$K_L - K_S$ mass difference 
with Lattice QCD at physical masses

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$K^0 - \bar{K}^0$ Mixing and $\Delta m_K$

$K^0(S = -1)$ and $\bar{K}^0(S = +1)$ mix through second order weak interactions:

$$i \frac{d}{dt} \begin{pmatrix} K^0(t) \\ \bar{K}^0(t) \end{pmatrix} = (M - \frac{i}{2} \Gamma) \begin{pmatrix} K^0(t) \\ \bar{K}^0(t) \end{pmatrix}, \quad (1)$$

Long-lived ($K_L$) and short-lived ($K_S$) are the two eigenstates:

$$K_S \approx \frac{K^0 - \bar{K}^0}{\sqrt{2}}, \quad K_L \approx \frac{K^0 + \bar{K}^0}{\sqrt{2}}. \quad (2)$$

$$\Delta m_K \equiv m_{K_L} - m_{K_S} = 2ReM_{0\bar{0}}$$
Physics Motivation

\[ \Delta m_K \equiv m_{K_L} - m_{K_S} = 2 \text{Re} M_{00} \]

- This quantity is:
  - **Tiny, sensitive to new physics**: FCNC via 2nd order weak interaction, precisely measured
    \[ \Delta m_{K,\text{exp}} = 3.483(6) \times 10^{-12} \text{ MeV} \]
  - Significant contribution from scale of \( m_c \) (GIM mechanism)
  - **Appears difficult to compute from QCD perturbation theory**: strong coupling at \( m_c \) scale; significant contributions from NNLO
    

- Lattice QCD:
  - from first principles
  - non-perturbative
  - systematic errors (FV, finite \( a \), etc) could be controlled
From Correlators to $\Delta m^\text{lat}_K$

- $\Delta m_K$ is given by:
  \[ \Delta m_K \equiv m_{K_L} - m_{K_S} = 2P \sum_n \frac{\langle \bar{K}^0 | H_W | n \rangle \langle n | H_W | K^0 \rangle}{m_K - E_n} \]  

- What we measure on lattice are:
  \[ G(t_1, t_2, t_i, t_f) \equiv \langle 0 | T \{ \bar{K}^0(t_f) H_W(t_2) H_W(t_1) K^0(t_i) \} | 0 \rangle \]  

\[ \rightarrow G(\delta) = N_K^2 e^{-m_K(t_f - t_i)} \sum_n \langle \bar{K}^0 | H_W | n \rangle \langle n | H_W | K^0 \rangle e^{(m_K - E_n)\delta} \]
Extract $\Delta m_K$ from Double-integrated Correlators

- The double-integrated correlator is defined as:

$$A \equiv \frac{1}{2!} \sum_{t_2 = t_a}^{t_b} \sum_{t_1 = t_a}^{t_b} \langle 0 | T \{ \bar{K}^0(t_f) H_W(t_2) H_W(t_1) K^0(t_i) \} | 0 \rangle$$  \hspace{1cm} (5)

- If we insert a complete set of intermediate states, we find:

$$A = N_K^2 e^{-m_K(t_f - t_i)} \sum_n \frac{\langle \bar{K}^0 | H_W | n \rangle \langle n | H_W | K^0 \rangle}{m_K - E_n} \left\{ -T + \frac{e^{(m_K - E_n) T} - 1}{m_K - E_n} \right\}$$  \hspace{1cm} (6)

with $T \equiv t_b - t_a + 1$. 

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Extract $\Delta m_K$ from Single-integrated Correlators

The single-integrated correlator is defined as:

$$A^s(t, T) \equiv \frac{1}{2!} \sum_{t_1 = t - T}^{t + T} \langle 0 | T \{ \bar{K}^0(t_f) H_W(t_1) H_W(t) K^0(t_i) \} | 0 \rangle \quad (7)$$

If we insert a complete set of intermediate states, we find:

$$A^s = N_K^2 e^{-m_K (t_f - t_i)} \sum_n \frac{\langle \bar{K}^0 | H_W | n \rangle \langle n | H_W | K^0 \rangle}{m_K - E_n} \left( -1 + e^{(m_K - E_n)(T+1)} \right)$$

$$\quad (8)$$
Subtraction of the light states

- Either Double- or Single-integrated Method requires subtraction of the terms from light states:

\[
A = N_K^2 e^{-m_K(t_f-t_i)} \sum_n \frac{\langle \bar{K}^0 | H_W | n \rangle \langle n | H_W | K^0 \rangle}{m_K - E_n} \left\{ -T + \frac{e^{(m_K-E_n)T} - 1}{m_K - E_n} \right\}
\]

(9)

\[
A^s = N_K^2 e^{-m_K(t_f-t_i)} \sum_n \frac{\langle \bar{K}^0 | H_W | n \rangle \langle n | H_W | K^0 \rangle}{m_K - E_n} \left\{ -1 + e^{(m_K-E_n)(T+1)} \right\}
\]

(10)

- For \(|n\rangle\) (in our case \(|0\rangle, |\pi\rangle, |\eta\rangle, |\pi\rangle\) with \(E_n < m_K\) or \(E_n \sim m_K\): the exponential terms will be significant. We can:
  - freedom of adding \(c_s \bar{s} d, c_p \bar{s} \gamma^5 d\) operators to the weak Hamiltonian
  - Here we choose:
    \[
    \langle 0 | H_W - c_p \bar{s} \gamma_5 d | K^0 \rangle = 0, \langle \eta | H_W - c_s \bar{s} d | \bar{K}^0 \rangle = 0
    \]
  - subtract contributions from other states \(|\pi\rangle, |\pi\pi\rangle\) explicitly
Operators of $\Delta m^\text{lat}_K$ calculation

- The $\Delta S = 1$ effective Weak Hamiltonian:

$$H_W = \frac{G_F}{\sqrt{2}} \sum_{q,q'=u,c} V_{qd} V^*_{q's} (C_1 Q_{1qq'} + C_2 Q_{2qq'})$$  \quad (11)$$

where the $Q_{iqq'}$ are current-current opearators, defined as:

$$Q_{1qq'} = (\bar{s}_i \gamma^\mu (1 - \gamma^5) d_i)(\bar{q}_j \gamma^\mu (1 - \gamma^5) q'_j)$$

$$Q_{2qq'} = (\bar{s}_i \gamma^\mu (1 - \gamma^5) d_j)(\bar{q}_j \gamma^\mu (1 - \gamma^5) q'_i)$$

- There are four states need to subtracted: $|0\rangle$, $|\pi\pi\rangle$, $|\eta\rangle$, $|\pi\rangle$. We add $c_s \bar{s}d$, $c_p \bar{s}\gamma^5 d$ operators to weak operators to make:

$$\langle 0| Q_i - c_p \bar{s}\gamma_5 d |K^0\rangle = 0, \langle \eta| Q_i - c_s \bar{s}d |K^0\rangle = 0$$  \quad (12)$$

$$Q'_i = Q_i - c_{pi} \bar{s}\gamma_5 d - c_{si} \bar{s}d$$  \quad (13)$$
Diagrams in the Calculation of $\Delta m_{K}^{\text{lat}}$

- For contractions among $Q_i$, there are four types of diagrams to be evaluated.

- In addition, there are "mixed" diagrams from the contractions between the $c_s \bar{s}d$ $c_p \bar{s}\gamma^5 d$ operators and $Q_i$ operators.
Finite lattice spacing effects

Short distance correction?

Figure: Different cases about physics on lattice with respect to energy scales. The shaded area represents where the contributions are important.
Finite lattice spacing effects

Quadratic divergences as the two $H_W$ approach each other: cutoff effect $\propto (1/a)^2$ needs short-distance correction.

GIM mechanism + LL structure removes both quadratic and logarithmic divergences:
$\sim (m_c a)^2$
Finite lattice spacing effects

- Ultraviolet divergences as the two $H_W$ approach each other: $\sim (1/a)^2$
- GIM mechanism $\rightarrow$ up minus charm quark propagators (for valence charm we used $am_c \simeq 0.31$)

16$^3 \times 32$ lattice: $Q_1 Q_1$ correlator amplitude reduction by a factor of 10 after introducing valence charm with mass 863 MeV (Jianglei Yu’s PhD thesis, 2014).

(a) Without charm: $\sim 1$
(b) With charm: $\sim 0.1$
Finite lattice spacing effects

- Ultraviolet divergences as the two $H_W$ approach each other: $\sim (1/a)^2$
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16$^3 \times 32$ lattice: $Q_1 Q_1$ correlator amplitude reduction by a factor of 10 after introducing valence charm with mass 863 MeV (Jianglei Yu’s PhD thesis, 2014).
Finite lattice spacing effects

Thus in our calculation of $\Delta m_K$, GIM mechanism + LL structure removes both quadratic and logarithmic divergences:

- short distance contribution greatly suppressed.
- Major contribution to $\Delta m_K$ from scale $\sim m_c$
Operator Renormalizations

- Renormalization of Lattice operator $Q_{1,2}$ in 3 steps:

$$C_{i}^{\text{lat}} = C_{a}^{\overline{\text{MS}}} (1 + \Delta r)^{\text{RI} \rightarrow \overline{\text{MS}}}_{ab} Z^{\text{lat} \rightarrow \text{RI}}_{bi}$$

- Non-perturbative Renormalization: from lattice to RI-SMOM

$$Z^{\text{lat} \rightarrow \text{RI}} = \begin{bmatrix} 0.6266 & -0.0437 \\ -0.0437 & 0.6266 \end{bmatrix}$$  \hspace{1cm} (14)

- Perturbation theory: from RI-SMOM to $\overline{\text{MS}}$


$$\Delta r^{\text{RI} \rightarrow \overline{\text{MS}}} = 10^{-3} \times \begin{bmatrix} -2.28 & 6.85 \\ 6.85 & -2.28 \end{bmatrix}$$  \hspace{1cm} (15)

- Use Wilson coefficients in the $\overline{\text{MS}}$ scheme


$$C^{\overline{\text{MS}}} = 10^{-3} \times \begin{bmatrix} -0.260 & 1.118 \end{bmatrix}$$  \hspace{1cm} (16)
Status of RBC-UKQCD Calculations of $\Delta m_k$

  Development of techniques and exploratory calculation on a $16^3 \times 32$ lattice with unphysical masses ($m_\pi = 421 \text{MeV}$) including only connected diagrams
Status of RBC-UKQCD Calculations of $\Delta m_k$

- "Long-distance contribution of the $K_L - K_S$ mass difference", 
  Development of techniques and exploratory calculation on a $16^3 \times 32$ lattice with unphysical masses ($m_\pi = 421$ MeV) including only connected diagrams

- "$K_L - K_S$ mass difference from Lattice QCD"
  Phys. Rev. Lett. 113(2014), 112003
  All diagrams included on a $24^3 \times 64$ lattice with unphysical masses
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- "The $K_L - K_S$ Mass Difference"
  All diagrams included on a $64^3 \times 128$ lattice with physical mass on 59 configurations: $\Delta m_k = (5.5 \pm 1.7_{\text{stat}}) \times 10^{-12} \text{MeV}$
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- Here I present an update of the analysis methods used and results having smaller statistical errors with 152 configurations.
Details of the Calculation

- $64^3 \times 128 \times 12$ lattice with Möbius DWF and the Iwasaki gauge action with physical pion mass (136 MeV) and $a^{-1} = 2.36\text{GeV}$

<table>
<thead>
<tr>
<th>$N_f$</th>
<th>$\beta$</th>
<th>$am_l$</th>
<th>$am_h$</th>
<th>$\alpha = b + c$</th>
<th>$L_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2+1</td>
<td>2.25</td>
<td>0.0006203</td>
<td>0.02539</td>
<td>2.0</td>
<td>12</td>
</tr>
</tbody>
</table>

- Data:
  - Sample AMA Correction and Super-jackknife Method

<table>
<thead>
<tr>
<th>data type</th>
<th>CG stop residual</th>
</tr>
</thead>
<tbody>
<tr>
<td>sloppy</td>
<td>$1e-4$</td>
</tr>
<tr>
<td>exact</td>
<td>$1e-8$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Data Set</th>
<th># of Sloppy</th>
<th># of Correction</th>
<th># of Type 1&amp;2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total</td>
<td>116</td>
<td><strong>36</strong></td>
<td><strong>36</strong></td>
</tr>
</tbody>
</table>

-Disconnected Type4 diagrams: save left- and right-pieces separately and use multiple source-sink separation for fitting.
Update of the results
2-point and 3-point results preliminary

- Meson masses are consistent with physical values

<table>
<thead>
<tr>
<th>$m_\pi$</th>
<th>$m_K$</th>
<th>$m_\eta$</th>
<th>$m_{\pi\pi, I=0}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0574(1)</td>
<td>0.2104(1)</td>
<td>0.258(16)</td>
<td>0.1138(5)</td>
</tr>
<tr>
<td>135.5(2)</td>
<td>496.5(2)</td>
<td>609.9(37.8)</td>
<td>268.5(1.3)</td>
</tr>
</tbody>
</table>

- $c'_s$'s and $c'_p$'s will be multiplied by the "mixing" diagrams and the errors from $c'_s$'s and $c'_p$'s will be carried all along.

<table>
<thead>
<tr>
<th>$c_{s1,\eta}$</th>
<th>$c_{s2,\eta}$</th>
<th>$c_{p1, vac}$</th>
<th>$c_{p2, vac}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2.13(33) \times 10^{-4}$</td>
<td>$-3.16(25) \times 10^{-4}$</td>
<td>$1.472(2) \times 10^{-4}$</td>
<td>$2.807(2) \times 10^{-4}$</td>
</tr>
<tr>
<td>$\langle \pi \pi I=0</td>
<td>Q'_1</td>
<td>K^0 \rangle$</td>
<td>$\langle \pi \pi I=0</td>
</tr>
<tr>
<td>$-8.7(1.5) \times 10^{-5}$</td>
<td>$9.5(1.5) \times 10^{-5}$</td>
<td>$7.7(2.5) \times 10^{-4}$</td>
<td>$-4.1(1.6) \times 10^{-4}$</td>
</tr>
</tbody>
</table>
Double-integrated correlators \textit{preliminary}

Fitting range: 10:20

All diagrams, uncorrelated fit

\[ \Delta m_K = 8.1(1.2) \times 10^{-12} \text{MeV} \]

\[ A = N_K^2 e^{-m_K(t_f-t_i)} \sum_n \frac{\langle K^0 | H_W | n \rangle \langle n | H_W | \bar{K}^0 \rangle}{m_K - E_n} \left\{ -T + \frac{e^{(m_K-E_n)T}}{m_K - E_n} - 1 \right\} \]

(17)
Single-integrated correlators \textbf{preliminary}

\[
G(\delta) = N_K^2 e^{-m_K(t_f-t_i)} \sum_n \langle \bar{K}^0 | H_W | n \rangle \langle n | H_W | K^0 \rangle e^{(m_K-E_n)\delta}
\]  \hspace{1cm} (18)

Check: Unintegrated $\rightarrow \langle 0 | Q'_i | K^0 \rangle = 0$, $\langle \eta | Q'_i | K^0 \rangle = 0 \rightarrow$ Subtract $\langle \pi | Q'_i | K^0 \rangle$

Next step: integrate and obtain $\Delta m_K$

Note: Need to add back contributions to $\Delta m_K$ from subtracted states.

\[Q'_i = Q_i - c_{pi} \bar{s} \gamma_5 d - c_{si} \bar{s} d\]
Single-integrated correlators: All diagrams, uncorrelated, preliminary

(a) unintegrated results with $\pi$ subtraction

(b) After integrating to large $T$, converged

Choosing $T=10$, as the integration upper limit:

$$\Delta m_K = 6.9(0.6) \times 10^{-12}\text{MeV}$$
Sources of Error

- **Statistical Error**
  - Less statistics for large operator separation

- **Systematic errors:**
  - **Finite-volume corrections**: small compared to statistical errors
    - "Effects of finite volume on the $K_L - K_S$ mass difference"

$$\Delta m_K(FV) = -0.22(7) \times 10^{-12} \text{MeV} \quad (19)$$

- **Discretization effects** are the largest sources of systematic error
  - $O(a)$: No contributions from DWF; Insure that no $O(a)$ error is introduced by lattice summation.(please see next slide)
  - $O(a^2)$: No short distance correction needed due to GIM cancellation Instead, $\sim (m_c a)^2$
Systematic errors

- **Discretization effects** are the largest source of systematic error:
  - $\mathcal{O}(a)$: No corrections needed: integrand’s boundary values goes to zero

- $\mathcal{O}(a^2)$:
  - Heavy charm quark, $\sim (m_c a)^2$ gives 25% Extrapolation needed.
  - Another estimate based on HVP calculation is $\sim 15\%$
Results *preliminary*

- Using single-integration method, we could:
  1. Manually **avoid including noise around zero** for large enough operator separations
  2. Smaller error in subtraction: $e^{-(E_n - m_K)t}$ rather than $\frac{1}{E_n - m_K} e^{-(E_n - m_K)t}$

- Δ$m_K$ values obtained from 2 analysis methods

<table>
<thead>
<tr>
<th>Method</th>
<th>Double-int</th>
<th>Single-int</th>
</tr>
</thead>
<tbody>
<tr>
<td>Δ$m_K/10^{-12}$ MeV</td>
<td>8.1(1.2)_{stat}</td>
<td>6.9(0.6)_{stat}</td>
</tr>
</tbody>
</table>

consistent within uncertainties
Conclusion and Outlook

- Our **preliminary** result based on 152 configurations is
  \[
  \Delta m_K = 6.7(0.6)_{stat}(1.7)_{sys} \times 10^{-12} \text{MeV}
  \]
  to be compared to the experimental value
  \[
  (\Delta m_K)^{exp} = 3.483(6) \times 10^{-12} \text{MeV}
  \]

Outlook

- Better estimate of the discretization error:
  Continue the calculation of $\Delta m_K$ on Summit:
  - On finer lattice ($96^3 \times 192, \, a^{-1} = 2.8 \text{ GeV}$) $\rightarrow$ smaller $m_c a$.
  - Continue the check of the measurement on lattice and data analysis(coefficients and renormalization factors), though the code was checked by Jianglei, Ziyuan and myself before.
Thanks for your attention!
Finite lattice spacing error effects

- Ultraviolet divergences as the two $H_W$ approach each other:

$$\int_{m_u}^{a^{-1}} d^4 p \gamma^\mu (1 - \gamma_5) \frac{p - m_u}{p^2 + m_u^2} \gamma^\nu (1 - \gamma_5) \frac{p - m_u}{p^2 + m_u^2} \propto (1/a)^2 \quad (20)$$

- GIM mechanism removes both quadratic and logarithmic divergences → charm quark propagators (for valence charm we used $a m_c \simeq 0.31$)

$$\int d^4 p \gamma^\mu (1 - \gamma_5)(\frac{p - m_u}{p^2 + m_u^2} - \frac{p - m_c}{p^2 + m_c^2}) \gamma^\nu (1 - \gamma_5)(... - ...) \quad (21)$$

$$\int d^4 p \gamma^\mu (1 - \gamma_5)(\frac{p(m_c^2 - m_u^2)}{(p^2 + m_u^2)(p^2 + m_c^2)}) \gamma^\nu (1 - \gamma_5)(... - ...) \quad (22)$$

And "short distance" now coming from $\sim 1/m_c$, with $\sim (m_c a)^2$

finite lattice spacing error relevant, rather than $\sim (a^{-1})^2$ divergence.
Finite lattice spacing effects

- Ultraviolet divergences as the two $H_W$ approach each other: $\sim (1/a)^2$
- GIM mechanism $\rightarrow$ charm minus up quark propagators (for valence charm we used $am_c \approx 0.31$)
  removes both quadratic and logarithmic divergences: $\sim m_c^2$

Figure: GIM effect in the QCD-free case on lattice quadratic $m_c$ dependence.

(a) Unintegrated correlators  
(b) $m_c$ dependence
Finite lattice spacing effects

- **GIM mechanism** \(\rightarrow\) 64 lattice charm quark propagators (for valence charm we used \(a m_c \approx 0.31\))

Similar behavior

(a) Without QCD

(b) With Iwasaki gauge action

Figure: GIM effect on \(64^3 \times 128\) lattice.