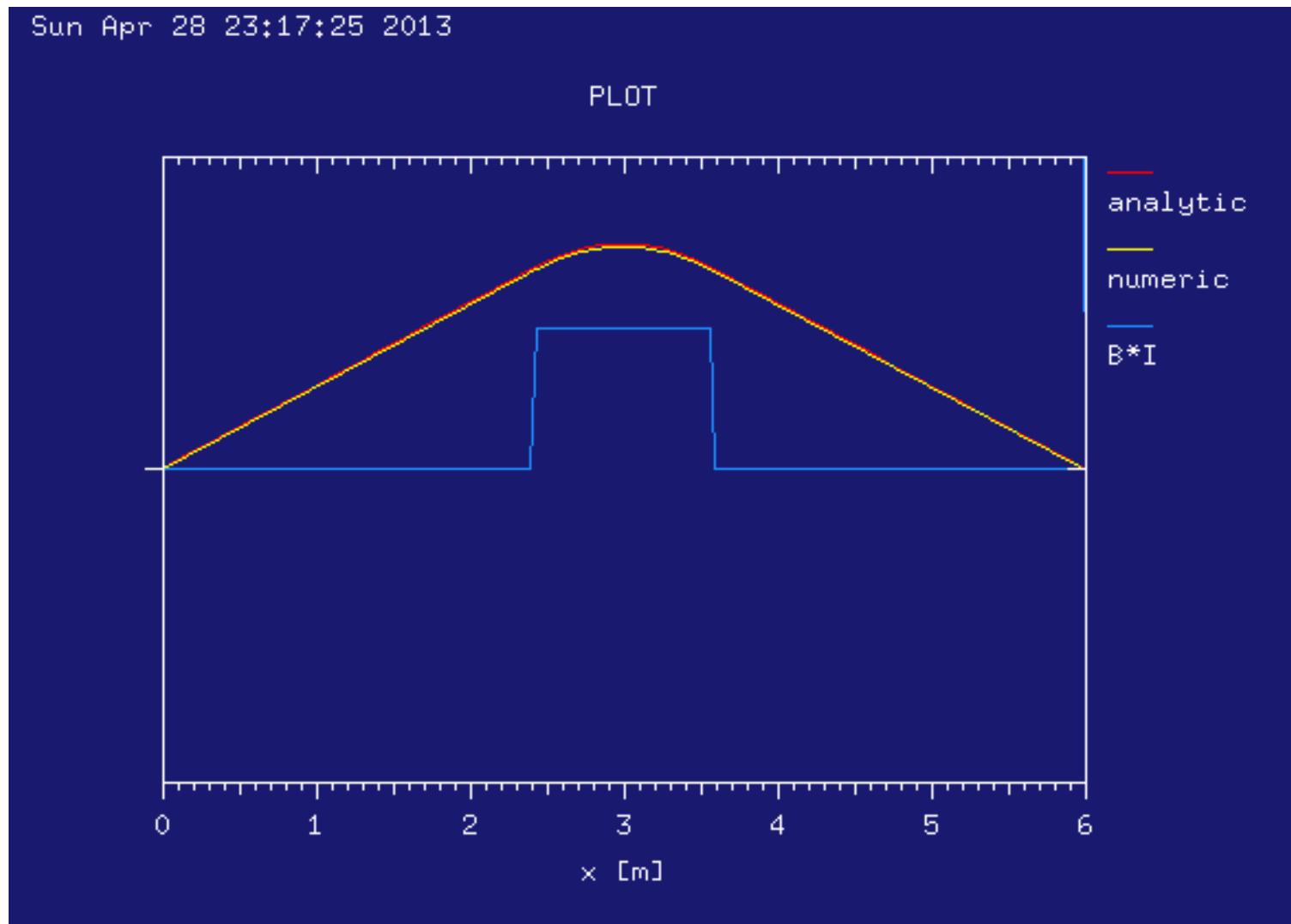
$$\varphi_m = \arctan\left(\frac{\alpha\omega}{-\rho\omega^2 + T\left(\frac{m\pi}{L}\right)^2}\right).$$

$$F_m := \int_0^L I_0 B_n(z) \sin\left(\frac{m\pi}{L}z\right) dz, \quad Y_m(z) := \sin\left(\frac{m\pi}{L}z\right) \quad \omega_m = 2\pi f_m = \frac{m\pi}{L} \sqrt{\frac{T}{\rho}}.$$

$$u(z, t) = \frac{2}{L} \sum_m \frac{F_m Y_m(z) q_m(t)}{\sqrt{[\rho(\omega_m^2 - \omega^2)]^2 + (\alpha\omega)^2}}.$$



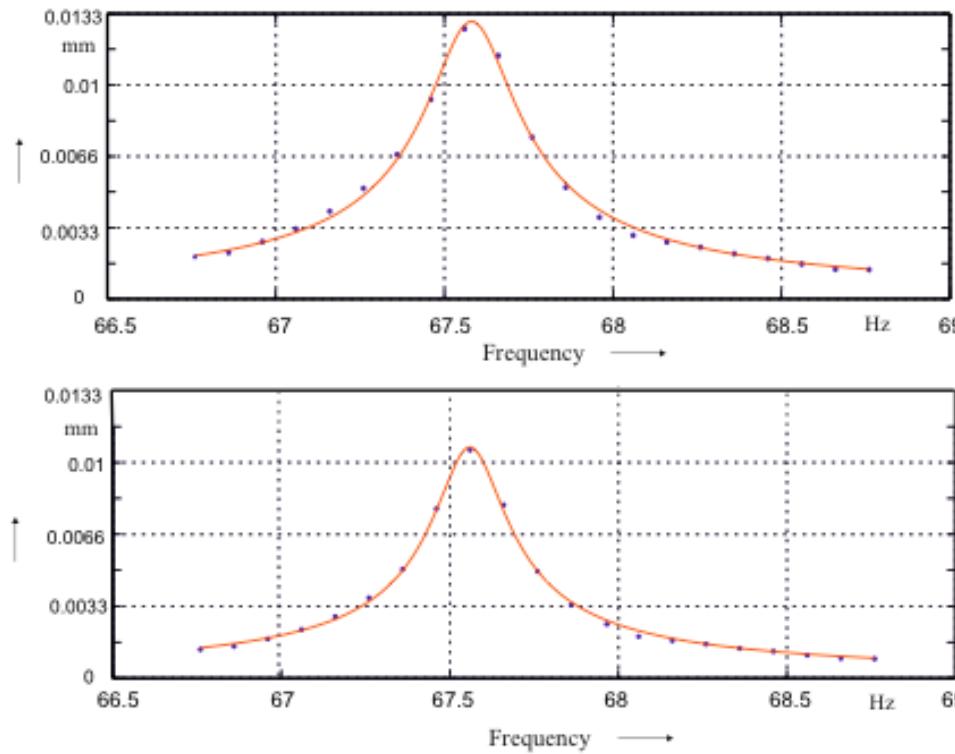
Check 2: Numerical simulation (FDTD) and the Steady State Solution



Solution of the Wave Equation II

Check 1: Behavior around the first natural resonance: Are the fit parameters physically meaningful?

$$f(\omega) = \frac{a}{\sqrt{[(b^2 - \omega^2)]^2 + (c\omega)^2}}.$$



Solution of the Wave Equation III

Check 2: Known longitudinal field or oscillation profiles

$$B_n(z) = \sum_n C_n \sin\left(\frac{n\pi}{L}z\right),$$

$$C_n = \frac{1}{L} \int_0^L B_n(z) \sin\left(\frac{n\pi}{L}z\right) dz.$$

$$u(z) \sim \sum_m \frac{C_m}{\sqrt{[\rho(\omega_m^2 - \omega^2)]^2 + (\alpha\omega)^2}} \sin\left(\frac{m\pi}{L}z\right).$$

Ideas welcome on how to measure the longitudinal profile of the oscillation wire (30 phototransistors?)



Solution of the Wave Equation IV

The slackline

$$u(z) \sim \frac{1}{\sqrt{(\rho\omega^2)^2 + (\alpha\omega)^2}} \sum_m C_m \sin\left(\frac{m\pi}{L}z\right) = \frac{1}{\sqrt{(\rho\omega^2)^2 + (\alpha\omega)^2}} B_n(z).$$

The hard-edge (model) magnet

$$\begin{aligned} B_0 \int_{\frac{L}{2}-\varepsilon}^{\frac{L}{2}+\varepsilon} \sin\left(\frac{m\pi}{L}z\right) dz &= B_0 \left[-\frac{L}{m\pi} \cos\left(\frac{m\pi}{L}z\right) \right]_{\frac{L}{2}-\varepsilon}^{\frac{L}{2}+\varepsilon} \\ &= -\frac{B_0 L}{m\pi} \left[\cos\left(\frac{m\pi}{L}\left(\frac{L}{2} + \varepsilon\right)\right) - \cos\left(\frac{m\pi}{L}\left(\frac{L}{2} - \varepsilon\right)\right) \right] \\ &= -\frac{B_0 L}{m\pi} \left[-2 \sin\left(\frac{m\pi}{2}\right) \sin(\varepsilon) \right], \end{aligned}$$



Test Cases

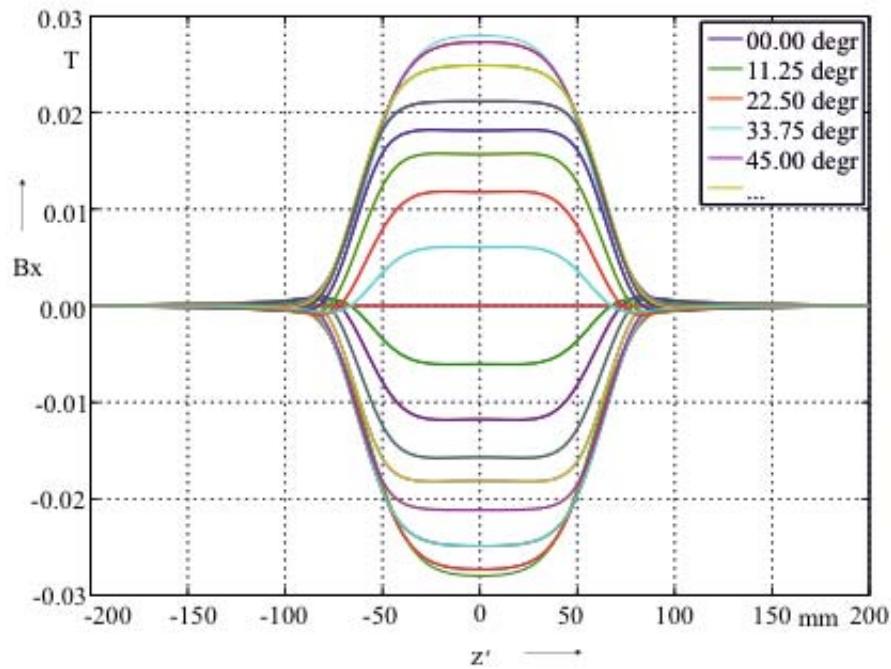


Air coil: Academic worst case

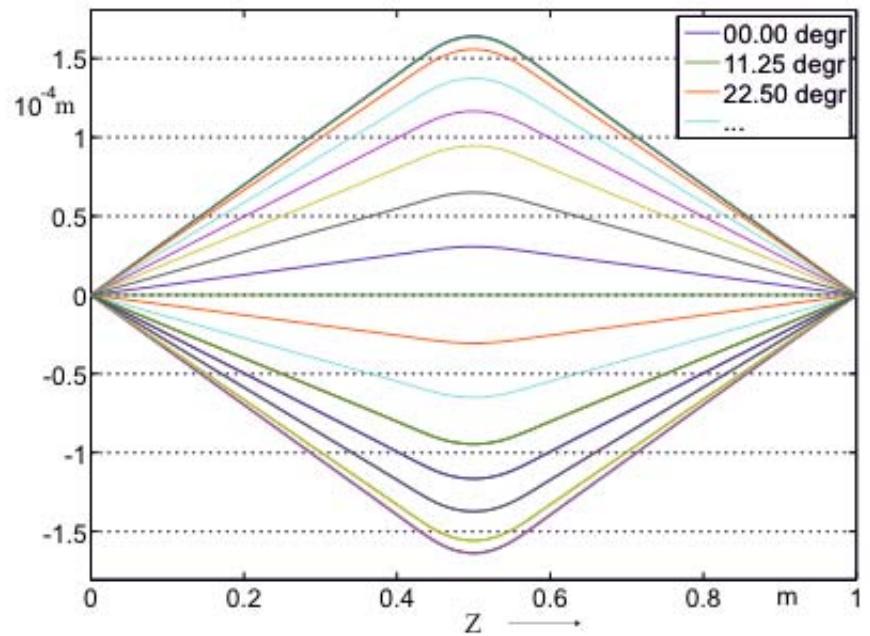


LEP-IL-QS
The “blue” magnet

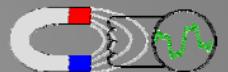
The Air Coil



Longitudinal field distribution

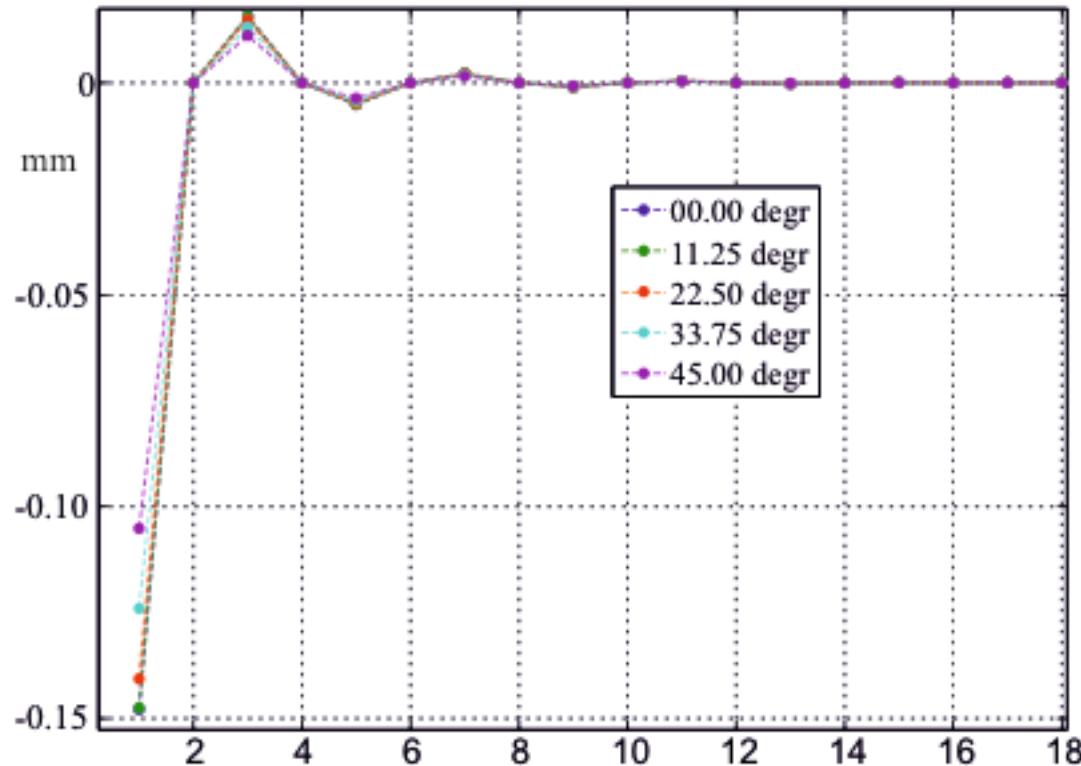


Longitudinal shape of the wire oscillation

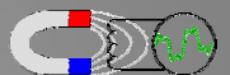
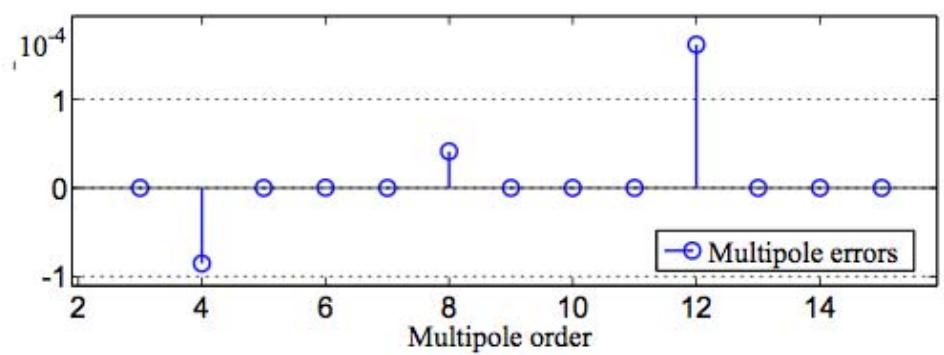
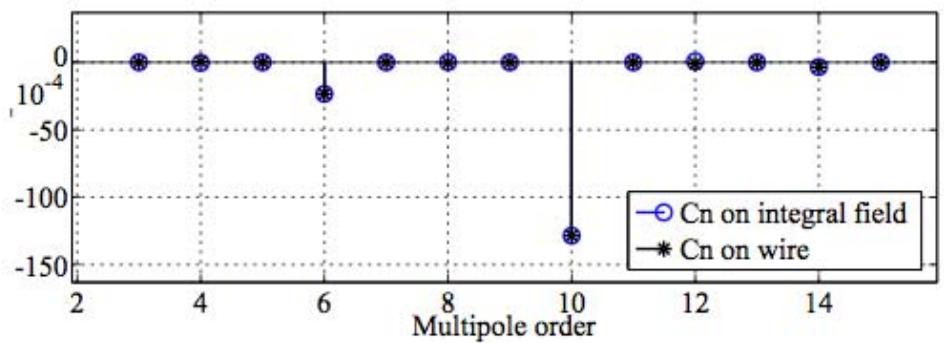
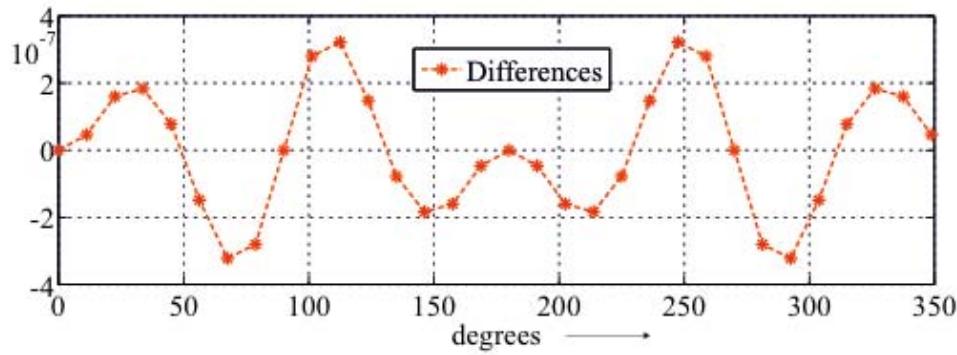
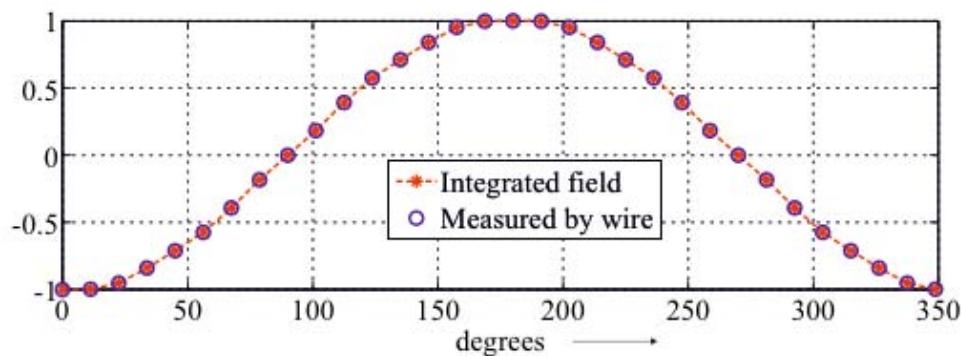


Convergence of the Modal Amplitude Functions

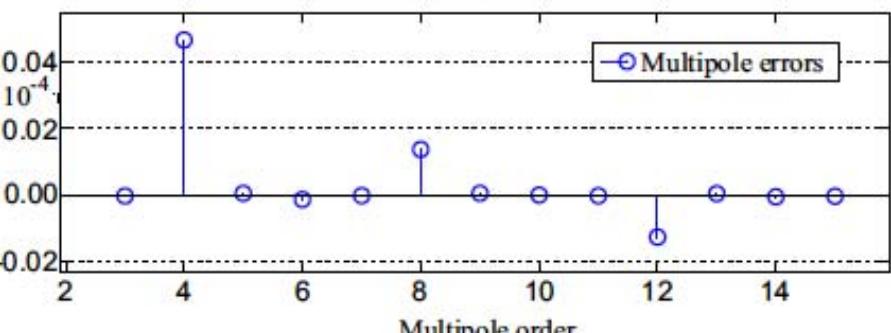
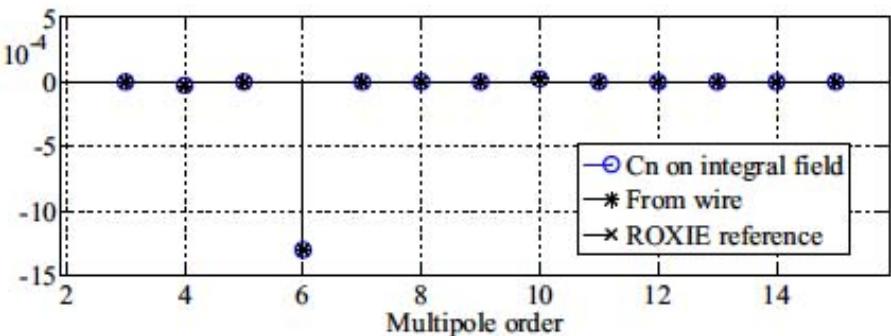
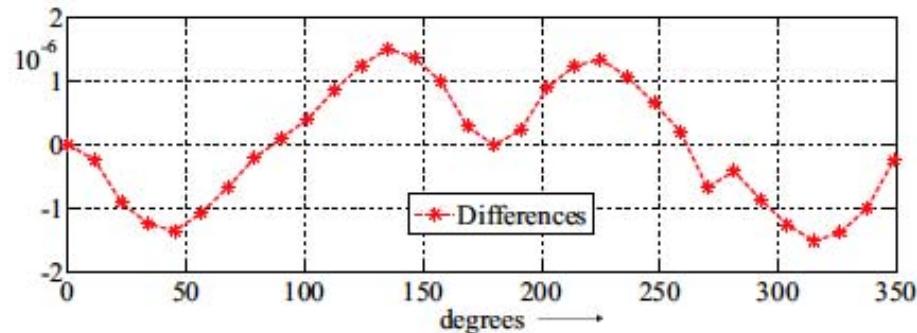
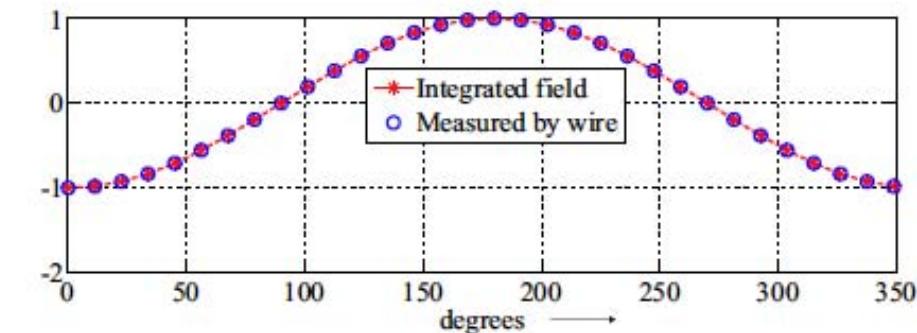
$$\mathcal{U}_m := \frac{2I_0}{L} \frac{\int_0^L B_x(z) \sin\left(\frac{m\pi}{L}z\right) dz}{\sqrt{[\rho(\omega_m^2 - \omega^2)]^2 + (\alpha\omega)^2}}$$



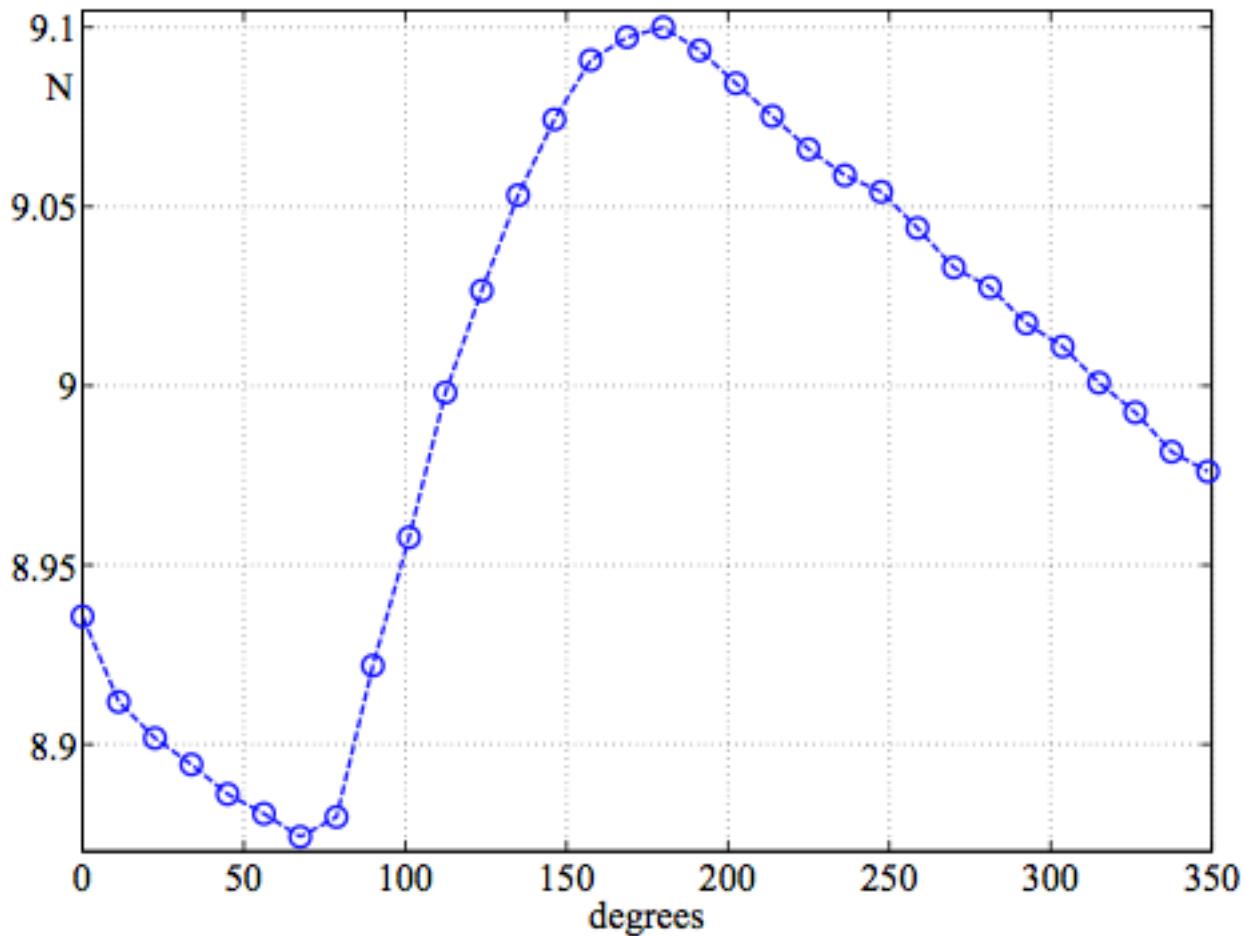
Results for the Air Coil



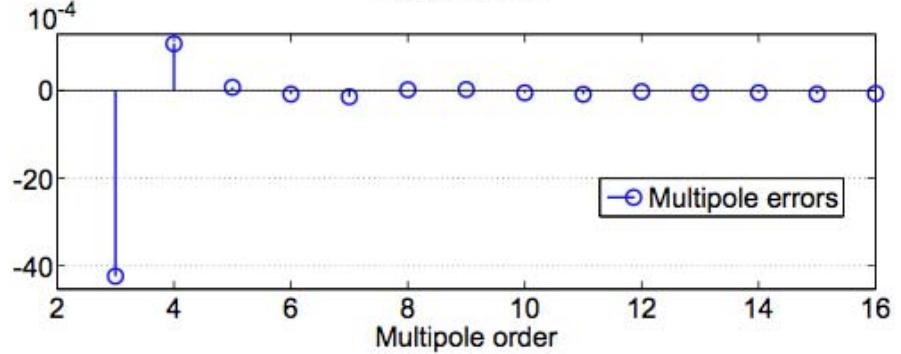
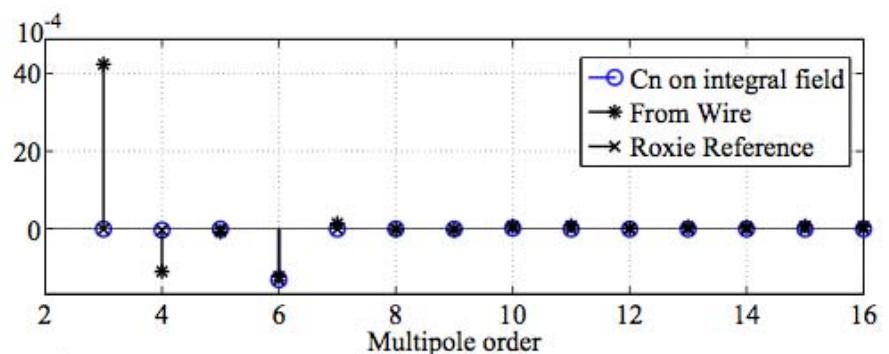
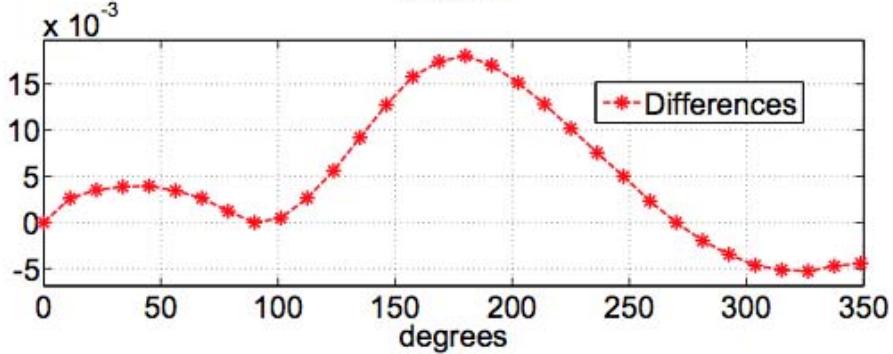
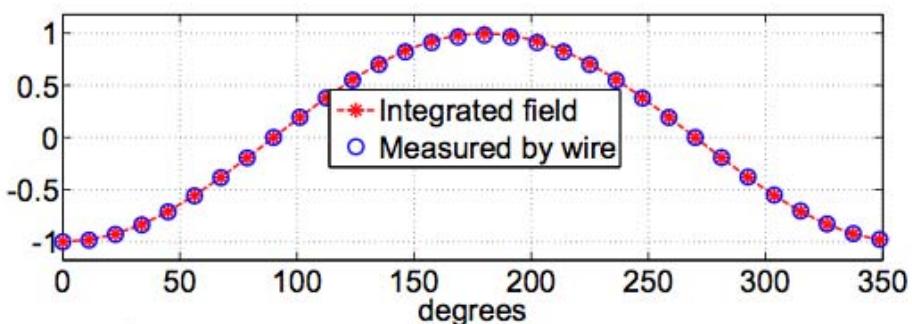
Results for the Blue Magnet



Variation of Wire Tension due to Stage Misalignment



Results of the Measurements with Stage Misalignment



Conclusion

- Because we measure the oscillation amplitude only at one point, we make an intrinsic error caused by the varying end fields as we move along the circular trajectory
 - The method is exact for the hard-edge (model) magnet and consequently for small-aperture magnets excited by rare-earth material
 - The intrinsic error can be estimated when the numerical model is available
 - Effects from stage misalignment are much larger than the intrinsic error
 - We would be exact if it was possible to measure the shape of the wire oscillation
 - Using the vibrating wire technique for extracting the modes of the longitudinal field profile is not sufficiently accurate
 - In particular it would fail for the hard-edge model magnet (Gibbs phenomenon, that is, no local convergence of the Fourier series for non-smooth functions)

