Exclusive Study of \((g - 2)_\mu\) HVP

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Outline

- Muon $g - 2$ Experiment
  - Motivation from muon $g - 2$
  - Tensions in $\pi\pi$ Scattering
  - Error Budget and LQCD Strategy

- Correlation Function Spectrum & Overlap
  - Lattice Parameters
  - GEVP Spectrum & Overlaps
  - $\pi\pi$ Scattering Phase Shift
  - $4\pi$ Correlation Functions

- Bounding Method and the Muon HVP
  - Correlation Function Reconstruction
  - (Improved) Bounding Method
  - Results

- Conclusions/Outlook
Introduction
Muon Anomalous Magnetic Moment Experiment

High-precision experiment of spin precession relative to momentum direction in storage ring

Anomalous frequency $\omega_a = \frac{g-2}{2} \frac{eB}{m} = a_\mu \frac{eB}{m}$

Sensitive to new physics, and also discrepant with experiment!
Fermilab Muon $g - 2$ Experiment

Experiment has come a long way (and so has theory!)

Aiming for a $4 \times$ improvement in uncertainty over the BNL result
Muon $g - 2$ Theory Error Budget

<table>
<thead>
<tr>
<th>Contribution</th>
<th>Value $\times 10^{10}$</th>
<th>Uncertainty $\times 10^{10}$</th>
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<tbody>
<tr>
<td>QED</td>
<td>11 658 471.895</td>
<td>0.008</td>
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<tr>
<td>EW</td>
<td>15.4</td>
<td>0.1</td>
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<tr>
<td>HVP LO</td>
<td>692.5</td>
<td>2.7</td>
</tr>
<tr>
<td>HVP NLO</td>
<td>-9.84</td>
<td>0.06</td>
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<td>HVP NNLO</td>
<td>1.24</td>
<td>0.01</td>
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<tr>
<td>Hadronic light-by-light</td>
<td>10.5</td>
<td>2.6</td>
</tr>
<tr>
<td>Total SM prediction</td>
<td>11 659 181.7</td>
<td>3.8</td>
</tr>
<tr>
<td>BNL E821 result</td>
<td>11 659 209.1</td>
<td>6.3</td>
</tr>
<tr>
<td>Fermilab E989 target</td>
<td></td>
<td>$\approx$ 1.6</td>
</tr>
</tbody>
</table>

Experiment-Theory difference is $27.4(7.3) \implies 3.7\sigma$ tension!

Target measurement:

Hadronic Vacuum Polarization (HVP)

$\implies$

Lattice results have larger uncertainty, but systematically improve

$\implies$

Dispersive (“R-ratio”) results more precise, but static

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[1904.09479[hep-lat]] $a_\mu \times 10^{10}$
Ratio data for $ee \rightarrow \pi\pi$ exclusive channel, $\sqrt{s} = 0.6 - 0.9$ GeV region
Tension between most precise measurements (BABAR/KLOE)
R-ratio $a^{HVP}_\mu$ uncertainty $<$ difference in this channel

Avoid tension by computing precise lattice-only estimate of $a^{HVP}_\mu$
Use lattice QCD to inform experiment, resolve discrepancy
Exclusive Channels in the HVP

\[ C(t) = \frac{1}{3} \sum_i \langle \bar{\psi} \gamma_i \psi \rangle_t \langle \bar{\psi} \gamma_i \psi \rangle_0 \]

Correlator has large statistical error in long-distance region, but contributions from high energy states are exponentially suppressed.

Long distance correlator dominated by two-pion states, but overlap of vector current with two-pion states is minimal.
Exclusive Channels in the HVP

\[ C(t) = \frac{1}{3} \sum_i \langle [\bar{\psi} \gamma_i \psi]_t \vert [\bar{\psi} \gamma_i \psi]_0 \rangle \]
\[ \approx \sum_n \left| \langle \Omega \vert \bar{\psi} \gamma_i \psi \vert n \rangle \right|^2 e^{-E_n t} \]

Correlator has large statistical error in long-distance region, but contributions from high energy states are exponentially suppressed.

Long distance correlator dominated by two-pion states, but overlap of vector current with two-pion states is minimal.

Strategy:

- Construct & measure operators that overlap strongly with \( \pi \pi \) states
- Correlate these operators with the local vector current
- \( a_{HVP}^\mu \) computed by integrating with time-momentum representation kernel,
  \[ a_{HVP}^\mu = \sum_t w_t C(t) \] [D. Bernecker & H. Meyer, 1107.4388 [hep-lat]]
Computation Setup
Ensemble Details

Computed on 2 + 1 flavor Möbius Domain Wall Fermions for valance and sea, $M_\pi$ at physical value on all ensembles

Computations using distillation setup

- $24^3$ and $32^3$ ($+48^3$) ensembles → infinite volume limit
- $48^3$ and $64^3$ ($+96^3$) ensembles → continuum limit

Compare results of explicit calculation of finite volume results to Luscher + Gounaris-Sakurai prediction [H. Meyer, 1107.4388[hep-lat]]

Not presented here, see [C. Lehner, Lattice 2018]
Operators

Operators constructed in $I = 1, P$-wave channel to impact upon $HVP_{\mu}$

Designed to have strong overlap with specific target states, but all operators unavoidably couple to all states in HVP spectrum

Vector current operators:

- Local $O_{J\mu} = \sum_x \bar{\psi}(x) \gamma_\mu \psi(x), \mu \in \{1, 2, 3\}$
- Smeared $O_{j\mu} = \sum_{xyz} \bar{\psi}(x)f(x - z)\gamma_\mu f(z - y)\psi(y)$

2$\pi$ operators with $O_n$ given by $\vec{p}_\pi \in \frac{2\pi}{L} \times \{(1, 0, 0), (1, 1, 0), (1, 1, 1), (2, 0, 0)\}$

$$O_n = \left| \sum_{xyz} \bar{\psi}(x)f(x - z)e^{-i\vec{p}_\pi \cdot \vec{z}}\gamma_5 f(z - y)\psi(y) \right|^2$$

Also test two $4\pi$ operators with $\vec{p}_\pi = \frac{2\pi}{L} \times (1, 0, 0)$:

$$O_{4\pi} = \left| \sum_{xyz} \bar{\psi}(x)f(x - z)e^{-i\vec{p}_\pi \cdot \vec{z}}\gamma_5 f(z - y)\psi(y) \right|^2 \left| \sum_{xy} \bar{\psi}(x)f(x - y)\gamma_5 \psi(y) \right|^2$$

Correlators arranged in a $N \times N$ symmetric matrix:

$$\begin{array}{c|cccc}
\otimes & O_{J\mu} & O_{j\mu} & O_{2\pi} & O_{4\pi} \\
O_{J\mu} & C_{J\mu J\mu} & C_{J\mu j\mu} & C_{J\mu 2\pi} & C_{J\mu 4\pi} \\
O_{j\mu} & C_{j\mu J\mu} & C_{j\mu j\mu} & C_{j\mu 2\pi} & C_{j\mu 4\pi} \\
O_{2\pi} & C_{2\pi 2\pi} & C_{2\pi 2\pi} & C_{2\pi 2\pi} & C_{2\pi 4\pi} \\
O_{4\pi} & C_{4\pi 4\pi} & C_{4\pi 4\pi} & C_{4\pi 4\pi} & C_{4\pi 4\pi} \\
\end{array} \rightarrow C(t)$$
Generalized EigenValue Problem (GEVP)

Generalized EigenValue Problem to estimate overlap with vector current & energies

\[ C(t) V = C(t + \delta t) V \Lambda(\delta t) \]

\[ \Lambda_{nn}(\delta t) \sim e^{+E_n\delta t}, \quad V_{im} \propto \langle \Omega | O_i | m \rangle \]

\( C(t) \) is the matrix of correlation functions from previous slide
Compute at fixed \( \delta t \), vary \( t \): plateau for large \( t \)

From result, reconstruct exponential dependence of local vector correlation function

\[ C_{ij}^{\text{latt.}}(t) = \sum_{n}^{N} \langle \Omega | O_i | n \rangle \langle n | O_j | \Omega \rangle e^{-E_n t} \]

In theory, infinite number of states contribute to correlation function
In practice, only finite \( N \) necessary to model correlation function
\[ \implies \text{finite GEVP basis is sufficient} \]
GEVP Results - $J_\mu + 2\pi$ Operators only

6-operator basis on 48I ensemble: local+smeared vector, $4 \times (2\pi)$

Data points from solving GEVP at fixed $\delta t$

$$C(t_0) V = C(t_0 + \delta t) V \Lambda(\delta t), \quad \Lambda_{nn}(\delta t) \sim e^{\pm E_n \delta t}$$

Excited state contaminations decay as $t_0, \delta t \to \infty$

moving right on plot $\implies$ asymptote to lowest states’ spectrum & overlaps

Statistics+systematics; Left: Spectrum; Right: Overlap with local vector current
Compute $\pi\pi$ scattering phase shifts in $l = 1$ channel from spectrum
Statistics + systematics

Compare to simple Breit-Wigner parametrization and pheno (courtesy of M.Bruno)
Good agreement with pheno for 32ID, 48I, 64I
24ID: remnant excited state contaminations, still to be removed

Scattering phase shift results to appear as part of series of papers by RBC+UKQCD
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Code builds a text representation of operators by performing tensor products and irrep decompositions of lattice operators with arbitrary spin & momentum.

This has resulted in a world-first computation of $4\pi$ to $4\pi$ correlation functions in $I = 1$ channel.
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4π Contractions
Contractions cont...
$4\pi$ Contractions cont... cont...
Extra 4\pi states could appear with overlap to local vector current
Breakdown of formalism for FVC could occur at 4\pi threshold
Results unaffected by inclusion of 4\pi operators, but states resolvable
GEVP Results - $4\pi$ Operators

Extra $4\pi$ states could appear with overlap to local vector current
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GEVP Results - $4\pi$ Operators

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Results unaffected by inclusion of $4\pi$ operators, but states resolvable
Overlap of $4\pi$ states with local vector current unresolvable
Overlap of states with $4\pi$ operator significant
$\Rightarrow$ $4\pi$ state safely negligible in local vector current
$\Rightarrow$ Will be neglected in all of following analysis
Correlator Reconstruction and Bounding
Plotted: (weight kernel) × (correlation function); integral → $a^HVP_{\mu}$

GEVP results to reconstruct long-distance behavior of
local vector correlation function needed to compute connected HVP

Explicit reconstruction good estimate of correlation function at long-distance,
missing excited states at short-distance

More states $\implies$ better reconstruction, can replace $C(t)$ at shorter distances
Improved Bounding Method

Use known results in spectrum to make a precise estimate of upper & lower bound on $a^\text{HVP}_\mu$ [RBC (2017)]

\[
\tilde{C}(t; t_{\text{max}}, E) = \begin{cases} 
C(t) & t < t_{\text{max}} \\
C(t_{\text{max}})e^{-E(t-t_{\text{max}})} & t \geq t_{\text{max}}
\end{cases}
\]

Upper bound: $E \leq E_0$, lowest state in spectrum

Lower bound: $E \geq \log\left[\frac{C(t_{\text{max}})}{C(t_{\text{max}}+1)}\right]

BMW Collaboration [K.Miura, Lattice2018] takes $E \to \infty$

With good control over lower states in spectrum from exclusive reconstruction, improve bounding method [RBC/UKQCD 2018 (CL@KEK Feb 2018)]:

Replace $C(t) \to C(t) - \sum_{n}^{N} |c_n|^2 e^{-E_nt}$ and apply bounding procedure for $a_\mu - \delta a_\mu$

$\implies$ Long distance convergence now $\propto e^{-E_{N+1}t}$, lower bound falls faster

$\implies$ Smaller overall contribution from neglected states

After bounding, add back $\delta a_\mu = \sum_{t=t_{\text{max}}}^{\infty} w_t \sum_{n}^{N} |c_n|^2 e^{-E_nt}$
Bounding Method Results - 48I

No bounding method:
Bounding method $t_{\text{max}} = 3.3$ fm, no reconstruction:

$$a_{\mu}^{HVP} = 638(21)$$
$$a_{\mu}^{HVP} = 626.5(8.6)$$

Bounding method gives factor of 3 improvement over no bounding method
Bounding Method Results - 48I

No bounding method:
Bounding method $t_{max} = 3.3$ fm, no reconstruction:
Bounding method $t_{max} = 3.0$ fm, 1 state reconstruction:

Bounding method gives factor of 3 improvement over no bounding method

$\sum_{t}^{T/2} w(t)C(t)/10^{-10}$

PRELIMINARY

$a^{HVP}_{\mu} = 638(21)$

$a^{HVP}_{\mu} = 626.5(8.6)$

$a^{HVP}_{\mu} = 627.5(7.7)$
Bounding Method Results - 48I

No bounding method:
Bounding method $t_{\text{max}} = 3.3$ \text{fm}, no reconstruction: $a_{HVP} = 638(21)$
Bounding method $t_{\text{max}} = 3.0$ \text{fm}, 1 state reconstruction: $a_{HVP} = 626.5(8.6)$
Bounding method $t_{\text{max}} = 2.9$ \text{fm}, 2 state reconstruction: $a_{HVP} = 627.5(7.7)$

Bounding method $t_{\text{max}} = 2.9$ \text{fm}, 3 state reconstruction:

Bounding method gives factor of 3 improvement over no bounding method
No bounding method: 
Bounding method $t_{\text{max}} = 3.3$ fm, no reconstruction: $a_{\text{HVP}}^\mu = 638(21)$  
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Bounding method $t_{\text{max}} = 2.9$ fm, 2 state reconstruction: $a_{\text{HVP}}^\mu = 627.5(7.7)$  
Bounding method $t_{\text{max}} = 2.2$ fm, 3 state reconstruction: $a_{\text{HVP}}^\mu = 629.1(5.7)$

Bounding method gives factor of 3 improvement over no bounding method
No bounding method: 
\( a^{HVP}_\mu = 638(21) \)

Bounding method \( t_{\text{max}} = 3.3 \text{ fm} \), no reconstruction: 
\( a^{HVP}_\mu = 626.5(8.6) \)

Bounding method \( t_{\text{max}} = 3.0 \text{ fm} \), 1 state reconstruction: 
\( a^{HVP}_\mu = 627.5(7.7) \)

Bounding method \( t_{\text{max}} = 2.9 \text{ fm} \), 2 state reconstruction: 
\( a^{HVP}_\mu = 629.1(5.7) \)

Bounding method \( t_{\text{max}} = 2.2 \text{ fm} \), 3 state reconstruction: 
\( a^{HVP}_\mu = 628.0(4.2) \)

Bounding method \( t_{\text{max}} = 1.8 \text{ fm} \), 4 state reconstruction: 
\( a^{HVP}_\mu = 626.2(3.9) \)

Bounding method gives factor of 3 improvement over no bounding method

Improving the bounding method increases gain to factor of 5, including systematics

Improvement should make all-lattice computation of \( a^{HVP}_\mu \) competitive with R-ratio by 2020
Update to RBC-UKQCD calculation including exclusive study in preparation

⇒ on target for precision improvement on $a_{\mu}^{HVP}$ at $5 \times 10^{-10}$ level

Further reduction will require full RBC-UKQCD program of computations

Work on the exclusive channel study using bounding method has led to
world-first estimation of finite volume corrections to $a_{\mu}^{HVP}$ at physical $M_\pi$

Complete analysis with full suite of systematic improvements ongoing

⇒ precision improvement $\times 10$ over original, target error on $a_{\mu}^{HVP}$ at $1 \times 10^{-10}$

Compare to dispersive $(3 - 5) \times 10^{-10}$

Aaron S. Meyer  Section: Correlator Reconstruction and Bounding 33/ 35
Conclusions
Conclusions

Pion scattering exclusive study poised to improve theory precision in $(g - 2)_\mu$:

- Dispersive approaches have unresolved tension in $\pi\pi$ scattering region, circumvented by LQCD calculation
- Computed $2\pi \rightarrow 4\pi$, $4\pi \rightarrow 4\pi$ correlation functions to show explicitly that $4\pi$ state has negligible effect on HVP at physical $M_\pi$
- Study of exclusive channels able to significantly reduce statistical uncertainty on an all-lattice computation of $a^{HVP}_\mu$
  \[ \implies \text{expect to reach precision of } O(5 \times 10^{-10}) \text{ by the end of year} \]
  \[ \implies \text{target } O(1 \times 10^{-10}) \text{ for all-lattice calculation} \]
- Part of ongoing lattice study to address all lattice systematics in RBC+UKQCD HVP computation (see [C.Lehner, Lattice 2019])
- New data on $64^3$ ensemble being analyzed
- Paper in progress; posting planned before end of year

Thank you!
BACKUP
Error Budget

<table>
<thead>
<tr>
<th>Term</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_{\mu_{ud}, \text{conn, isospin}}$</td>
<td>$202.9(1.4)<em>{S}(0.2)</em>{C}(0.1)<em>{V}(0.2)</em>{A}(0.2)_{Z}$</td>
</tr>
<tr>
<td>$a_{\mu_{s}, \text{conn, isospin}}$</td>
<td>$27.0(0.2)<em>{S}(0.0)</em>{C}(0.1)<em>{A}(0.0)</em>{Z}$</td>
</tr>
<tr>
<td>$a_{\mu_{c}, \text{conn, isospin}}$</td>
<td>$3.0(0.0)<em>{S}(0.1)</em>{C}(0.0)<em>{Z}(0.0)</em>{M}$</td>
</tr>
<tr>
<td>$a_{\mu_{uds}, \text{disc, isospin}}$</td>
<td>$-1.0(0.1)<em>{S}(0.0)</em>{C}(0.0)<em>{V}(0.0)</em>{A}(0.0)_{Z}$</td>
</tr>
<tr>
<td>$a_{\mu_{\text{QED, conn}}}$</td>
<td>$0.2(0.2)<em>{S}(0.0)</em>{C}(0.0)<em>{V}(0.0)</em>{A}(0.0)<em>{Z}(0.0)</em>{E}$</td>
</tr>
<tr>
<td>$a_{\mu_{\text{QED, disc}}}$</td>
<td>$-0.2(0.1)<em>{S}(0.0)</em>{C}(0.0)<em>{V}(0.0)</em>{A}(0.0)<em>{Z}(0.0)</em>{E}$</td>
</tr>
<tr>
<td>$a_{\mu_{\text{SIB}}}$</td>
<td>$0.1(0.2)<em>{S}(0.0)</em>{C}(0.0)<em>{V}(0.0)</em>{A}(0.0)<em>{Z}(0.0)</em>{E_{48}}$</td>
</tr>
<tr>
<td>$a_{\mu_{\text{uds, isospin}}}$</td>
<td>$231.9(1.4)<em>{S}(0.2)</em>{C}(0.1)<em>{V}(0.3)</em>{A}(0.2)<em>{Z}(0.0)</em>{M}$</td>
</tr>
<tr>
<td>$a_{\mu_{\text{QED, SIB}}}$</td>
<td>$0.1(0.3)<em>{S}(0.0)</em>{C}(0.2)<em>{V}(0.0)</em>{A}(0.0)<em>{Z}(0.0)</em>{E}(0.0)<em>{E</em>{48}}$</td>
</tr>
<tr>
<td>$a_{\mu_{R-\text{ratio}}}$</td>
<td>$460.4(0.7)<em>{RST}(2.1)</em>{RSY}$</td>
</tr>
<tr>
<td>$a_{\mu}$</td>
<td>$692.5(1.4)<em>{S}(0.2)</em>{C}(0.2)<em>{V}(0.3)</em>{A}(0.2)<em>{Z}(0.0)</em>{E}(0.0)<em>{E</em>{48}}$</td>
</tr>
</tbody>
</table>

TABLE I. Individual and summed contributions to $a_{\mu}$ multiplied by $10^{10}$. The left column lists results for the window method with $t_0 = 0.4\text{ fm}$ and $t_1 = 1\text{ fm}$. The right column shows results for the pure first-principles lattice calculation. The respective uncertainties are defined in the main text. [Blum et al., (2018)]

Full program of computations to reduce uncertainties:

- Reduce statistical uncertainties on light connected contribution
- Compute QED contribution
- Improve lattice spacing determination
- Finite volume and continuum extrapolation study
First constrain the p-wave phase shift from our $L = 6.22$ fm physical pion mass lattice:

\[ E_{\rho} = 0.766(21) \text{ GeV (PDG 0.77549(34) GeV)} \]
\[ \Gamma_{\rho} = 0.139(18) \text{ GeV (PDG 0.1462(7) GeV)} \]

[Lehner, Mainz 2018]
Predicts $|F_\pi(s)|^2$:

We can then also predict matrix elements and energies for our other lattices; successfully checked!

[Lehner, Mainz 2018]
Finite Volume Corrections on the Lattice

Complete error budget needs extrapolation to infinite volume

FV shift can be measured directly from results of exclusive study

\[ \implies \text{First time this shift resolved from zero at physical } M_\pi! \]
\[ \implies \text{Previous bound at } 10(26) \times 10^{-10}, M_\pi = 146 \text{ MeV} [1805.04250[\text{hep-lat}]] \]

Can compare FV shift predictions from phenomenological estimations:
Gounaris-Sakurai-Lüscher, proposed by H. Meyer
and scalar QED

\[
a_H^{\mu VP}(L = 6.2 \text{ fm}) - a_H^{\mu VP}(L = 4.7 \text{ fm}) = \begin{cases} 
21.6(6.3) \times 10^{-10} & \text{LQCD} \\
20(3) \times 10^{-10} & \text{GSL} \\
12.2 \times 10^{-10} & \text{sQED} 
\end{cases}
\]

Good agreement with GSL in range of energies probed by LQCD