Lattice X Intensity Frontier Workshop
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Non-perturbative matching of 3/4-flavor Wilson coefficients with position-space correlators

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N_f in Weak Hamiltonian

$$H_W = \Sigma_i\, w_i^{4f}(\mu)\, O_i^{4f}(\mu)$$

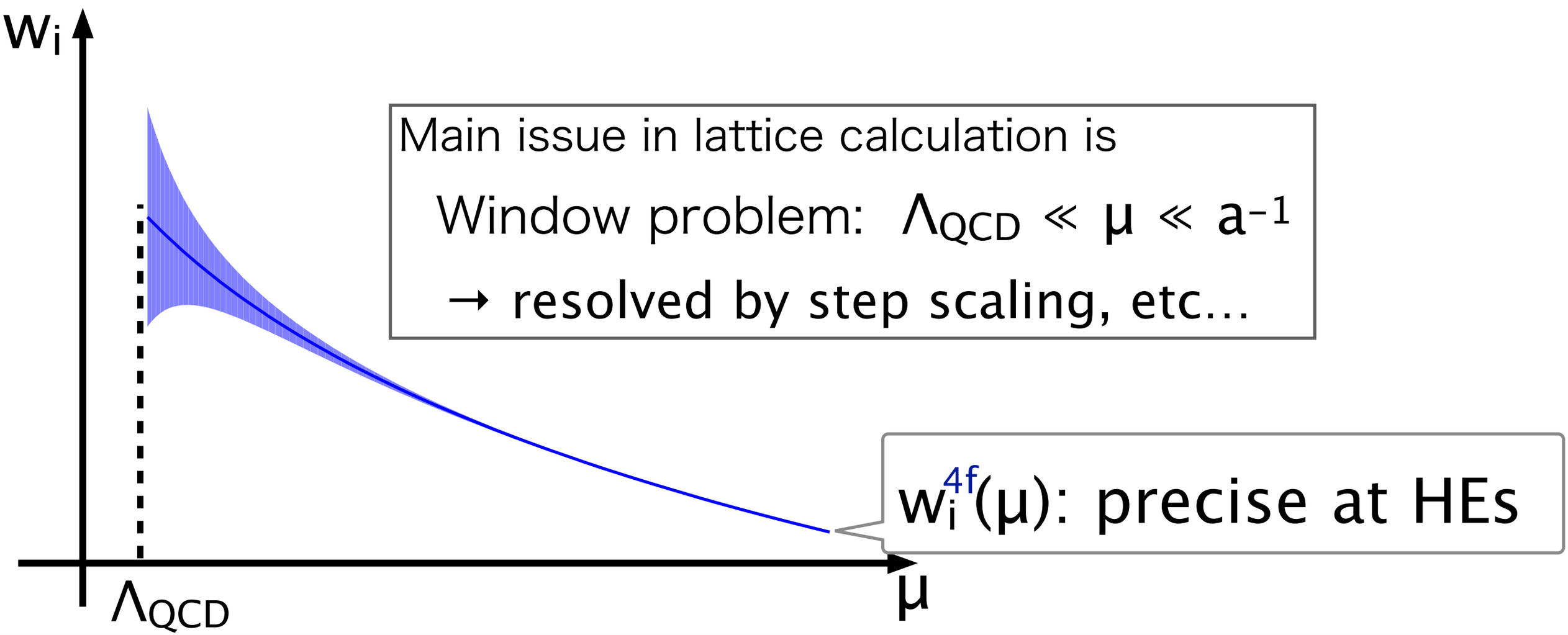
$$= \Sigma_i\, w_i^{3f}(\mu)\, O_i^{3f}(\mu)$$

$$= \ldots$$

We can use either 3f or 4f for WMEs

WMEs w/ 4-flavor operators

$$\langle f|H_W|i\rangle = \Sigma_i \underbrace{w_i^{4f}(\mu)}_{\text{pQCD}} \underbrace{\langle f|O_i^{4f}(\mu)|i\rangle}_{\text{LQCD}}$$



Main issue in lattice calculation is

Window problem: $\Lambda_{QCD} \ll \mu \ll a^{-1}$

→ resolved by step scaling, etc…

WMEs w/ 3-flavor operators

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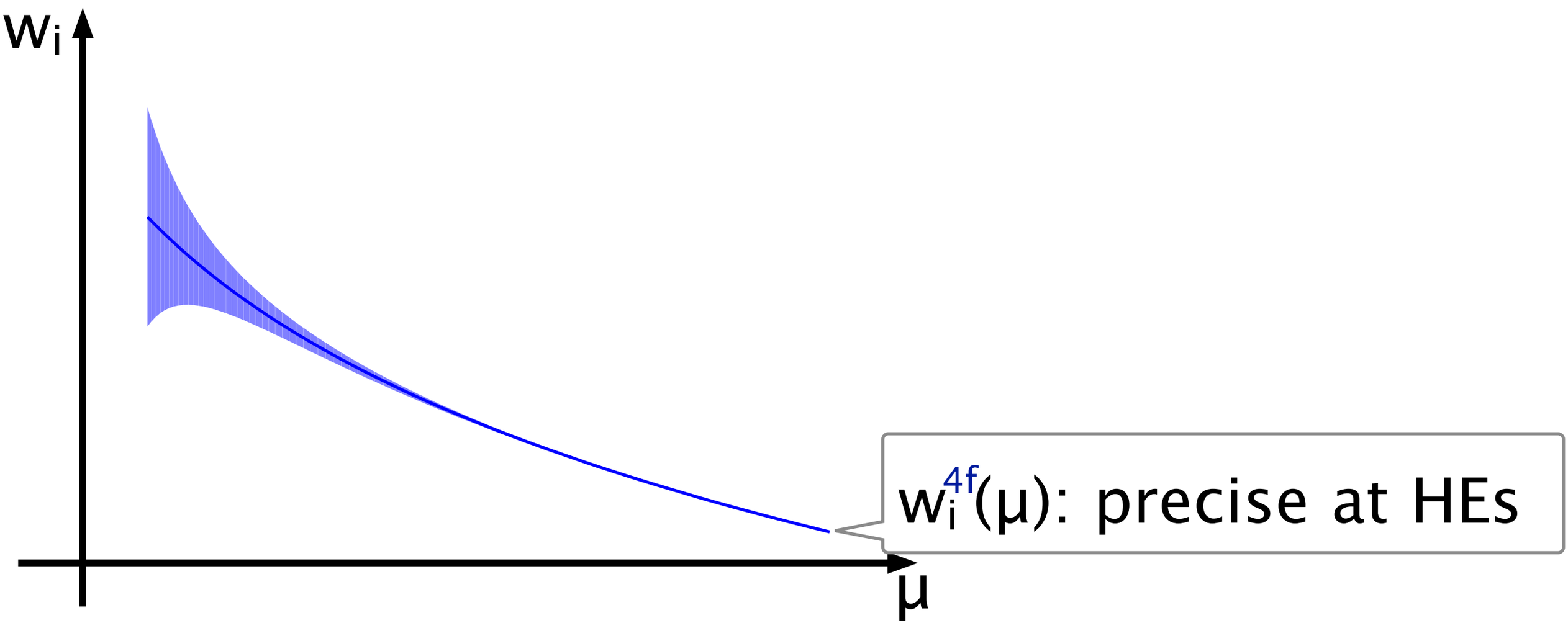

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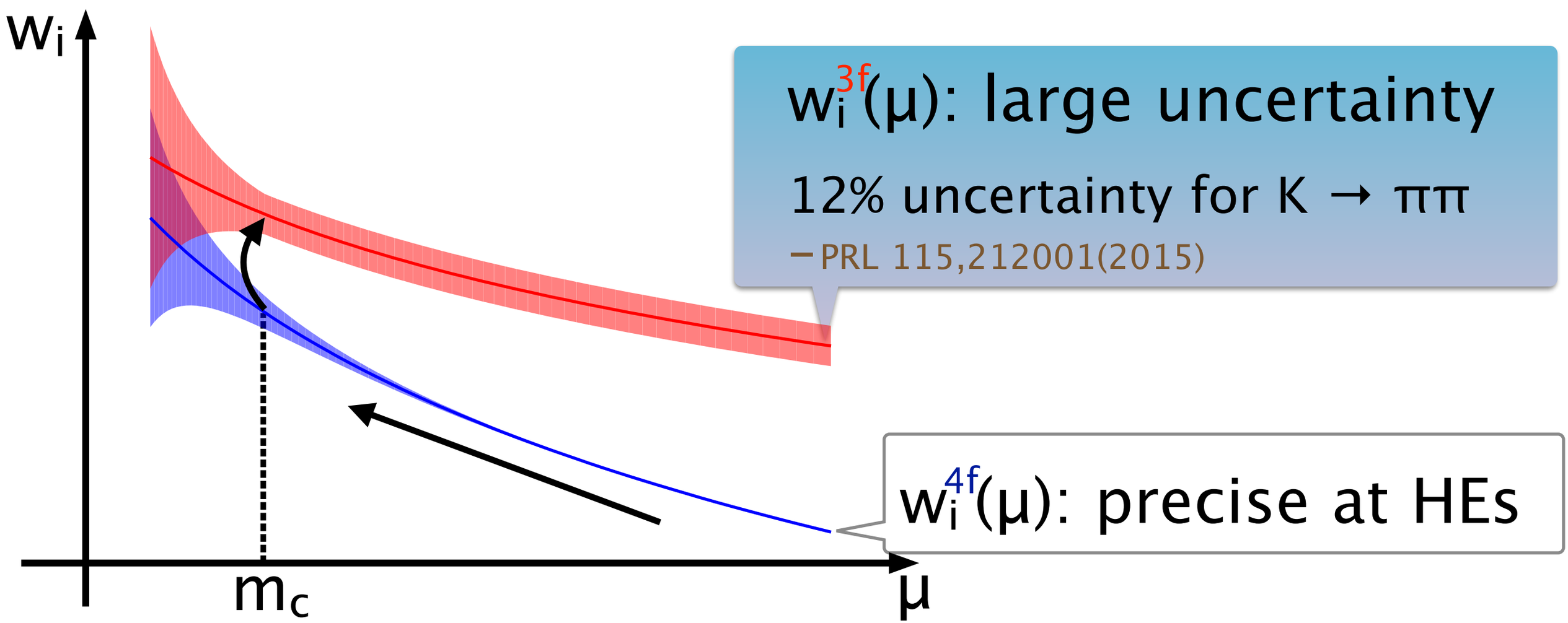
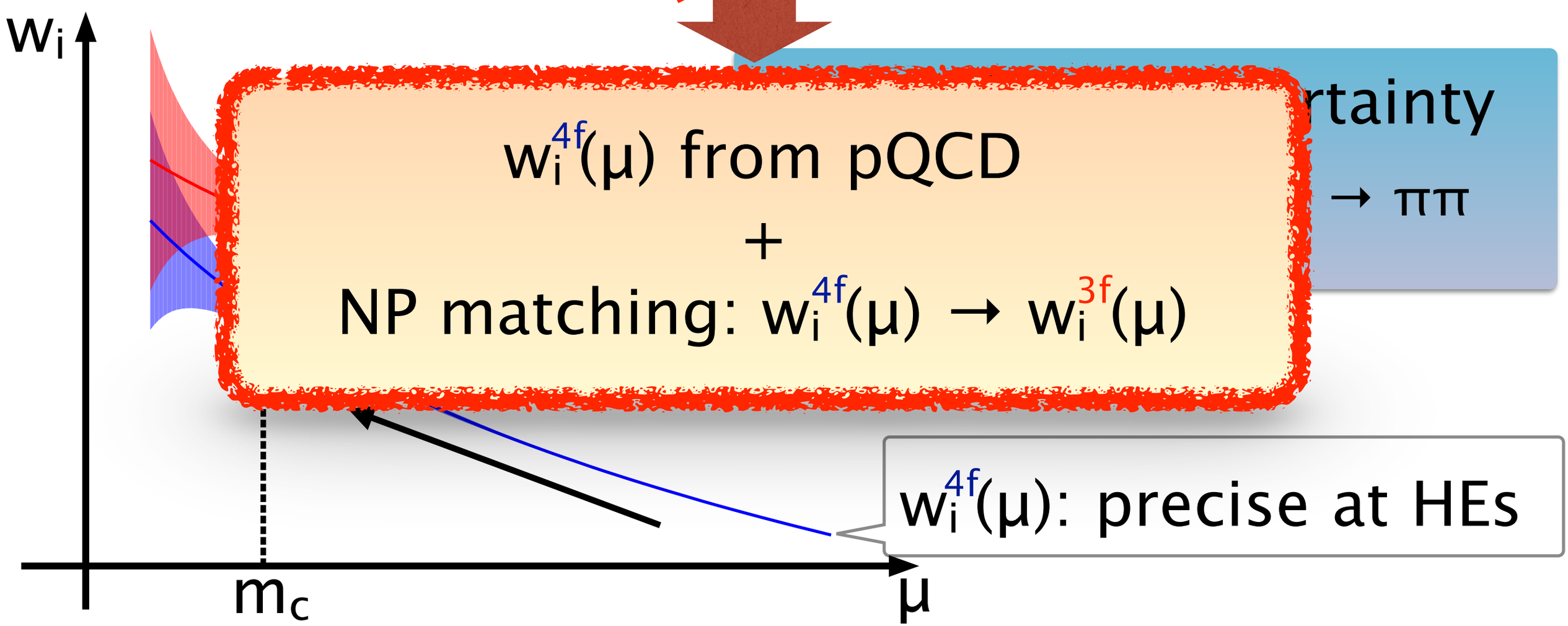


WMEs w/ 3-flavor operators

$$\langle f|H_W|i\rangle = \Sigma_i \, w_i^{3f}(\mu) \, \langle f|O_i^{3f}(\mu)|i\rangle$$

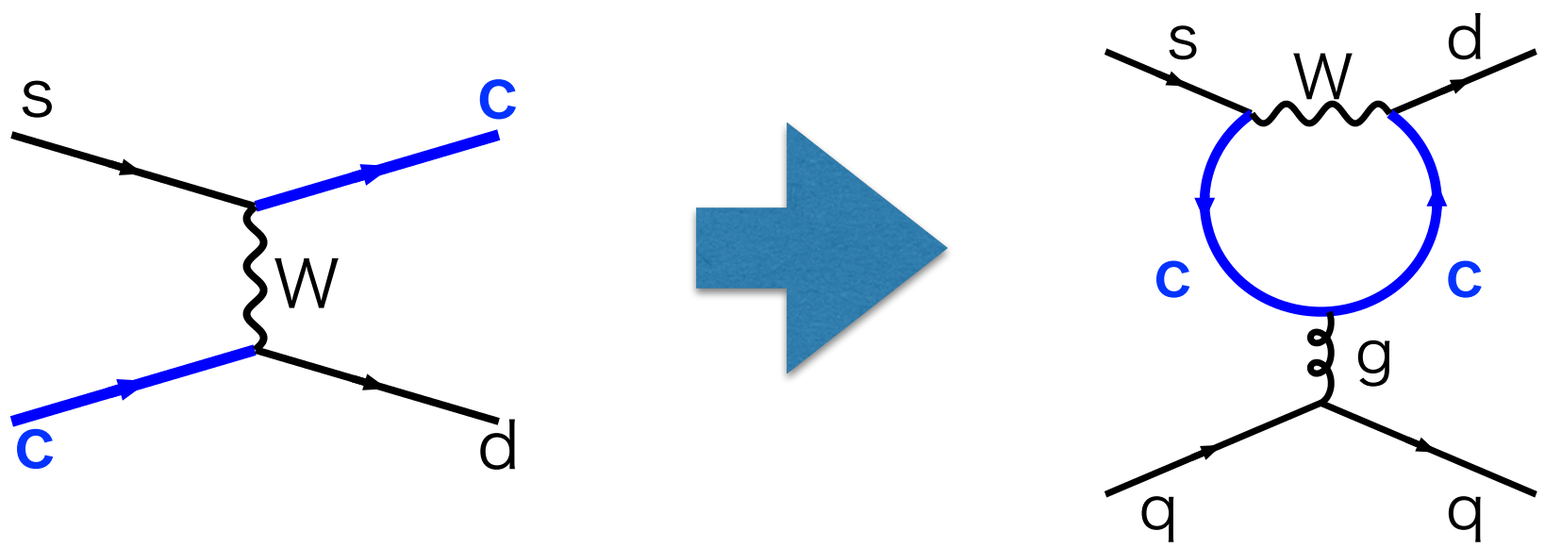
~~pQCD~~ LQCD



$w_i^{3f}(\mu) \neq w_i^{4f}(\mu)$?

- Of course sea charm effects $\Rightarrow w_i^{3f}(\mu) \neq w_i^{4f}(\mu)$
  - Maybe small difference → neglect in this work
- If O_i^{4f} contains charm…

Ex)



  - O_i^{4f}'s w/ charm turn to a combination of O_i^{3f}'s in $\mu \ll m_c$
  - $w_i^{3f} = w_i^{4f} + \Sigma_j M_{ij} w_j^{4f,c}$

$K \to \pi\pi$ by RBC/UKQCD (2015)

- 2+1 DWF / Iwasaki + DSDR gauge action
- a^{-1} = 1.38 GeV

  ⇒ too coarse to introduce charm

  ⇒ 3-flavor operators for MEs

  & perturbative 3/4-flavor matching

  ⇒ 12% systematic uncertainty

- NP matching (obtained from finer lattices) is desired

Outline

- Introduction
- NP matching strategy
  - Two-point functions in position space
  - Gauge invariant
- Technique for reducing discretization errors
  - Average over spheres
  - Enables an appropriate $a \to 0$ limit
- Result of exploratory calculation
  - $16^3 \times 32$
  - Statistical error significant

NP 4f–3f matching in position Sp.

- Redefinition:
  - $O_i^{3f} \longrightarrow Q_i$ $\qquad w_i^{3f} \longrightarrow w_i^{3f}$
  - $O_i^{4f} \longrightarrow (Q_i, P_\alpha)$ $\qquad w_i^{4f} \longrightarrow (w_i^{4f}, w_\alpha^{4f,c})$

Q_i : charmless

P_α : includes charm

- $P_\alpha \xrightarrow{\text{LDs}} \Sigma_i T_{\alpha i} Q_i$

$$H_W = \Sigma_i w_i^{3f} Q_i = \Sigma_i w_i^{4f} Q_i + \Sigma_\alpha w_\alpha^{4f,c} P_\alpha$$

$$\longrightarrow \Sigma_i (w_i^{4f} + \Sigma_\alpha w_\alpha^{4f,c} T_{\alpha i}) Q_i$$

$$\Rightarrow \underline{w_i^{3f} = (w_i^{4f} + \Sigma_\alpha T^T{}_{i\alpha} w_\alpha^{4f,c})}$$

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  Q_i : charmless

  P_α : includes charm

- $P_\alpha \xrightarrow{\text{LDs}} \Sigma_i T_{\alpha i} Q_i$
  - $\langle P_\alpha(x)\, \bar{O}(y)^\dagger \rangle \xrightarrow{|x-y| \gg 1/m_c} \Sigma_i T_{\alpha i} \langle Q_i(x)\, \bar{O}(y)^\dagger \rangle$
  - $\bar{O}$: any operator

  $\Rightarrow$ We consider

  $$\bar{O}(x) = Q_i(x) \quad i = 1, 2, \ldots$$

NP 4f–3f matching in position Sp.

$$\underbrace{\langle P_\alpha(x)\, Q_i(y)^\dagger \rangle}_{G^{PQ}_{\alpha i}(x-y)} = \Sigma_j\, T_{\alpha j}\, \underbrace{\langle Q_j(x)\, Q_i(y)^\dagger \rangle}_{G^{QQ}_{ji}(x-y)} \qquad (\,|x-y| \gg 1/m_c\,)$$

$$M_{i\alpha} = T_{\alpha i} = \Sigma_j\, (G^{QQ}(x-y))^{-1}_{ij}\, G^{QP}_{j\alpha}(x-y)$$

$$w_i^{3f} = w_i^{4f} + \Sigma_\alpha\, M_{i\alpha}\, w_\alpha^{4f,c}$$

- ☆ Gauge invariant & free from contact terms

$\Rightarrow$ can prevent mixing with irrelevant operators

Independence of Q_i

- $M_{i\alpha} = \Sigma_j (G^{QQ}(x-y))^{-1}_{ij} G^{QP}_{j\alpha}(x-y)$
- Inverse matrix $(G^{QQ}(x-y))^{-1}_{ij}$ exists

  ONLY IF <u>Q_i's are independent with each other</u>
  - Ex: $\Delta S = 1$ weak operators not the case!

Type	Q_i
current-current	$Q_1 = (\bar{s}_\alpha d_\alpha)_L (\bar{u}_\beta u_\beta)_L$ $Q_2 = (\bar{s}_\alpha d_\beta)_L (\bar{u}_\beta u_\alpha)_L$
QCD penguin	$Q_3 = (\bar{s}_\alpha d_\alpha)_L \sum_q^{3f} (\bar{q}_\beta q_\beta)_L$ $Q_4 = (\bar{s}_\alpha d_\beta)_L \sum_q^{3f} (\bar{q}_\beta q_\alpha)_L$ $Q_5 = (\bar{s}_\alpha d_\alpha)_L \sum_q^{3f} (\bar{q}_\beta q_\beta)_R$ $Q_6 = (\bar{s}_\alpha d_\beta)_L \sum_q^{3f} (\bar{q}_\beta q_\alpha)_R$
EW penguin	$Q_7 = \frac{3}{2}(\bar{s}_\alpha d_\alpha)_L \sum_q^{3f} e_q (\bar{q}_\beta q_\beta)_R$ $Q_8 = \frac{3}{2}(\bar{s}_\alpha d_\beta)_L \sum_q^{3f} e_q (\bar{q}_\beta q_\alpha)_R$ $Q_9 = \frac{3}{2}(\bar{s}_\alpha d_\alpha)_L \sum_q^{3f} e_q (\bar{q}_\beta q_\beta)_L$ $Q_{10} = \frac{3}{2}(\bar{s}_\alpha d_\beta)_L \sum_q^{3f} e_q (\bar{q}_\beta q_\alpha)_L$

3 relations among Q_i's

(Fierz sym. + simple comb.)

→ 7 independent operators

Representations of Weak Oprs.

- Representations of SU(3) x SU(3) chiral symmetry (n_L, n_R)
- 7 independent Q_i's
  - 1 in (27,1) representation
  - 4 in (8,1) representation
  - 2 in (8,8) representation

[RBC/UKQCD 2003]

- 4 independent P_α's
  - $(\bar{s}d)_L\,(\bar{c}c)_{L/R}$ (color-mixed and unmixed contractions)
  - all in (8,1)
- Only (8,1) associated with charm decoupling

$$P_\alpha^{(8,1)} \to \Sigma_i\, T_{\alpha i}^{(8,1)} Q_i^{(8,1)}$$

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Color trivialization by Fierz trf.

- Def: $(\bar{s}d)_L(\bar{q}q)_{R/L} = \bar{s}\gamma_\mu(1-\gamma_5)d \cdot \bar{q}\gamma_\mu(1\pm\gamma_5)q$

- Left-Left operators

$$(\bar{s}_\alpha d_\beta)_L(\bar{q}_\beta q_\alpha)_L = (\bar{s}_\alpha q_\alpha)_L(\bar{q}_\beta d_\beta)_L$$

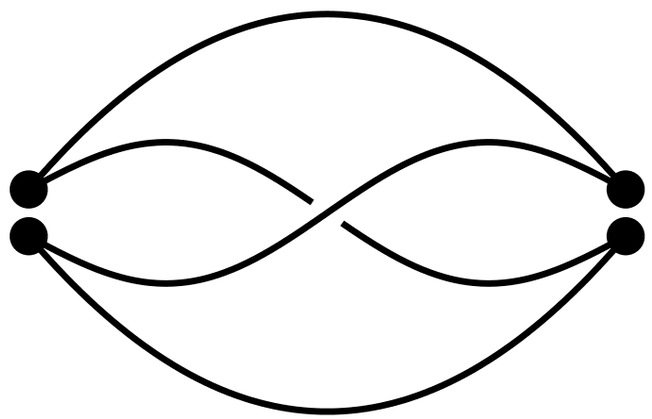
- Left-Right operators

$$(\bar{s}_\alpha d_\beta)_L(\bar{q}_\beta q_\alpha)_R = -2\bar{s}_\alpha(1+\gamma_5)q_\alpha \cdot \bar{q}_\beta(1-\gamma_5)d_\beta$$

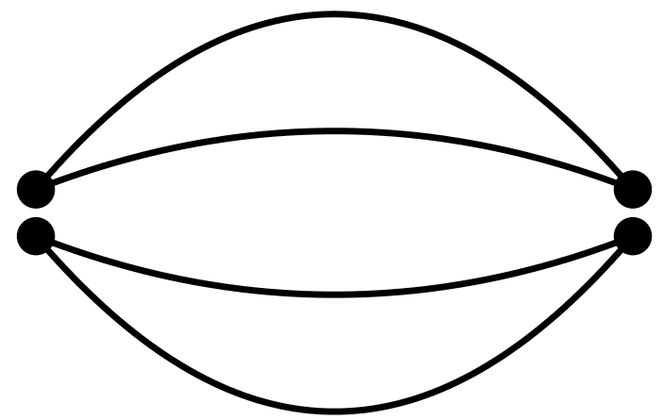
Contractions

- 4/3-flavor matching should be independent of m_{ud} & m_s

  ⇒ Calculate w/ SU(3) valence quarks + 1 heavier quark



6 contractions



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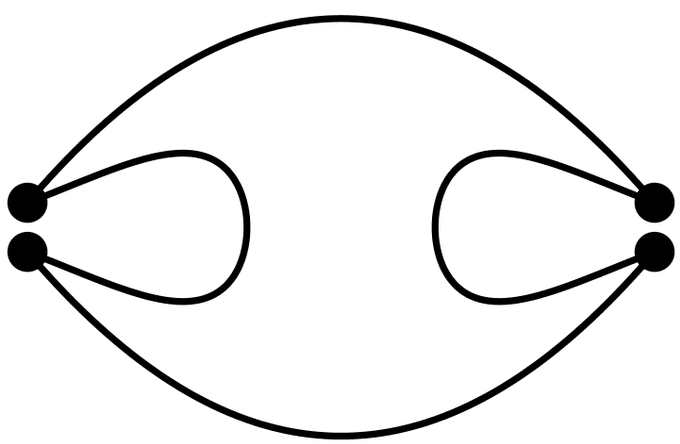
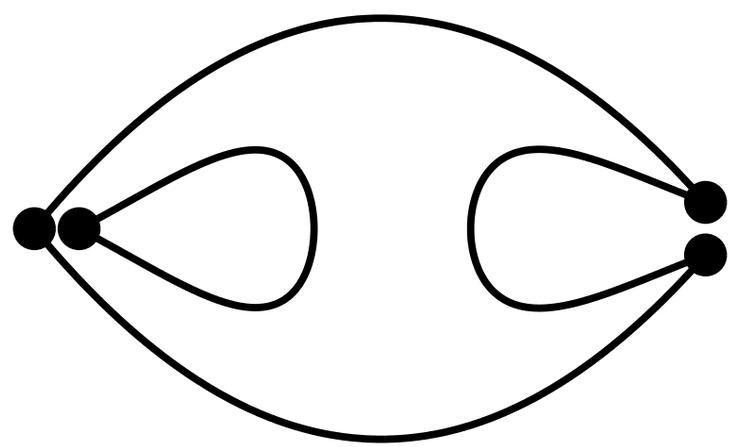
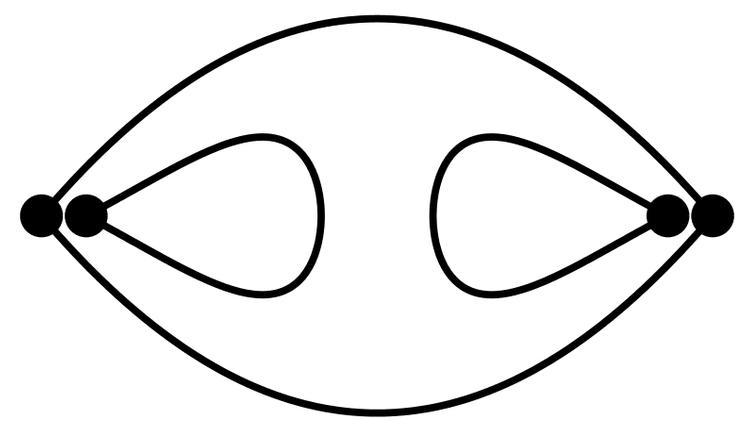


18 contractions



32 contractions



18 contractions

Subtraction of power divergence

- Loop diagram can contain power divergence
  - from power divergence of operators

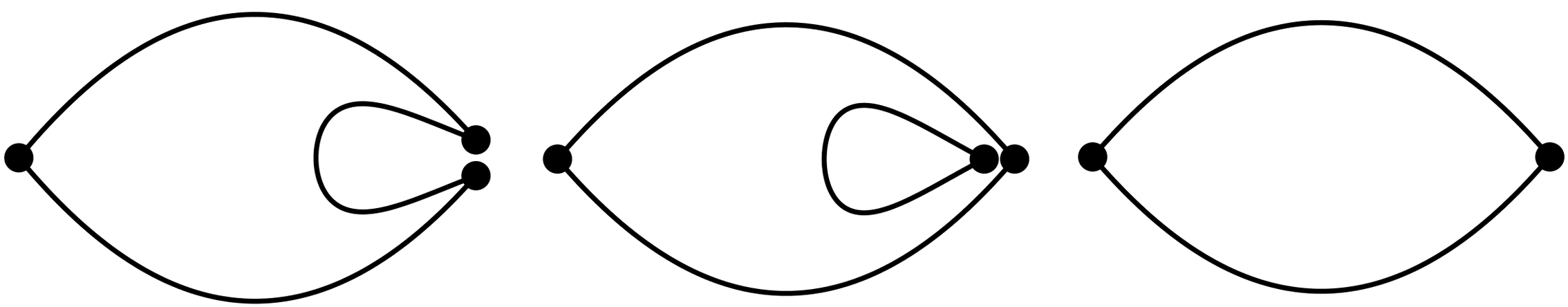
$$O_i \sim \frac{m_q}{a^2}$$

  - Eliminate by redefining

$$O_i' = O_i - C_- \bar{s}(1-\gamma_5)d - C_+ \bar{s}(1+\gamma_5)d$$

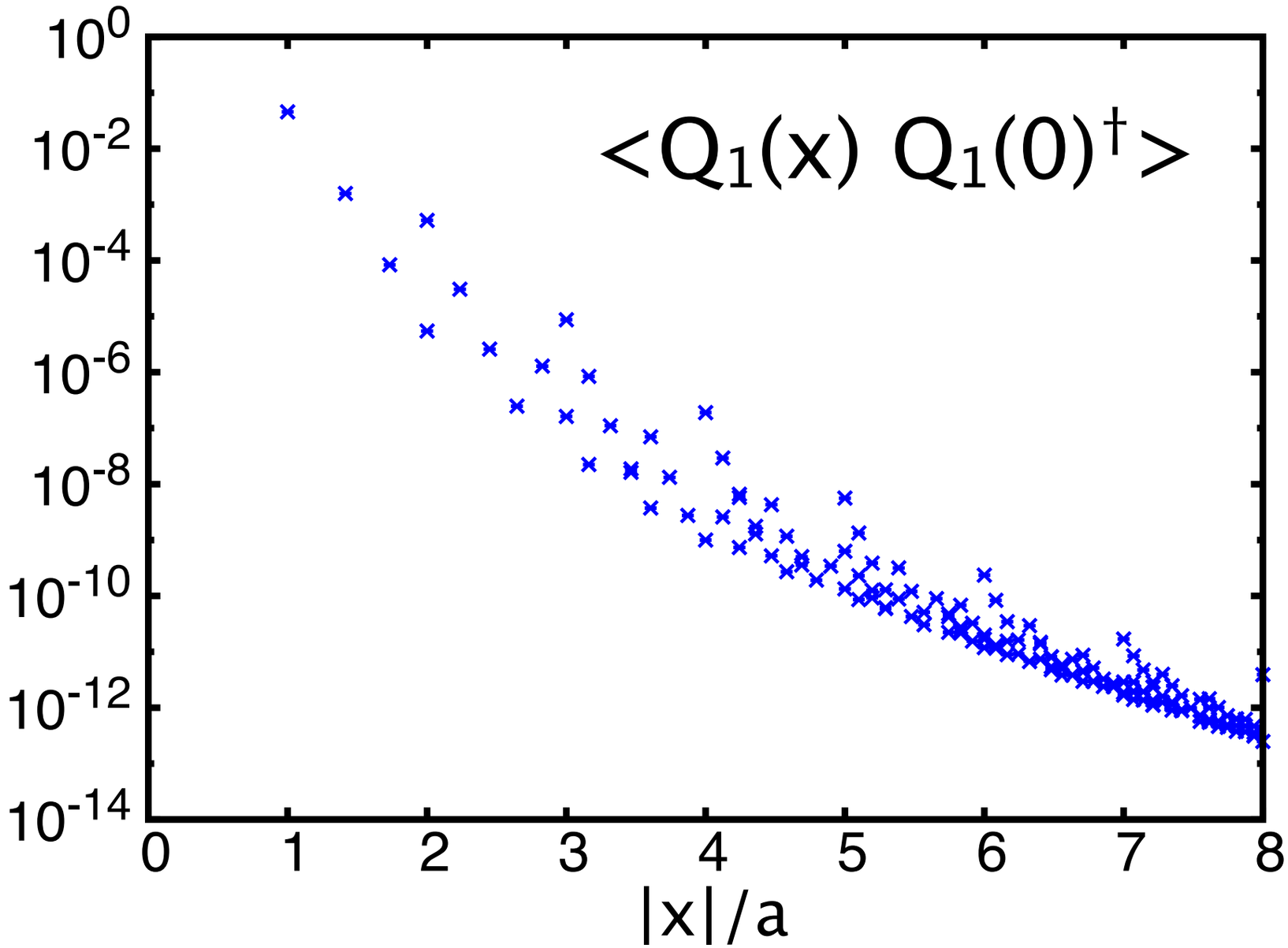
with a condition

$$\left\langle \bar{s}(1 \pm \gamma_5)d(x) \cdot O_i'(y)^\dagger \right\rangle \Big|_{x-y=x_0} = 0$$



12 contractions 12 contractions 3 contractions

Representative result



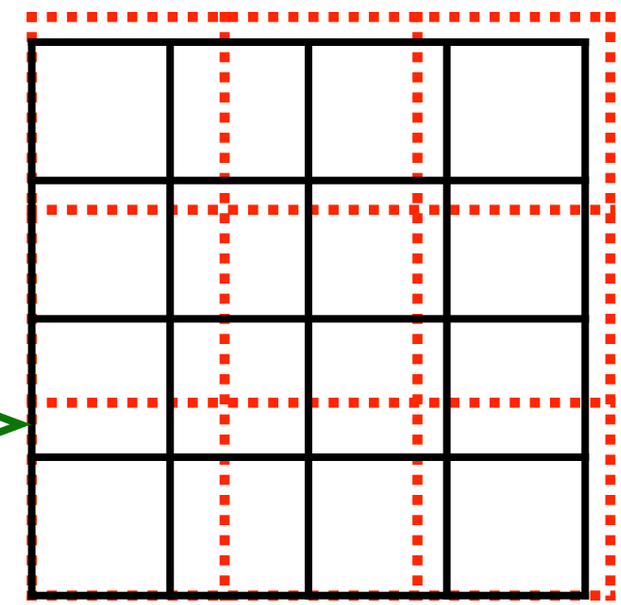
- $16^3 \times 32$
- $a^{-1} = 1.78$ GeV
- $m_{ud}^{val} = m_s^{val} = m_s^{sea}$
- Plotted in lattice units
- Unrenormalized

- Different lattice points distinguished ($(1,1,1,1) \neq (0,0,0,2)$)

How to a → 0?

- Our final goal: continuum limit of $M_{i\alpha}$
  - to apply it to our K → ππ result on 1.38 GeV lattice
- a-dependence of $M_{i\alpha}$ may depend on x

  - Need lattice cite w/ a mutual physical distance for each a for a → 0 limit
  - No or few such mutual distance



- We propose an idea to take continuum limit in a more appropriate way
  - Averaging correlators over sphere to get O(4)-symmetric ones at any physical distance |x|
  - Example for Z_m [MT & N. Christ, PRD99,014515]

Average over spheres

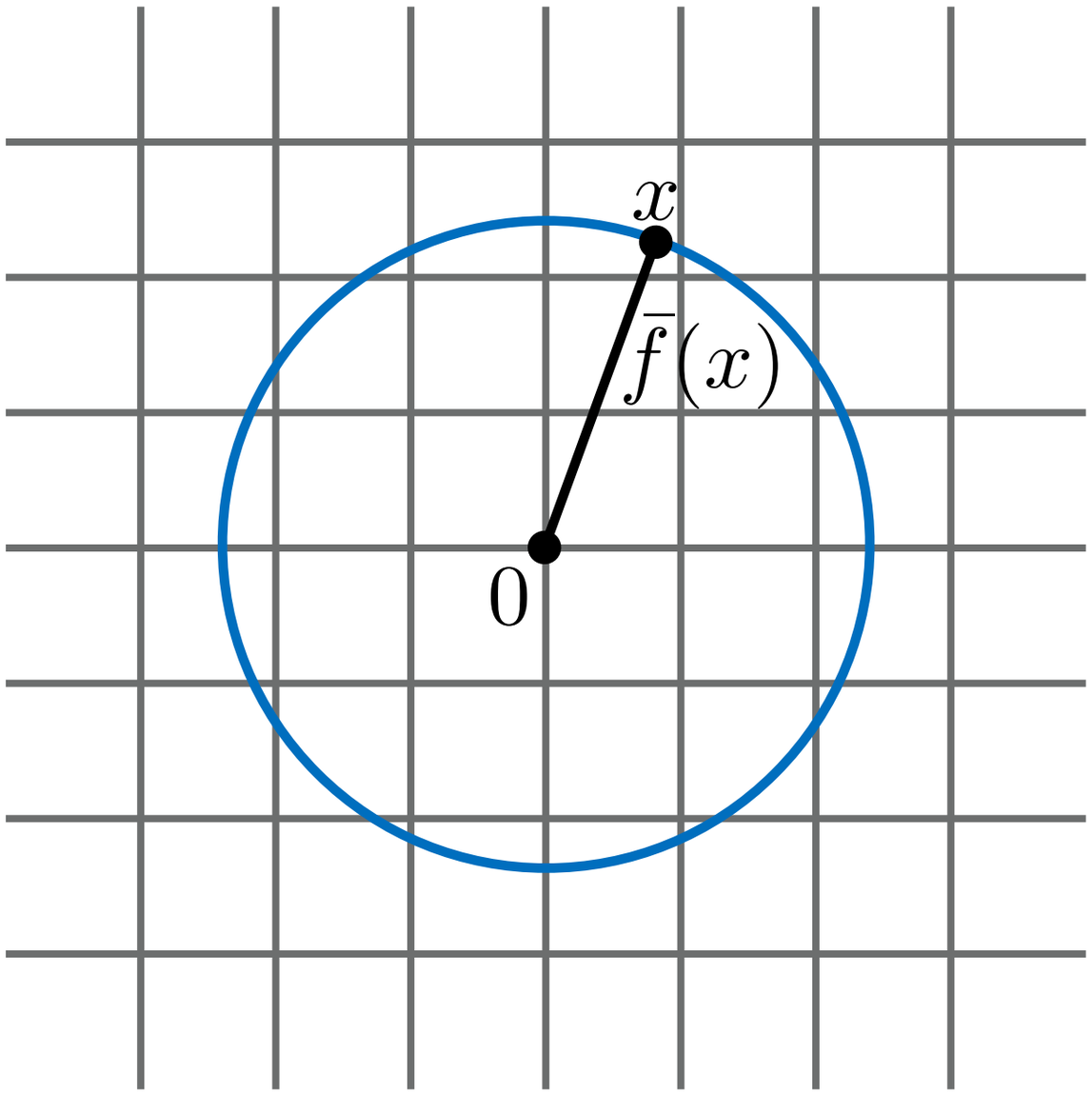
- Evaluate the value of a quantity at each 4d point from values at lattice points, with a guideline

$$\bar{f}(x) = \eta(f^{\mathrm{lat}}; x)$$

※ details in following slides

- Take the average over the sphere for each distance |x|

$$\hat{f}(|x|) = \frac{1}{2\pi^2} \oint_{S^3(|x|)} \mathrm{d}\Omega\, \bar{f}(x)$$



Potential $O(a^1)$ error (1-dim)

- Defs:
  - f_n: lattice value at site n
  - $f(x)$: "continuum limit" : $f_n = f(an) + O(a^2)$

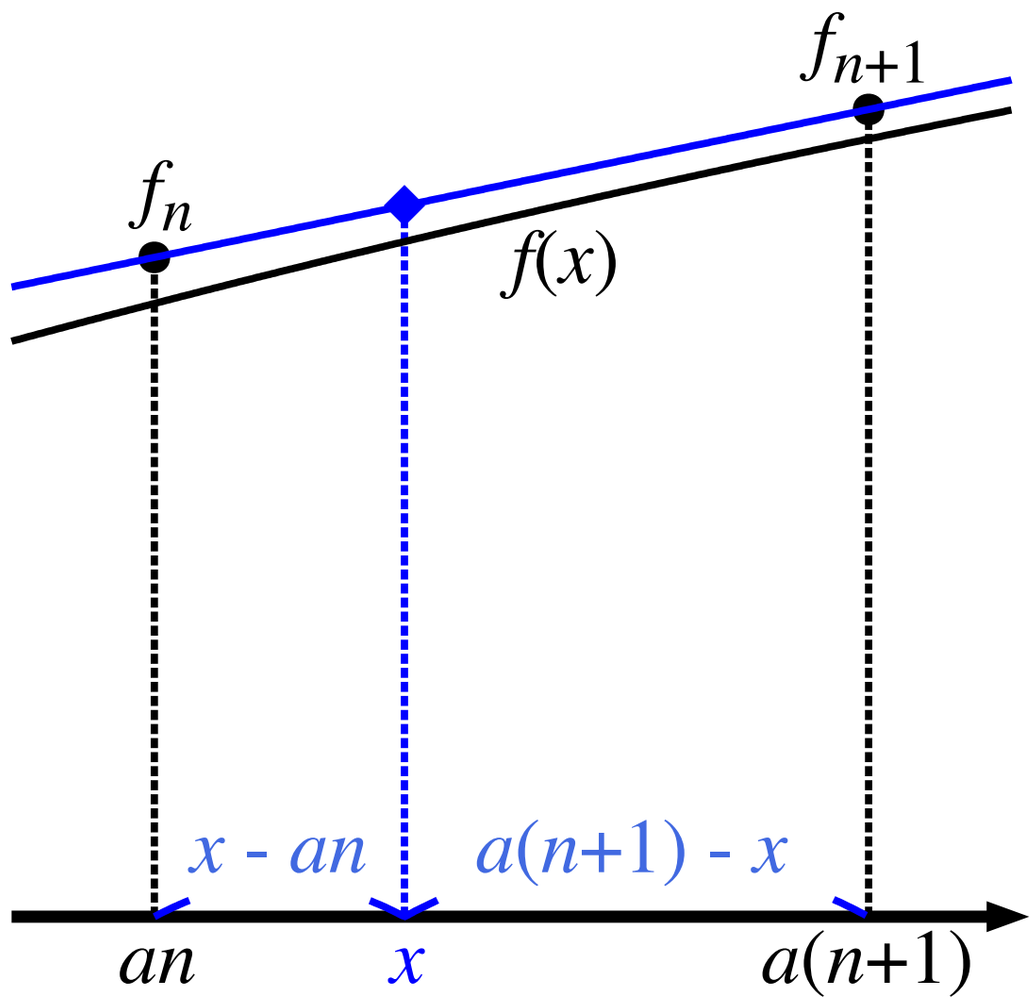
- Estimation $\bar{f}(x)$ should satisfy
  - $\bar{f}(x) = f(x) + O(a^2)$

- Potential $O(a^1)$ error in $\bar{f}(x)$
  - $f_n = f(an) + O(a^2)$

    $= f(x) + \underbrace{f'(x)\cdot(an-x)}_{O(a^1)} + O(a^2)$
  - $\bar{f}(x)$ is calculated using f_n's $\Rightarrow$ $O(a^1)$ can appear
  - Balanced combination needed

Evaluation of $\bar{f}(x)$ (1-dim)

- Linear interpolation

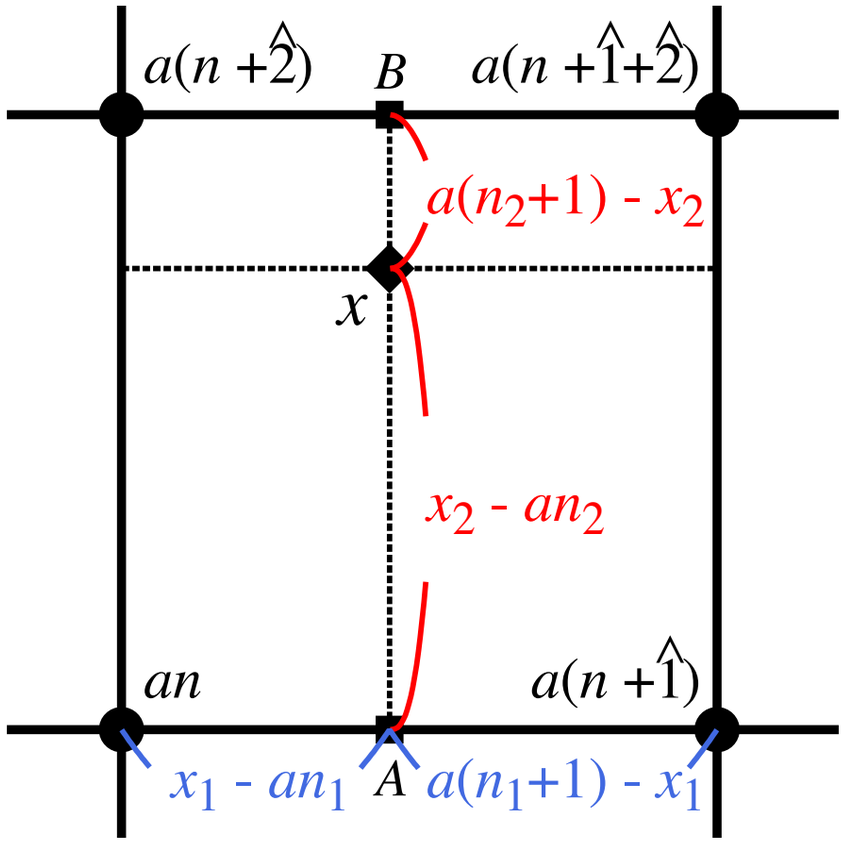


$$\bar{f}(x) = \frac{(a(n+1)-x)f_n + (x-an)f_{n+1}}{a} = f(x) + O(a^2)$$

Accurate up to $O(a^2)$

Evaluation of $\bar{f}(x)$ (2-dim)

- Bilinear interpolation



$$\bar{f}(x) = \frac{(a(n_2+1) - x_2)\bar{f}(A) + (x_2 - an_2)\bar{f}(B)}{a}$$

$$= a^{-2}\begin{pmatrix} a(n_1+1) - x_1 & x_1 - an_1 \end{pmatrix} \begin{pmatrix} f_n & f_{n+\hat{2}} \\ f_{n+\hat{1}} & f_{n+\hat{1}+\hat{2}} \end{pmatrix} \begin{pmatrix} a(n_2+1) - x_2 \\ x_2 - an_2 \end{pmatrix}$$

$$= f(x) + O(a^2)$$

Evaluation of $\bar{f}(x)$ (4-dim)

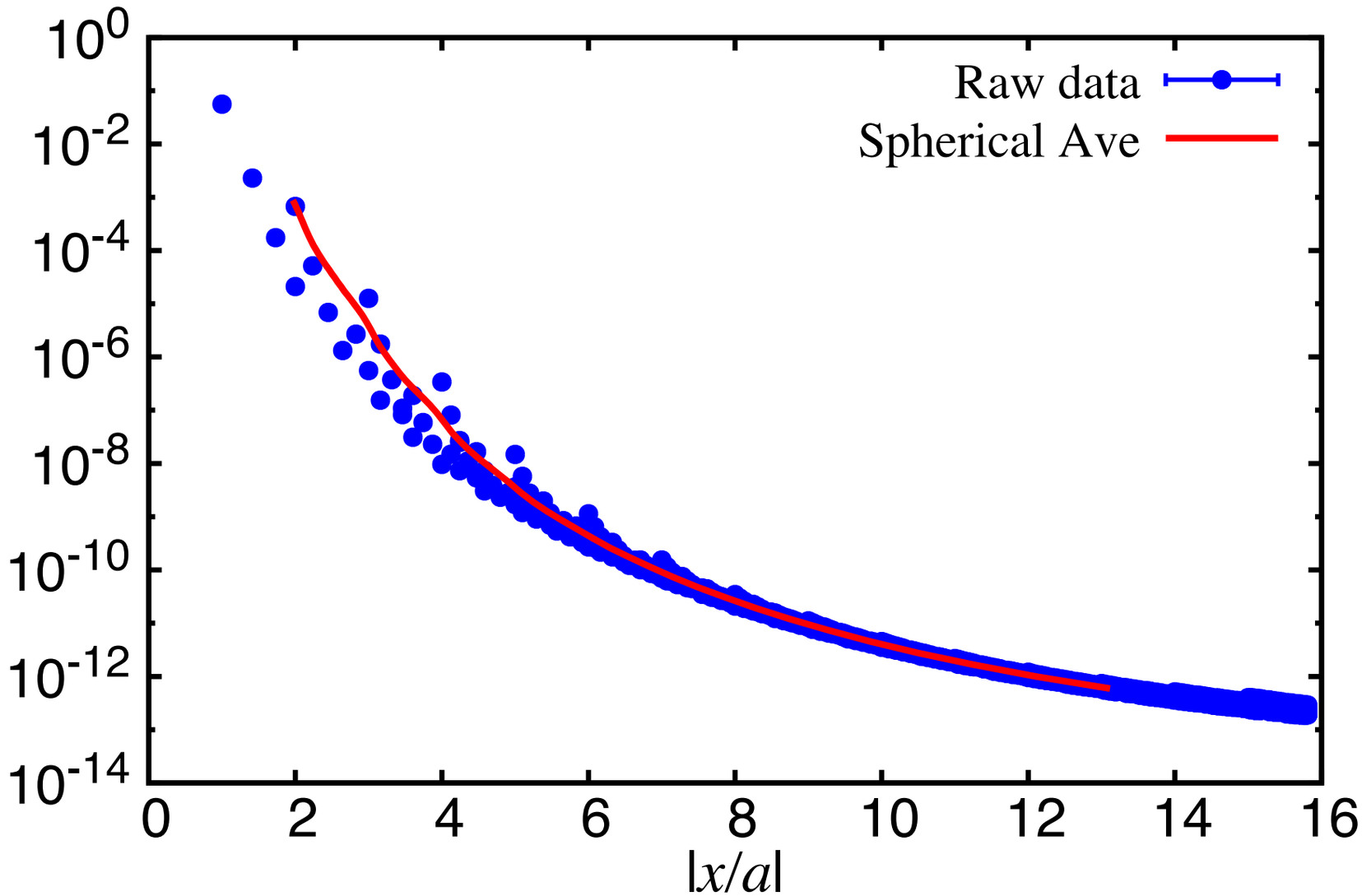
- Quadrilinear interpolation

$$\bar{f}(x) = a^{-4} \sum_{i,j,k,l=0}^{1} \Delta_{1,i}\Delta_{2,j}\Delta_{3,k}\Delta_{4,l} f_{n+i\hat{1}+j\hat{2}+k\hat{3}+l\hat{4}}$$

$$\Delta_{\mu,i} = |a(n_\mu + 1 - i) - x_\mu|$$

– Accurate up to $O(a^2)$

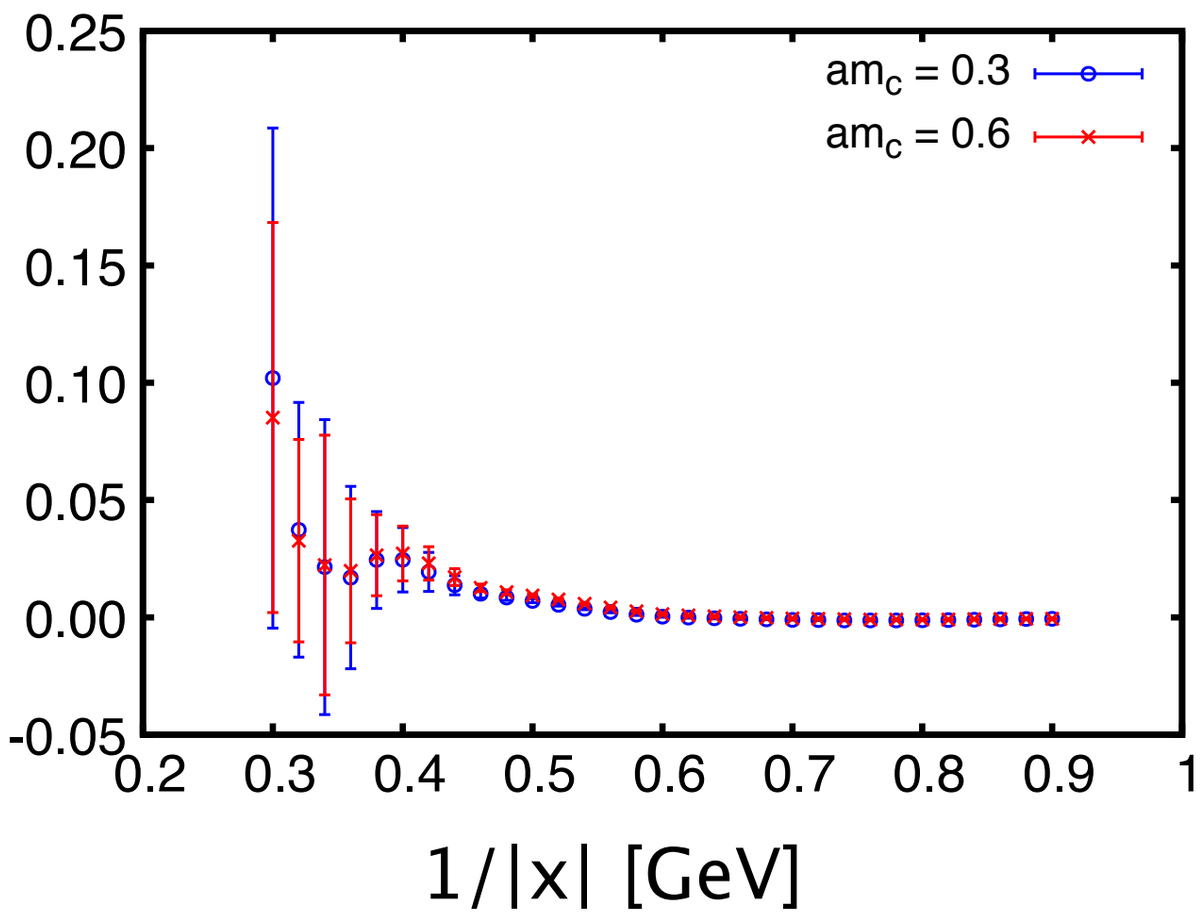
Spherical Ave. for 2pt func



Results for M_{ij}

$$M_{i\alpha} = \Sigma_j \, (G^{QQ}(x))^{-1}_{ij} \, G^{QP}_{j\alpha}(x)$$

– should be independent of x at LDs $|x| \gg 1/m_c$



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- $a^{-1} = 1.78$ GeV
- 88 confs in 3,500 MD time
- $m_{ud}^{val} = m_s^{val} = m_s^{sea}$
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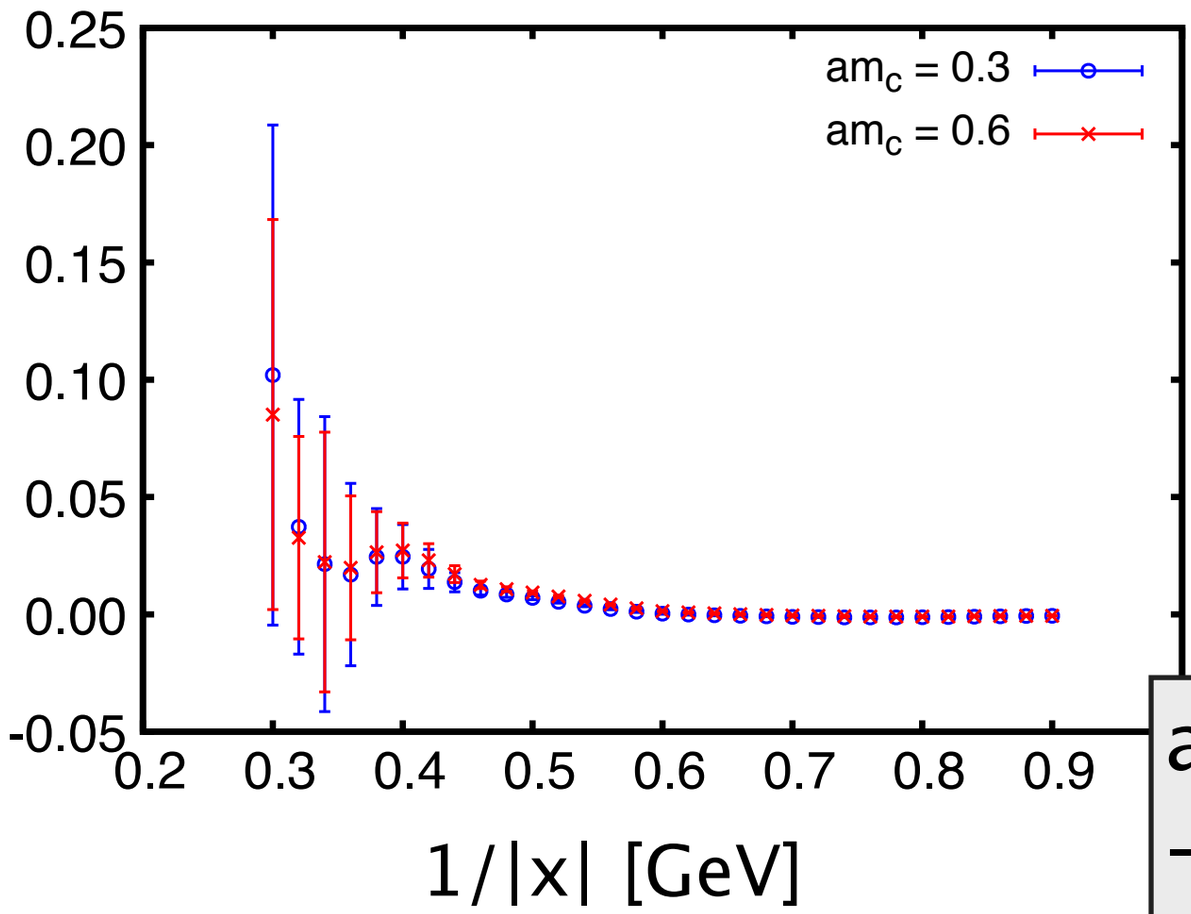

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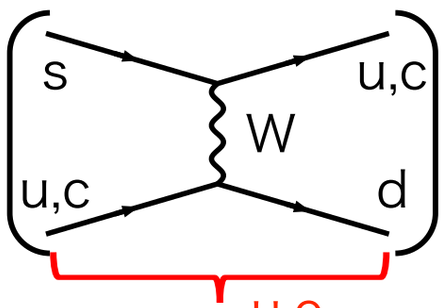
around $1/|x| = 0.4$ GeV…
– Large statistical error
– Less clear plateau

Significance of matching

$$w_i^{3f} = w_i^{4f} + \Sigma_\alpha M_{i\alpha}\, w_\alpha^{4f,c}$$

- Naive estimation of $w_\alpha^{4f,c}$

w 

= O(1)

$Q_2^{u,c} = (su)_L\,(ud)_L,\ (sc)_L\,(cd)_L$

w(others) = $O(\alpha_s)$

  - $w_2^{4f,c} = O(1)$
  - $w_{1,3,4}^{4f,c} = O(\alpha_s)$

- M_{i2} : most sensitive

$$P_1^{(8,1)} = Q_1^c$$

$$P_2^{(8,1)} = Q_2^c$$

$$P_3^{(8,1)} = (\bar{s}_i d_i)_L (\bar{c}_j c_j)_R$$

$$P_4^{(8,1)} = (\bar{s}_i d_j)_L (\bar{c}_j c_i)_R$$

$M_{i\alpha}$ at $|x-y|^{-1} = 400$ MeV

$$w_i^{3f} = w_i^{4f} + \Sigma_\alpha M_{i\alpha} w_\alpha^{4f,c}$$

i \ α	1	2	3	4
1	$-4.0(3.1) \times 10^{-3}$	$2.01(65) \times 10^{-2}$	$-3.8(3.5) \times 10^{-3}$	$2.7(1.2) \times 10^{-2}$
2	$1.1(1.1) \times 10^{-3}$	$-4.6(3.3) \times 10^{-3}$	$-3(11) \times 10^{-4}$	$-5.4(2.6) \times 10^{-3}$
3	$-1.02(51) \times 10^{-2}$	$3.0(1.7) \times 10^{-2}$	$-1(36) \times 10^{-4}$	$1.1(1.1) \times 10^{-2}$
4	$5.4(2.3) \times 10^{-3}$	$-1.47(83) \times 10^{-2}$	$10(21) \times 10^{-4}$	$-6.7(6.0) \times 10^{-3}$

- Need to improve statistics

$M_{i\alpha}$ at $|x-y|^{-1} = 400$ MeV

$$w_i^{3f} = w_i^{4f} + \Sigma_\alpha M_{i\alpha} w_\alpha^{4f,c}$$

α

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Multiplied by $w_2^{4f,c} \sim 1$

- Need to improve statistics

Main error source

- $G^{QQ}(x-y)^{-1}$ at $|x-y|^{-1}$ = 400 MeV

	1	2	3	4
1	$2.958(44) \times 10^8$	$9.0(1.4) \times 10^6$	$-4(10) \times 10^6$	$-3.4(5.0) \times 10^6$
2	$9.0(1.4) \times 10^6$	$2.510(10) \times 10^8$	$3.6(3.6) \times 10^6$	$2.5(1.5) \times 10^6$
3	$-4(10) \times 10^6$	$3.6(3.6) \times 10^6$	$6.06(11) \times 10^8$	$-2.296(63) \times 10^8$
4	$-3.4(5.0) \times 10^6$	$2.5(1.5) \times 10^6$	$-2.296(63) \times 10^8$	$1.488(34) \times 10^8$

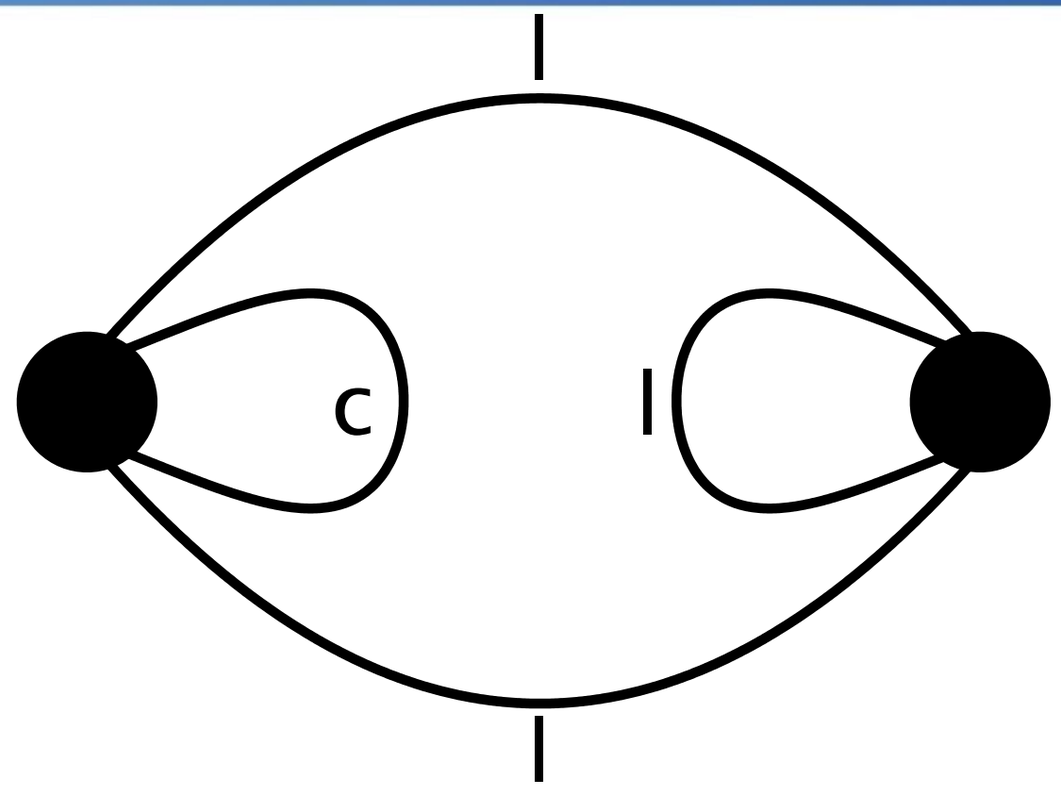
  - Good precision

- $G^{QP}(x-y)$ at $|x-y|^{-1}$ = 400 MeV

	1	2	3	4
1	$-1.4(1.1) \times 10^{-11}$	$6.9(2.2) \times 10^{-11}$	$-1.3(1.2) \times 10^{-11}$	$9.3(4.0) \times 10^{-11}$
2	$4.9(4.6) \times 10^{-12}$	$-2.1(1.3) \times 10^{-11}$	$-9(42) \times 10^{-13}$	$-2.5(1.2) \times 10^{-11}$
3	$-8.4(7.7) \times 10^{-12}$	$3.2(2.2) \times 10^{-11}$	$4.9(5.5) \times 10^{-12}$	$6(18) \times 10^{-12}$
4	$2.28(94) \times 10^{-11}$	$-4.8(3.7) \times 10^{-11}$	$1.4(1.6) \times 10^{-11}$	$-3.3(4.6) \times 10^{-11}$

  - Worse precision

$G^{QP}(x-y)$



- A2A propagator either c or l
  - Pure random noise sources for this tentative calc.
  - A2A l may be improved by using low modes & deflated CG
    Deflated CG w/ $m_l = m_s$: 2,3x faster
  - A2A may be improved by sparse noise sources

    [RBC/UKQCD 2016, HVP]

Summary

- Purpose

$$w_i^{4f}, w_\alpha^{4f} \xrightarrow{\text{NP method}} w_i^{3f} \quad (\text{for } K \to \pi\pi)$$

- Idea

$$w_i^{3f} Q_i \to w_i^{4f} Q_i + w_\alpha^{4f,c} P_\alpha$$

$$\longrightarrow \quad w_i^{3f} = w_i^{4f} + \sum_{j,\alpha} (G^{QQ}(x-y))^{-1}_{ij} G^{QP}_{j\alpha}(x-y) w_\alpha^{4f,c}$$

- Gauge invariant — privent mixing w/ irrelevant Oprs.
- Large statistical error in an exploratory calculation on 16^{-1} lattice at a^{-1} = 1.78 GeV
  - Trying to improve using Lanczos A2A, sparse noise, ...
- Main calculation will be at a^{-1} = 2.35 GeV, 3.15 GeV, ...

Previous effort in mom Sp.

- Condition

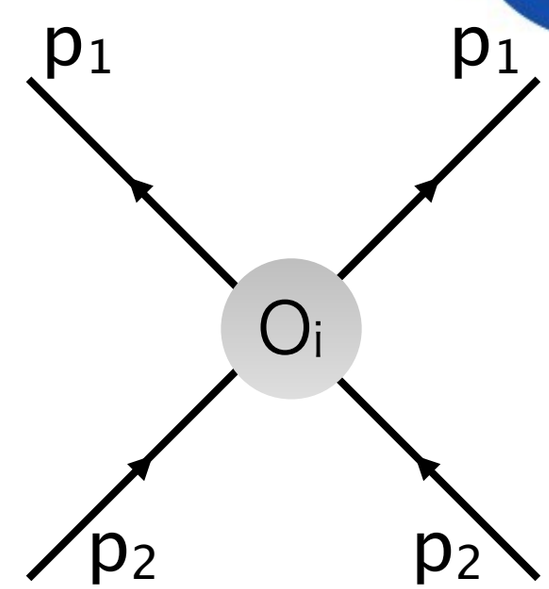
$$P^{abcd}_{\alpha\beta\gamma\delta}\ \Lambda^{abcd}_{\alpha\beta\gamma\delta}(O_i^{3f}(\mu);p_1,p_2)\ w_i^{3f}(\mu)$$

$$\|$$

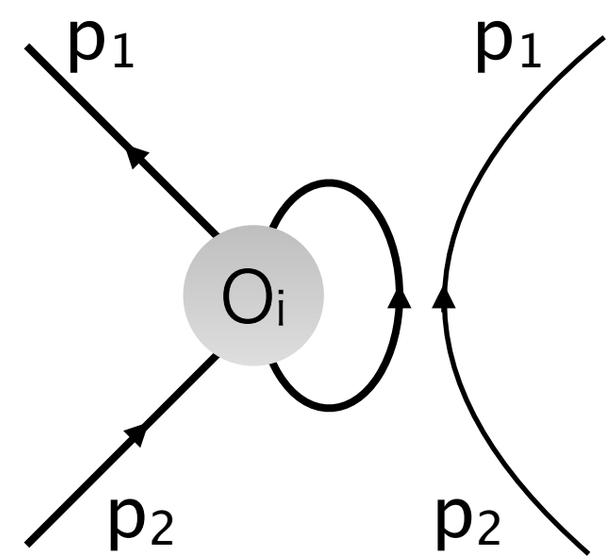
$$P^{abcd}_{\alpha\beta\gamma\delta}\ \Lambda^{abcd}_{\alpha\beta\gamma\delta}(O_i^{4f}(\mu);p_1,p_2)\ w_i^{4f}(\mu)$$

G-fixed amputated Green's function

flavor, color and spin projector



- Condition valid in $|p_{1,2}| \ll m_c$
- Statistical error
  - $|p_{1,2}|$ = 1.2 GeV → 10%
  - $|p_{1,2}|$ = 0.6 GeV → 50%
-



Why mom procedure so bad?

- Gauge fixing
  - Large Gribov noise
    - Gauge condition does not have a unique solution on the gauge orbit
    - Gauge-dependent quantities have some ambiguity
  - Mixing with gauge-noninvariant operators
- Off-shell condition
  - Mixing with operators that vanish by EoM

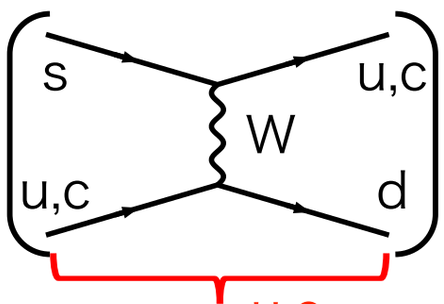
☆ All significant at small $p_{1,2}$

☆ Position-space procedure is free from all of these

Significance of matching

$w_i^{3f} = w_i^{4f} + \Sigma_\alpha M_{i\alpha} w_\alpha^{4f,c}$

- Naive estimation of w_i^{4f}, $w_\alpha^{4f,c}$

w 

= O(1)

$Q_2^{u,c} = (su)_L\,(ud)_L,\ (sc)_L\,(cd)_L$

w(others) = $O(\alpha_s)$

- $w_{1,2}^{4f}$, $w_2^{4f,c}$ = O(1)
- $w_{3,4}^{4f}$, $w_{1,3,4}^{4f,c}$ = $O(\alpha_s)$

- M_{i2} : most sensitive

$$Q_1^{(8,1)} = -\frac{1}{\sqrt{10}} Q_1^u + \frac{1}{\sqrt{10}} Q_2^u + \frac{2}{\sqrt{10}} Q_3$$

$$Q_2^{(8,1)} = \frac{1}{\sqrt{2}} Q_1^u - \frac{1}{\sqrt{2}} Q_2^u$$

$$Q_3^{(8,1)} = Q_5$$

$$Q_4^{(8,1)} = Q_6$$

$$P_1^{(8,1)} = Q_1^c$$

$$P_2^{(8,1)} = Q_2^c$$

$$P_3^{(8,1)} = (\bar{s}_i d_i)_L (\bar{c}_j c_j)_R$$

$$P_4^{(8,1)} = (\bar{s}_i d_j)_L (\bar{c}_j c_i)_R$$