Higher-Loop Calculations of the UV to IR Evolution of Gauge Theories with Infrared Fixed Points

Robert Shrock

YITP, Stony Brook University

BNL Forum, 5/2/2013

Outline

- Renormalization-group flow from UV to IR; types of IR behavior; role of an exact or approximate IR fixed point

- Higher-loop calculations of UV to IR evolution, including IR zero of β and anomalous dimension γ_m of fermion bilinear

- Some comparisons with lattice measurements of γ_m

- Higher-loop calcs. of UV to IR evolution for supersymmetric gauge theory

- Study of scheme-dependence in calculation of IR fixed point

- Conclusions

This talk contains material from

- T. A. Ryttov and R. Shrock, "Higher-Loop Corrections to the Infrared Evolution of a Gauge Theory with Fermions", Phys. Rev. D 83, 056011 (2011), arXiv:1011.4542

- T. A. Ryttov and R. Shrock, "Comparison of Some Exact and Perturbative Results for a Supersymmetric SU(N_c) Gauge Theory", Phys. Rev. D 85, 076009 (2012), arXiv:1202.1297

- T. A. Ryttov and R. Shrock, "Scheme Transformations in the Vicinity of an Infrared Fixed Point", Phys. Rev. D 86, 065032 (2012), arXiv:1206.2366

- T. A. Ryttov and R. Shrock, "An Analysis of Scheme Transformations in the Vicinity of an Infrared Fixed Point", Phys. Rev. D 86, 085005 (2012), arXiv:1206.6895

- R. Shrock, "Higher-Loop Structural Properties of the β Function in Asymptotically Free Vectorial Gauge Theories", Phys. Rev. D, in press, arXiv:1301.3209

- R. Shrock, "Higher-Loop Calculations of the Ultraviolet to Infrared Evolution of a Vectorial Gauge Theory in the Limit $N_c \to \infty$, $N_f \to \infty$ with N_f/N_c Fixed", arXiv:1302.5434

Renormalization-group Flow from UV to IR; Types of IR Behavior and Role of IR Fixed Point

Consider an asymptotically free, vectorial gauge theory with gauge group G and N_f massless fermions in representation R of G.

Asymptotic freedom $\Rightarrow$ theory is weakly coupled, properties are perturbatively calculable for large Euclidean momentum scale μ in deep ultraviolet (UV).

The question of how this theory flows from large μ in the UV to small μ in the infrared (IR) is of fundamental field-theoretic significance and motivates a detailed study of this evolution.

For some fermion contents, the theory may have an exact or approximate IR fixed point (zero of β), with a number of interesting consequences.

Denote running gauge coupling at scale μ as $g = g(\mu)$, and let $\alpha(\mu) = g(\mu)^2/(4\pi)$ and $a(\mu) = g(\mu)^2/(16\pi^2) = \alpha(\mu)/(4\pi)$.

The dependence of $\alpha(\mu)$ on μ is described by the β function

$$\beta_\alpha \equiv \frac{d\alpha}{dt} = -2\alpha \sum_{\ell=1}^{\infty} b_\ell \, a^\ell = -2\alpha \sum_{\ell=1}^{\infty} \bar{b}_\ell \, \alpha^\ell \ ,$$

where $t = \ln \mu$, $\ell =$ loop order of the coeff. b_ℓ, and $\bar{b}_\ell = b_\ell/(4\pi)^\ell$.

Coeffs. b_1 and b_2 in β are indep. of regularization/renormalization scheme, while b_ℓ for $\ell \geq 3$ are scheme-dep.

Asymptotic freedom means $b_1 > 0$, so $\beta < 0$ for small $\alpha(\mu)$, in neighborhood of UV fixed point (UVFP) at $\alpha = 0$.

As the scale μ decreases from large values, $\alpha(\mu)$ increases. Denote α_{cr} as minimum value for formation of bilinear fermion condensates and resultant spontaneous chiral symmetry breaking (SχSB).

Two generic possibilities for β and resultant UV to IR flow:

- β has no IR zero, so as μ decreases, $\alpha(\mu)$ increases, eventually beyond the perturbatively calculable region. This is the case for QCD.
- β has a IR zero, α_{IR}, so as μ decreases, $\alpha \longrightarrow \alpha_{IR}$. In this class of theories, there are two further generic possibilities: $\alpha_{IR} < \alpha_{cr}$ or $\alpha_{IR} > \alpha_{cr}$.

If $\alpha_{IR} < \alpha_{cr}$, the zero of β at α_{IR} is an exact IR fixed point (IRFP) of the renorm. group (RG); as $\mu \longrightarrow 0$ and $\alpha \longrightarrow \alpha_{IR}$, $\beta \longrightarrow \beta(\alpha_{IR}) = 0$, and the theory becomes exactly scale-invariant with nontrivial anomalous dimensions (Caswell, Banks-Zaks).

If β has no IR zero, or an IR zero at $\alpha_{IR} > \alpha_{cr}$, then as μ decreases through a scale Λ, $\alpha(\mu)$ exceeds α_{cr} and SχSB occurs, so fermions gain dynamical masses $\sim \Lambda$.

If SχSB occurs, then in low-energy effective field theory applicable for $\mu < \Lambda$, one integrates these fermions out, and β function becomes that of a pure gauge theory, which has no IR zero. Hence, if β has a zero at $\alpha_{IR} > \alpha_{cr}$, this is only an approx. IRFP of RG.

If α_{IR} is only slightly greater than α_{cr}, then, as $\alpha(\mu)$ approaches α_{IR}, since $\beta = d\alpha/dt \to 0$, $\alpha(\mu)$ varies very slowly as a function of the scale μ, i.e., there is approximately scale-invariant, i.e. dilatation-invariant behavior.

Let $\Lambda_* =$ scale where $\alpha(\mu)$ grows to a value only slightly less than α_{cr}. The approx. scale-invariant behavior occurs for $\Lambda < \mu < \Lambda_*$.

SχSB at Λ also breaks the approx. dilatation symmetry, plausibly leading to a resultant approx. NGB, the dilaton. This is not massless, since $\beta(\alpha_{cr})$ is small but nonzero.

Denote the n-loop β fn. as $\beta_{n\ell}$ and the IR zero of $\beta_{n\ell}$ as $\alpha_{IR,n\ell}$.

At the $n = 2$ loop level,

$$\alpha_{IR,2\ell} = -\frac{4\pi b_1}{b_2}$$

which is physical for $b_2 < 0$. One-loop coefficient b_1 is

$$b_1 = \frac{1}{3}(11C_A - 4N_f T_f)$$

(Gross, Wilczek, Politzer, 't Hooft)

where $C_A \equiv C_2(G)$ is quadratic Casimir invariant, $T_f \equiv T(R)$ is trace invariant. Focus here on $G = \mathrm{SU}(N)$.

Asymp. freedom requires $N_f < N_{f,b1z}$, where

$$N_{f,b1z} = \frac{11C_A}{4T_f}$$

e.g., for $R =$ fund. $N_f < (11/2)N$.

Two-loop coeff. b_2 is (with $C_f \equiv C_2(R)$)

$$b_2 = \frac{1}{3}\left[34C_A^2 - 4(5C_A + 3C_f)N_f\,T_f\right]$$

(Caswell, Jones)

For small N_f, $b_2 > 0$; b_2 decreases as fn. of N_f and vanishes with sign reversal at $N_f = N_{f,b2z}$, where

$$N_{f,b2z} = \frac{34C_A^2}{4T_f(5C_A + 3C_f)} .$$

For arbitrary G and R, $N_{f,b2z} < N_{f,b1z}$, so there is always an interval in N_f for which β has an IR zero, namely

$$I: \quad N_{f,b2z} < N_f < N_{f,b1z}$$

For $R =$ fund.

$$I: \quad \frac{34N^3}{13N^2 - 3} < N_f < \frac{11N}{2}$$

e.g., for SU(2), I is $5.55 < N_f < 11$; for SU(3), I is $8.05 < N_f < 16.5$. As $N \to \infty$, I is $2.62N < N_f < 5.5N$. (Here and below, evaluate expressions in $\mathbb{R}$, but understand that physical values of N_f are nonnegative integers.)

As N_f decreases from the upper to lower end of interval I, α_{IR} increases. Denote

$$N_f = N_{f,cr} \quad \text{at} \quad \alpha_{IR} = \alpha_{cr}$$

Value of $N_{f,cr}$ is of fundamental importance, since it separates the (zero-temp.) chirally symmetric and broken IR phases.

Longstanding effort to determine $N_{f,cr}$ for various N and R, using both continuum and lattice methods.

Approx. soln. of Schwinger-Dyson eq. for fermion propagator suggested $N_{f,cr} \sim 4N$. Given the strong-coupling involved, this is only rough estimate.

Higher-Loop Corrections to UV $\to$ IR Evolution of Gauge Theories

Because of this strong-coupling physics, one should calculate the IR zero in β, α_{IR}, and resultant value of γ evaluated at α_{IR} to higher-loop order. We have done this to 3-loop and 4-loop order in Ryttov and Shrock, PRD83, 056011 (2011) (related work by Gardi, Grunberg, Karliner, Pica, Sannino)

Although coeffs. in β at $\ell \geq 3$ loop order are scheme-dependent, results give a measure of accuracy of the 2-loop calc. of the IR zero of β, and similarly with γ evaluated at this IR zero.

We use $\overline{MS}$ scheme, for which β and γ have been calculated to 4-loops by Vermaseren, Larin, and van Ritbergen.

The value of higher-loop calculations has been amply shown in comparison of QCD predictions with experimental data, e.g., in $\overline{MS}$ scheme.

At 3-loop level, we use

$$b_3 = \frac{2857}{54}C_A^3 + T_f N_f \left[2C_f^2 - \frac{205}{9}C_A C_f - \frac{1415}{27}C_A^2\right] + (T_f N_f)^2 \left[\frac{44}{9}C_f + \frac{158}{27}C_A\right]$$

We find that $b_3 < 0$ for $N_f \in I$.

At this 3-loop level,

$$\beta = -\frac{\alpha^2}{2\pi}(b_1 + b_2 a + b_3 a^2)$$

so $\beta = 0$ away from $\alpha = 0$ at two values,

$$\alpha = \frac{2\pi}{b_3}\Big(- b_2 \pm \sqrt{b_2^2 - 4b_1 b_3}\ \Big)$$

Since $b_2 < 0$ and $b_3 < 0$, these are

$$\alpha = \frac{2\pi}{|b_3|}\Big(- |b_2| \mp \sqrt{b_2^2 + 4b_1|b_3|}\ \Big)$$

Soln. with $-$ sqrt is negative, hence unphysical; soln. with $+$ sqrt is $\alpha_{IR,3\ell}$.

N.B.: if a scheme had $b_3 > 0$ in I, then, since $b_2 \longrightarrow 0$ at lower end of I, $b_2^2 - 4b_1 b_3 < 0$, so this scheme would not have a physical $\alpha_{IR,3\ell}$ in this region.

We find that the value of the IR zero decreases when calculated at the 3-loop level, i.e.,

$$\alpha_{IR,3\ell} < \alpha_{IR,2\ell}$$

Proof:

$$\begin{aligned}\alpha_{IR,2\ell} - \alpha_{IR,3\ell} &= \frac{4\pi b_1}{|b_2|} - \frac{2\pi}{|b_3|}\left(-|b_2| + \sqrt{b_2^2 + 4b_1|b_3|}\ \right) \\ &= \frac{2\pi}{|b_2 b_3|}\left[2b_1|b_3| + b_2^2 - |b_2|\sqrt{b_2^2 + 4b_1|b_3|}\ \right]\end{aligned}$$

The expression in square brackets is positive if and only if

$$(2b_1|b_3| + b_2^2)^2 - b_2^2(b_2^2 + 4b_1|b_3|) > 0$$

This difference is equal to the pos.-def. quantity $4b_1^2 b_3^2$, which proves the inequality.

Since the existence of the IR zero in β at 2-loop level is scheme-independent, one may require that a scheme should maintain this property to higher-loop order, and hence that $b_3 < 0$ for $N_f \in I$. The inequality $\alpha_{IR,3\ell} < \alpha_{IR,2\ell}$ holds in all such schemes, and is thus more general than just for $\overline{MS}$ (Shrock, PRD in press, arXiv:1301.3209).

The 4-loop β function is

$$\beta = -\frac{\alpha^2}{2\pi}\left(b_1 + b_2 a + b_3 a^2 + b_4 a^3\right)$$

so β has three zeros away from the origin. We determine the smallest positive real zero as $\alpha_{IR,4\ell}$.

Going from 3-loop to 4-loop level, there is a slight change in the value of the IR zero, but this change is smaller than the decrease from 2-loops to 3-loops, so

$$\alpha_{IR,4\ell} < \alpha_{IR,2\ell}$$

Our result of smaller fractional change in value of IR zero of β at higher-loop order agrees with expectation that calculating to higher loop order should give more stable result.

Some numerical values of $\alpha_{IR,n\ell}$ at the $n = 2,\ 3,\ 4$ loop level for SU(2), SU(3) and fermions in fund. rep.

N	N_f	$\alpha_{IR,2\ell}$	$\alpha_{IR,3\ell}$	$\alpha_{IR,4\ell}$
2	7	2.83	1.05	1.21
2	8	1.26	0.688	0.760
2	9	0.595	0.418	0.444
2	10	0.231	0.196	0.200
3	10	2.21	0.764	0.815
3	11	1.23	0.578	0.626
3	12	0.754	0.435	0.470
3	13	0.468	0.317	0.337
3	14	0.278	0.215	0.224
3	15	0.143	0.123	0.126
3	16	0.0416	0.0397	0.0398

We have performed the corresponding higher-loop calculations for SU(N) gauge theories with N_f fermions in the adjoint, symmetric and antisymmetric rank-2 tensor representations. The general result $\alpha_{IR,3\ell} < \alpha_{IR,2\ell}$ applies. Details are in our papers.

The anomalous dimension $\gamma_m \equiv \gamma$ for the fermion bilinear is

$$\gamma = \sum_{\ell=1}^{\infty} c_\ell a^\ell = \sum_{\ell=1}^{\infty} \bar{c}_\ell \alpha^\ell$$

where $\bar{c}_\ell = c_\ell/(4\pi)^\ell$ is the ℓ-loop coeff. The one-loop coeff. $c_1 = 6C_f$ is scheme-independent, the c_ℓ with $\ell \geq 2$ are scheme-dependent and have been calculated up to 4-loop level in $\overline{MS}$ scheme.

It is of interest to calculate γ at the exact IRFP in IR-conformal phase and the approx. IRFP in phase with SχSB.

Denote γ calculated to n-loop ($n\ell$) level as $\gamma_{n\ell}$ and, evaluated at the n-loop value of the IR zero of β, as

$$\gamma_{IR,n\ell} \equiv \gamma_{n\ell}(\alpha = \alpha_{IR,n\ell})$$

N.B.: In the IR chirally symmetric phase, an all-order calc. of γ evaluated at an all-order calc. of α_{IR} would be an exact property of the theory

In the χ bk. phase, just as the IR zero of β is only an approx. IRFP, so also, the γ is only approx., describing the running of $\bar{\psi}\psi$ and the dynamically generated running fermion mass near the zero of β having large-momentum behavior $\Sigma(k) \sim \Lambda(\Lambda/k)^{2-\gamma}$.

In both phases, γ is bounded above as $\gamma < 2$. At 2-loop level we find $\gamma_{IR,2\ell} =$

$$\frac{C_f(11C_A - 4T_fN_f)[455C_A^2 + 99C_AC_f + (180C_f - 248C_A)T_fN_f + 80(T_fN_f)^2]}{12[-17C_A^2 + 2(5C_A + 3C_f)T_fN_f]^2}$$

Illustrative numerical values of $\gamma_{IR,n\ell}$ for SU(2) and SU(3) at the $n = 2,\ 3, 4$ loop level and fermions in the fund. rep.:

N	N_f	$\gamma_{IR,2\ell}$	$\gamma_{IR,3\ell}$	$\gamma_{IR,4\ell}$
2	8	0.752	0.272	0.204
2	9	0.275	0.161	0.157
2	10	0.0910	0.0738	0.0748
3	11	1.61	0.439	0.250
3	12	0.773	0.312	0.253
3	13	0.404	0.220	0.210
3	14	0.212	0.146	0.147
3	15	0.0997	0.0826	0.0836
3	16	0.0272	0.0258	0.0259

Equivalently, we show plots of γ as fn. of N_f for SU(2) and SU(3):

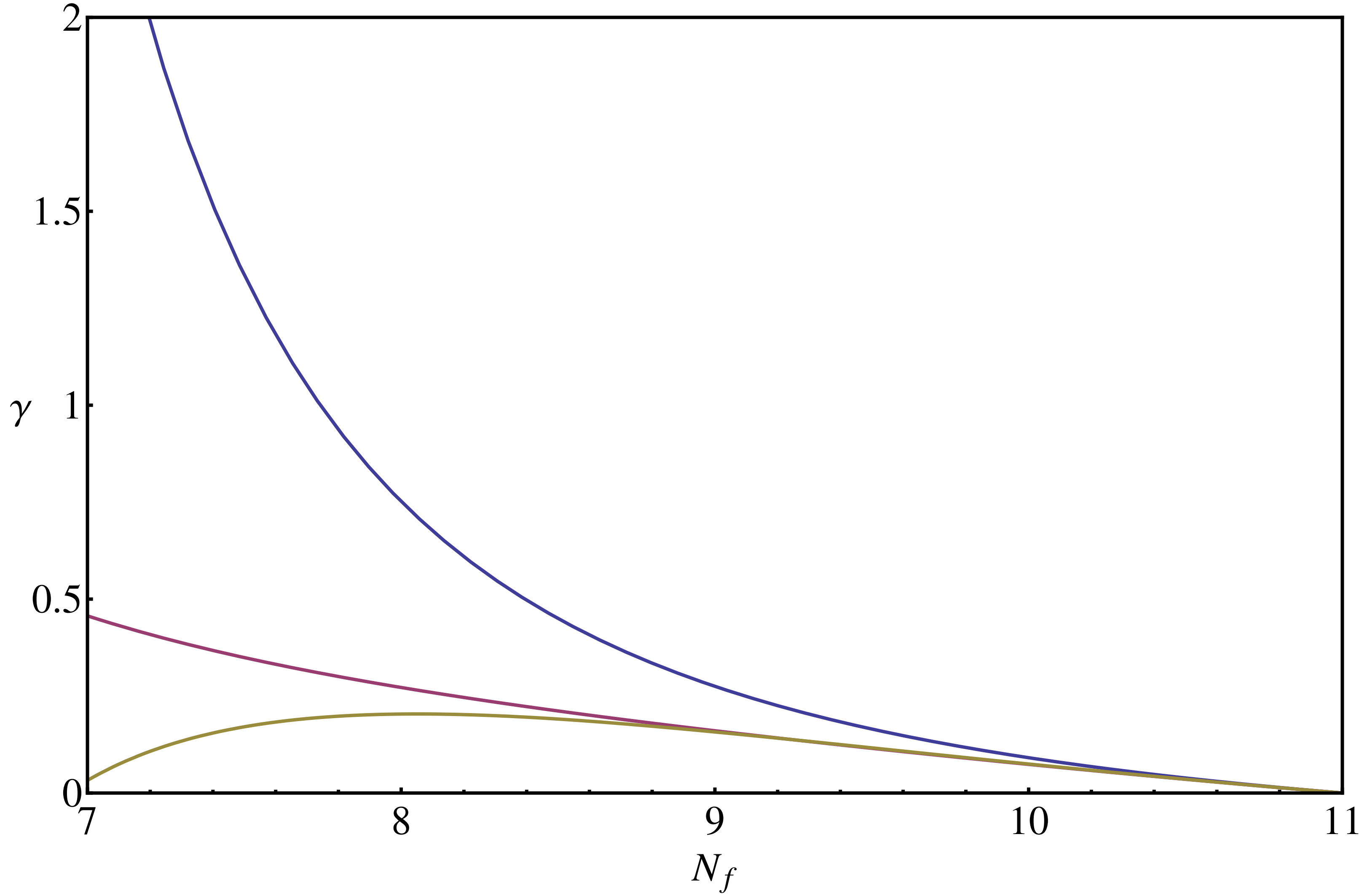

Figure 1: $\boldsymbol{n}$-loop anomalous dimension $\boldsymbol{\gamma_{IR,n\ell}}$ at $\boldsymbol{\alpha_{IR,n\ell}}$ for SU(2) with $\boldsymbol{N_f}$ fermions in fund. rep. (i) blue: $\boldsymbol{\gamma_{IR,2\ell}}$; (ii) red: $\boldsymbol{\gamma_{IR,3\ell}}$; (iii) brown: $\boldsymbol{\gamma_{IR,4\ell}}$.

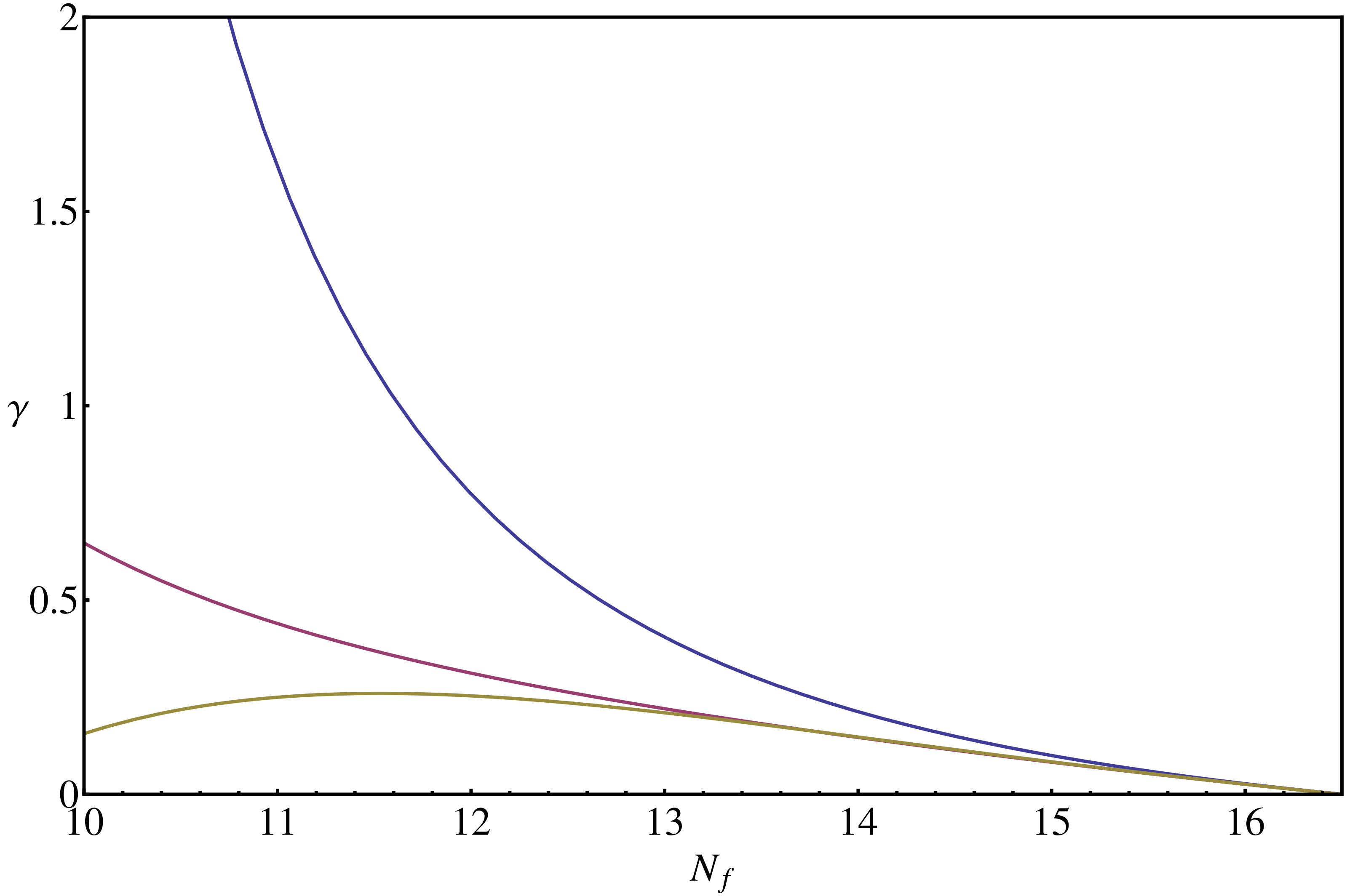

Figure 2: $\boldsymbol{n}$-loop anomalous dimension $\boldsymbol{\gamma_{IR,n\ell}}$ at $\boldsymbol{\alpha_{IR,n\ell}}$ for SU(3) with $\boldsymbol{N_f}$ fermions in fund. rep: (i) blue: $\boldsymbol{\gamma_{IR,2\ell}}$; (ii) red: $\boldsymbol{\gamma_{IR,3\ell}}$; (iii) brown: $\boldsymbol{\gamma_{IR,4\ell}}$.

A necessary condition for a perturb. calc. to be reliable is that higher-order contribs. do not modify the result too much.

One sees from the tables and figures that the 3-loop and 4-loop results are closer to each other for a larger range of N_f than the 2-loop and 3-loop results.

So our higher-loop calcs. of α_{IR} and γ allow us to probe the theory reliably down to smaller values of N_f and thus stronger couplings, closer to $N_{f,cr}$.

We have also performed these higher-loop calculations for larger fermion reps. R. In general, we find that, for a given N, R, and N_f, the values of $\gamma_{IR,n\ell}$ calculated to 3-loop and 4-loop order are smaller than the 2-loop value.

Comparisons with Lattice Measurements

Consider SU(3) with $N_f = 12$, $R =$ fund. rep. No consensus yet as to whether this theory is chirally symmetric or broken in the IR. Appelquist et al. (LSD); Deuzeman et al; Hasenfratz et al.; DeGrand et al.; Aoki et al. find IR-χ sym. while Jin and Mawhinney and Kuti et al. find SχSB.

For either case we can compare our γ calculation with the lattice measurements. From our table above,

$$\gamma_{IR,2\ell} = 0.77, \qquad \gamma_{IR,3\ell} = 0.31, \qquad \gamma_{IR,4\ell} = 0.25$$

Some lattice results (N.B.: some error estimates do not include all systematic uncertainties)

$\gamma = 0.414 \pm 0.016$ (Appelquist, Fleming, Lin, Neil, Schaich, PRD 84, 054501 (2011), arXiv:1106.2148, analyzing data of Kuti et al., PLB 703, 348 (2011), arXiv:1104.3124)

$\gamma \sim 0.35$ (DeGrand, PRD 84, 116901 (2011), arXiv:1109.1237, also analyzing data of Kuti et al.)

$0.2 \lesssim \gamma \lesssim 0.4$ (Fodor, Holland, Kuti, Nogradi, Schroeder, Wong (method-dep.), arXiv:1205.1878, arXiv:1211.3548, 1211.6164)

$\gamma = 0.4 - 0.5$ (Y. Aoki et al., (LatKMI) PRD 86, 054506 (2012), arXiv:1207.3060)

$\gamma = 0.27 \pm 0.03$ (Hasenfratz et al., PoS(Lattice 2012)034, arXiv:1207.7162)

So here the 2-loop value is larger than, and the 3-loop and 4-loop values closer to, these lattice measurements. Thus, our higher-loop calcs. of γ yield better agreement with these lattice measurements than two-loop calc.

The possibility that $\gamma \sim O(1)$ near the lower end of interval I remains.

We have carried out comparisons of our calculations with lattice measurements for SU(2) and also for SU(2) and SU(3) with larger fermion reps.

Further Higher-Loop Structural Properties of β

The UV to IR flow is controlled by β in the interval $\alpha = 0$ to $\alpha \to \alpha_{IR}$. In addition to $\alpha_{IR,n\ell}$, further structural properties of interest include the magnitude and location of the minimum in $\beta_{n\ell}$ and the derivative $\beta'_{IR,n\ell} \equiv d\beta_{n\ell}/d\alpha$ evaluated at $\alpha_{IR,n\ell}$.

In quasi-scale-invariant case where $\alpha_{IR} \gtrsim \alpha_{cr}$, dilaton mass relevant in dynamical EWSB models depends on how small β is for α near to α_{IR} and hence, at n-loop order, on $\beta'_{IR,n\ell}$, via the series expansion of $\beta_{n\ell}$ around $\alpha_{IR,n\ell}$,

$$\beta_{n\ell}(\alpha) = \beta'_{IR,n\ell}\,(\alpha - \alpha_{IR,n\ell}) + O\Big((\alpha - \alpha_{IR,n\ell})^2\Big)$$

We have calculated these structural properties analytically and numerically (Shrock, PRD, in press, arXiv:1301.3209).

We find, e.g., that $\beta'_{IR,n\ell}$ decreases as n increases from $n = 2$ to higher-loop order, as is evident in the following table:

N	N_f	$\beta'_{IR,2\ell}$	$\beta'_{IR,3\ell}$	$\beta'_{IR,4\ell}$
2	7	1.20	0.728	0.677
2	8	0.400	0.318	0.300
2	9	0.126	0.115	0.110
2	10	0.0245	0.0239	0.0235
3	10	1.52	0.872	0.853
3	11	0.720	0.517	0.498
3	12	0.360	0.2955	0.282
3	13	0.174	0.156	0.149
3	14	0.0737	0.0699	0.0678
3	15	0.0227	0.0223	0.0220
3	16	0.00221	0.00220	0.00220

Illustrative figures for SU(2) with $N_f = 8$ fermions and SU(3) with $N_f = 12$ fermions:

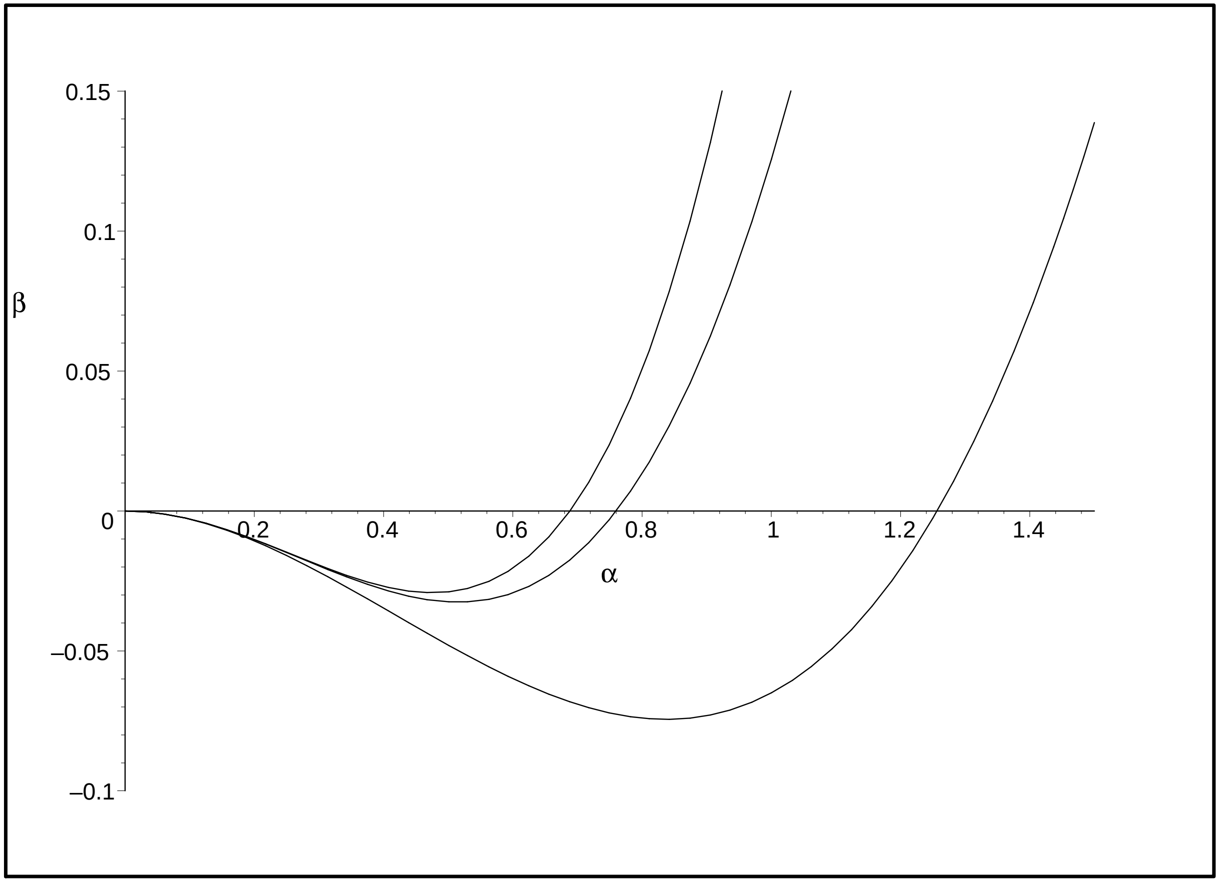

Figure 3: $\boldsymbol{\beta_{n\ell}}$ for SU(2), $\boldsymbol{N_f = 8}$, at $\boldsymbol{n = 2,\ 3,\ 4}$ loops. From bottom to top, curves are $\boldsymbol{\beta_{2\ell}}$, $\boldsymbol{\beta_{4\ell}}$, $\boldsymbol{\beta_{3\ell}}$.

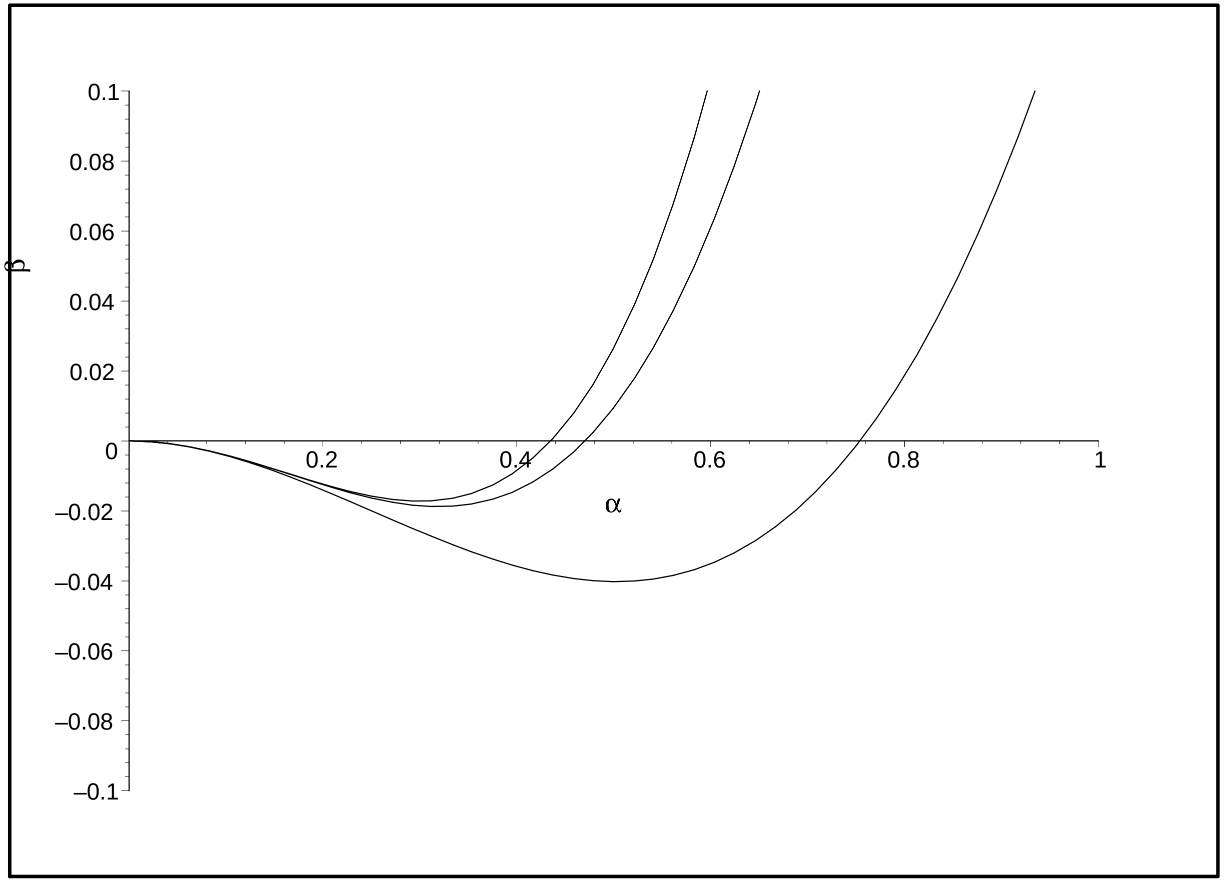

Figure 4: $\boldsymbol{\beta_{n\ell}}$ for SU(3), $\boldsymbol{N_f = 12}$, at $\boldsymbol{n = 2,\ 3,\ 4}$ loops. From bottom to top, curves are $\boldsymbol{\beta_{2\ell}}$, $\boldsymbol{\beta_{4\ell}}$, $\boldsymbol{\beta_{3\ell}}$.

Interesting property: for $R =$ fund. rep., $\alpha_{IR,n\ell}N$, $\gamma_{IR,n\ell}$, and other structural properties of $\beta_{n\ell}$ are similar in theories with different values of N and N_f if they have equal or similar values of $r = N_f/N$.

This motivates a study of the UV to IR evolution of an SU(N) gauge theory with N_f fermions in the fund. rep. in the 't Hooft-Veneziano limit $N \to \infty$, $N_f \to \infty$ with $r \equiv N_f/N$ fixed and $\alpha(\mu)N$ independent of N. We have carried out this study in arXiv:1302.5434.

We find that corrections to this limit are strongly suppressed, like $1/N^2$, leading to a rapid approach to the asymptotic expressions for various quantities.

This provides a unified quantitative understanding of the similarities in UV to IR evolution of SU(N) theories with different N and N_f but similar r.

Various quantities are calculated as fns. of r; for example, for $\gamma_{IR,n\ell}$:

r	$\gamma_{IR,2\ell}$	$\gamma_{IR,3\ell}$	$\gamma_{IR,4\ell}$
3.6	1.853	0.5201	0.3083
3.8	1.178	0.4197	0.3061
4.0	0.7847	0.3414	0.2877
4.2	0.5366	0.2771	0.2664
4.4	0.3707	0.2221	0.2173
4.6	0.2543	0.1735	0.1745
4.8	0.1696	0.1294	0.1313
5.0	0.1057	0.08886	0.08999
5.2	0.05620	0.05123	0.05156
5.4	0.01682	0.01637	0.01638

Higher-Loop Calculations of UV to IR Evolution for an $\mathcal{N} = 1$ Supersymmetric Gauge Theory

It is of interest to carry out a similar analysis in an asymptotically free $\mathcal{N} = 1$ supersymmetric gauge theory with vectorial chiral superfield content Φ_i, $\tilde{\Phi}_i$, $i = 1, ..., N_f$ in the R, $\bar{R}$ reps., respectively.

We have done this in Ryttov and Shrock, Phys. Rev. D 85, 076009 (2012), arXiv:1202.1297; and Shrock, Phys. Rev. D, in press, arXiv:1301.3209; arXiv:1302.5434.

An appeal of this analysis: exact results on the IR properties of the theory are known.

We find that the perturb. calc. slightly overestimates the value of $N_{f,cr}$, i.e., slightly underestimates the size of the IR-conformal phase.

Study of Scheme-Dependence in Calculation of IR Fixed Point

It is useful to study effects of scheme transformations on $\alpha_{IR,n\ell}$ for $n \geq 3$ where the coeffs. in β are scheme-dependent. We have done this in Ryttov and Shrock, PRD 86, 065032 (2012), arXiv:1206.2366; PRD 86, 085005 (2012), arXiv:1206.6895.

A scheme transformation (ST) is a map between α and α' or equivalently, a and a', where $a = \alpha/(4\pi)$ of the form

$$a = a' f(a')$$

with $f(0) = 1$ to keep UV properties unchanged. Write

$$f(a') = 1 + \sum_{s=1}^{s_{max}} k_s (a')^s = 1 + \sum_{s=1}^{s_{max}} \bar{k}_s (\alpha')^s \ ,$$

where $\bar{k}_s = k_s/(4\pi)^s$, and s_{max} may be finite or infinite. Then

$$\beta_{\alpha'} = -2\alpha' \sum_{\ell=1}^{\infty} b'_\ell (a')^\ell = -2\alpha' \sum_{\ell=1}^{\infty} \bar{b}'_\ell (\alpha')^\ell \ ,$$

where $\bar{b}'_\ell = b'_\ell/(4\pi)^\ell$.

We calculate the b'_ℓ as functions of the b_ℓ and k_s. At 1-loop and 2-loop, this yields the well-known results

$$b'_1 = b_1 \ , \quad b'_2 = b_2$$

We find

$$b'_3 = b_3 + k_1 b_2 + (k_1^2 - k_2) b_1 \ ,$$

$$b'_4 = b_4 + 2k_1 b_3 + k_1^2 b_2 + (-2k_1^3 + 4k_1 k_2 - 2k_3) b_1$$

$$b'_5 = b_5 + 3k_1 b_4 + (2k_1^2 + k_2) b_3 + (-k_1^3 + 3k_1 k_2 - k_3) b_2$$
$$+ (4k_1^4 - 11k_1^2 k_2 + 6k_1 k_3 + 4k_2^2 - 3k_4) b_1$$

etc. at higher-loop order.

We construct an explicit ST that, in the vicinity of a UVFP, to the 't Hooft scheme where $b'_\ell = 0$ for $\ell \geq 3$

We point out that a physically acceptable ST must satisfy several conditions, such as mapping a real positive α to a real positive α', which are easily satisfied in the vicinity of a UVFP, but can be quite restrictive at an IRFP.

For example, consider the ST (dependent on a parameter r)

$$a = \frac{\tanh(ra')}{r}$$

with inverse

$$a' = \frac{1}{2r} \ln \left(\frac{1 + ra}{1 - ra} \right)$$

This is acceptable for small a, but if $a > 1/r$, i.e., $\alpha > 4\pi/r$, it maps a real α to a complex α' and hence is physically unacceptable. For, say, $r = 8\pi$, this pathology can occur at a moderate $\alpha = 0.5$.

We have constructed several STs that are acceptable at an IRFP and have studied scheme dependence of the IR zero of $\beta_{n\ell}$ using these. For example,

$$a = \frac{\sinh(ra')}{r}$$

with inverse

$$a' = \frac{1}{r} \ln \left[ra + \sqrt{1 + (ra)^2} \, \right]$$

We find reasonably small scheme-dependence for moderate α_{IR}.

Conclusions

- Understanding the UV to IR evolution of an asymptotically free gauge theory and the nature of the IR behavior is of fundamental interest and can be relevant to exploring BSM physics

- Our higher-loop calculations give new information on this UV to IR flow and on determination of $\alpha_{IR,n\ell}$ and $\gamma_{IR,n\ell}$; valuable to compare and combine results from higher-loop continuum calcs. with lattice measurements

- Results on the limit $N \longrightarrow \infty$, $N_f \longrightarrow \infty$ with N_f/N fixed provide understanding of similarities in UV to IR flows in theories with different N and N_f but similar r.

- Higher-loop study of UV to IR flow for supersymmetric gauge theories gives further insights

- We have investigated effects of scheme-dependence of IR zero in higher-loop calculations and have pointed out that scheme transformations are subject to conditions that are easily satisfied at a UVFP but are a significant constraint at an IRFP