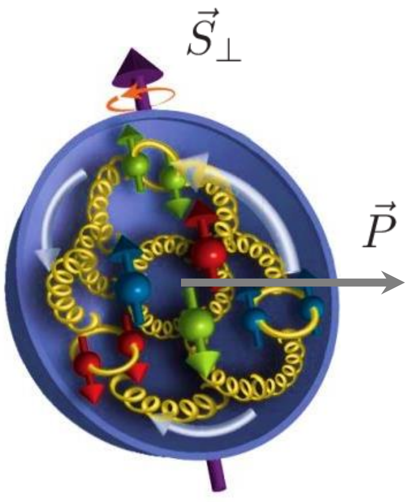

Transverse spin sum rule

Based on **[C.L., arXiv:2103.10100]**

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In short …

Nucleon spin decomposition

$$\langle \vec{J} \rangle = \sum_{a=q,g} \langle \vec{J}_a \rangle$$

Belinfante form

Longitudinal spin

$$\langle J_a^L \rangle = \frac{1}{2}\left[A_a(0) + B_a(0)\right]$$

[Ji, PRL78 (1997)]

$$\sum_a A_a(0) = 1 \qquad \sum_a B_a(0) = 0$$

$$A_q(0) = \int \mathrm{d}x\, x\, H_q(x,\xi,0)$$

$$B_q(0) = \int \mathrm{d}x\, x\, E_q(x,\xi,0)$$

Gravitational form factor | **GPD**

Transverse spin

$$\langle J_a^T \rangle = \frac{1}{2}\left[A_a(0) + \frac{p^0}{M} B_a(0)\right]$$

[Leader, PRD85 (2012)]

$$\langle J_a^T \rangle = \frac{p^0}{M} \frac{A_a(0) + B_a(0)}{2} \equiv \frac{p^0}{M} \langle S_a^T \rangle$$

[Ji-Yuan, PLB810 (2020)]

Both are correct !

[C.L., arXiv:2103.10100]

What do we mean by « spin »?

Originally « spin » was reserved to *intrinsic* angular momentum (AM),
to be distinguished from orbital AM

Nowadays « spin » refers more generally to internal AM,
i.e. AM about the center of the system

$$\vec{J} = \vec{R} \times \vec{P} + \vec{S}$$

Position of the center ($\vec{R}$)

Momentum of the system ($\vec{P}$)

→ The key question is: **what is the relativistic center of the system?**

Poorly addressed in the nucleon spin literature …
triggered a review of the subject [C.L., EPJC78 (2018)]

Option 1: Relativistic center of energy

Relativistic version of the center of inertial mass

[Fokker, *Relativiteitstheorie* (1929)]
[Born-Infeld, PRSLA150 (1935)]

$$R^\mu_E = \frac{1}{P^0}\int \mathrm{d}^3r\, r^\mu T^{00}$$

$$P^\mu = \int \mathrm{d}^3r\, T^{0\mu}$$

$$\Rightarrow \quad \vec{R}_E = t\,\frac{\vec{P}}{P^0} - \frac{\vec{K}}{P^0}$$

← Lorentz boost generator

Spin operator

$$\boxed{\vec{S}_E \equiv \vec{J} - \vec{R}_E \times \vec{P} = \frac{\vec{W}}{P^0}}$$

Pauli-Lubański pseudo-vector

$$W^\mu = \frac{1}{2}\,\epsilon^{\mu\alpha\beta\lambda} M_{\alpha\beta} P_\lambda$$

$$M^{\alpha\beta} = \int \mathrm{d}^3r\,\left[r^\alpha T^{0\beta} - r^\beta T^{0\alpha}\right]$$

Remark: These definitions coincide in the infinite-momentum frame with the corresponding light-front operators

Option 2: Relativistic center of mass

R^μ_E does not transform as a Lorentz four-vector

[Pryce, PRSLA195 (1948)]
[Møller, CDIASA5 (1949)]
[Fleming, PR137 (1965)]

Covariant four-position operator

$$R^\mu_M = \left(t + \frac{\vec{P}\cdot\vec{K}}{M^2}\right)\frac{P^\mu}{P^0} - \frac{P_\nu M^{\nu\mu}}{M^2}$$

$$\tau = \frac{M}{P^0}\left(t + \frac{\vec{P}\cdot\vec{K}}{M^2}\right)$$

Proper time

$$\Rightarrow \quad \vec{R}_M = \vec{R}_E - \frac{\vec{P}\times\vec{W}}{P^0 M^2}$$

Spin operator

$$\vec{S}_M \equiv \vec{J} - \vec{R}_M \times \vec{P} = \frac{P^0\vec{W} - \vec{P}W^0}{M^2}$$

Remarks:
- $\vec{R}_M = \vec{R}_E$ in the rest frame
- Relativistic center of mass can be considered as a physical point

(i.e. not just as a mere representative point)

Option 3: Relativistic center of spin

Canonical relations $\begin{cases} [R^i_X, R^j_X] = 0 \\ [S^i_X, S^j_X] = i\epsilon^{ijk} S^k_X \end{cases}$ not satisfied for $X = E, M$

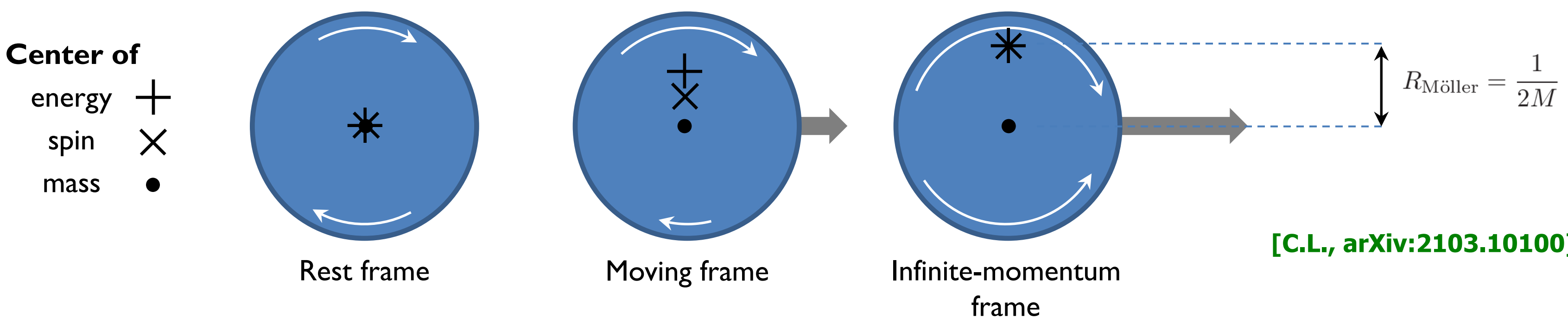
[Pryce, PRSLA180 (1935)]
[Pryce, PRSLA195 (1948)]
[Møller, CDIASA5 (1949)]
[Newton-Wigner, RMP21 (1949)]
[Bogolyubov-Logunov-Todorov, *Introduction to Axomatic Quantum Field Theory* (1975)]

Canonical operator
$$R^\mu_c = \frac{P^0 R^\mu_E + M R^\mu_M}{P^0 + M}$$

$$\vec{R}_c = \vec{R}_E - \frac{\vec{P} \times \vec{W}}{M P^0 (P^0 + M)} = \vec{R}_M + \frac{\vec{P} \times \vec{W}}{M^2 (P^0 + M)}$$

Spin operator
$$\vec{S}_c \equiv \vec{J} - \vec{R}_c \times \vec{P} = \frac{\vec{W}}{M} - \frac{\vec{P} W^0}{M(P^0 + M)}$$

Transversely polarized nucleon



[C.L., arXiv:2103.10100]

Comparison between expectation values

Let us denote rest-frame spin vector by $\frac{1}{2}\vec{s}$ with $\vec{s}^{\,2} = 1$

Longitudinal component

$$\langle S_E^L \rangle = \langle S_M^L \rangle = \langle S_c^L \rangle = \tfrac{1}{2} s_L$$

Explains why the question of the nucleon center did not draw much attention in the past

Transverse component

$$\langle S_E^T \rangle = \gamma^{-1}\, \tfrac{1}{2} s_T$$

Transverse part of a four-vector is subleading $\quad \gamma = p^0/M$

$$\langle S_M^T \rangle = \gamma\, \tfrac{1}{2} s_T$$

Transverse part of an antisymmetric rank-two tensor is leading

[Landau-Lifshitz, *Classical Theory of Fields* (1951)]

$$\langle S_c^T \rangle = \tfrac{1}{2} s_T$$

Frame-independent!

(simple AM composition crucial for the wavefunction formalism)

Expectation values for plane waves

$$\langle p,\vec{s}|\int \mathrm{d}^3r\, r^i O(r)|p,\vec{s}\rangle = \langle p,\vec{s}|O(0)|p,\vec{s}\rangle \underbrace{\int \mathrm{d}^3r\, r^i}_{\text{Ambiguous!}}$$

$$O(r) = e^{iP\cdot r}O(0)e^{-iP\cdot r}$$

Standard approach

[Jaffe-Manohar, NPB337 (1990)]
[Bakker-Leader-Trueman, PRD70 (2004)]
[Leader-C.L., PR541 (2014)]
[Lowdon-Chiu-Brodsky, PLB774 (2017)]

$$\langle \int \mathrm{d}^3r\, r^i O(r)\rangle_{\text{standard}} \equiv \lim_{\vec{\Delta}\to\vec{0}} \frac{\langle \vec{p}+\frac{\Delta}{2},\vec{s}|\int \mathrm{d}^3r\, r^i O(r)|\vec{p}-\frac{\Delta}{2},\vec{s}\rangle}{\langle p,\vec{s}|p,\vec{s}\rangle}$$

$$= \underbrace{\frac{\langle p,\vec{s}|O(0)|p,\vec{s}\rangle}{\langle p,\vec{s}|p,\vec{s}\rangle}(2\pi)^3 i\nabla^i\delta^{(3)}(\vec{0})}_{\text{Contribution from the center of wave packet}} + \underbrace{\frac{1}{2p^0}\left[-i\nabla^i_\Delta\langle\vec{p}+\tfrac{\Delta}{2},\vec{s}|O(0)|\vec{p}-\tfrac{\Delta}{2},\vec{s}\rangle\right]_{\vec{\Delta}=\vec{0}}}_{\text{Internal contribution}}$$

$$\equiv \langle \int \mathrm{d}^3r\, r^i O(r)\rangle_{\text{int}}$$

Transverse spin sum rules

$$\mathcal{S}^\mu = \left(\frac{\vec{p}\cdot\vec{s}}{M}, \vec{s} + \frac{\vec{p}(\vec{p}\cdot\vec{s})}{M(p^0+M)}\right)$$

$$\langle \int \mathrm{d}^3r\, r^i T_a^{0j}(r)\rangle_{\text{int}} = -\left(p^0\epsilon^{ij\alpha\beta} + p^i\epsilon^{j0\alpha\beta} + p^j\epsilon^{i0\alpha\beta}\right)\frac{\mathcal{S}_\alpha p_\beta}{2p^0 M}\,\frac{A_a(0)+B_a(0)}{2} - \frac{p^j\epsilon^{0i\alpha\beta}\mathcal{S}_\alpha p_\beta}{2M(p^0+M)}\,A_a(0)$$

Blue tems do not contribute to W^μ

Leader sum rule

[Leader, PRD85 (2012)]
[Leader-C.L., PR541 (2014)]

$$\langle J_a^k\rangle_{\text{Leader}} = \epsilon^{kij}\langle \int \mathrm{d}^3r\, r^i T_a^{0j}(r)\rangle_{\text{int}}$$

$$= \frac{s^k}{2}\,A_a(0) + \frac{p^0}{2M}\left(s^k - \frac{p^k(\vec{p}\cdot\vec{s})}{p^0(p^0+M)}\right)B_a(0)$$

Ji-Yuan sum rule

[Ji-Yuan, PLB810 (2020)]

$$\langle J_a^k\rangle_{\text{Ji-Yuan}} = \epsilon^{kij}\langle \int \mathrm{d}^3r\, r^i T_a^{0j}(r)\rangle_{\text{int}}\Big|_{\textit{without}\ \text{blue terms}}$$

$$= \frac{p^0}{M}\left(s^k - \frac{p^k(\vec{p}\cdot\vec{s})}{p^0(p^0+M)}\right)\frac{A_a(0)+B_a(0)}{2}$$

Interpreted as contributions from « center-of-mass motion »

(requires rigorous justification)

Phase-space approach

$$\langle\psi|O|\psi\rangle = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\,\mathrm{d}^3\mathcal{R}\,\rho_\psi(\vec{\mathcal{R}},\vec{p})\,\langle O\rangle_{\vec{\mathcal{R}},\vec{p}}$$

[C.L., EPJC78 (2018)]
[C.L., arXiv:2103.10100]

Nucleon Wigner distribution

$$\rho_\psi(\vec{\mathcal{R}},\vec{p}) = \int \mathrm{d}^3z\, e^{-i\vec{p}\cdot\vec{z}}\,\psi^*(\vec{\mathcal{R}}-\tfrac{\vec{z}}{2})\psi(\vec{\mathcal{R}}+\tfrac{\vec{z}}{2})$$

$$= \int \frac{\mathrm{d}^3q}{(2\pi)^3}\, e^{-i\vec{q}\cdot\vec{\mathcal{R}}}\,\tilde{\psi}^*(\vec{p}+\tfrac{\vec{q}}{2})\tilde{\psi}(\vec{p}-\tfrac{\vec{q}}{2})$$

$$\psi(\vec{r}) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\, e^{-i\vec{p}\cdot\vec{r}}\,\tilde{\psi}(\vec{p})$$

$$\tilde{\psi}(\vec{p}) = \frac{\langle p|\psi\rangle}{\sqrt{2p^0}}$$

System localized in phase-space

$$\langle O\rangle_{\vec{\mathcal{R}},\vec{p}} = \int \frac{\mathrm{d}^3\Delta}{(2\pi)^3}\, e^{i\vec{\Delta}\cdot\vec{\mathcal{R}}}\,\frac{\langle \bar{p}+\frac{\Delta}{2}|O|\bar{p}-\frac{\Delta}{2}\rangle}{\sqrt{4(\bar{p}^0)^2-(\Delta^0)^2}}\Bigg|_{\vec{\bar{p}}=\vec{p}}$$

$$\Rightarrow\quad \langle \int \mathrm{d}^3r\, r^i O(r)\rangle_{\vec{\mathcal{R}},\vec{p}} = \mathcal{R}^i\,\frac{\langle p|O(0)|p\rangle}{2p^0} + \frac{1}{2p^0}\left[-i\nabla^i_\Delta \langle \bar{p}+\tfrac{\Delta}{2}|O(0)|\bar{p}-\tfrac{\Delta}{2}\rangle\right]_{\vec{\Delta}=\vec{0}}$$

Remark: $\langle O\rangle_{\text{standard}} = \int \frac{\mathrm{d}^3\mathcal{R}}{(2\pi)^3\delta^{(3)}(\vec{0})}\,\langle O\rangle_{\vec{\mathcal{R}},\vec{p}}$ explains the origin of the delta contribution in standard approach

Phase-space approach

$$\langle \vec{R}_E \rangle = \vec{\mathcal{R}} + \frac{\vec{p} \times \vec{s}}{2p^0(p^0 + M)} \qquad \langle \vec{S}_E \rangle = \frac{M}{2p^0} \left(\vec{s} + \frac{\vec{p}(\vec{p} \cdot \vec{s})}{M(p^0 + M)} \right)$$

$$\langle \vec{R}_M \rangle = \vec{\mathcal{R}} - \frac{\vec{p} \times \vec{s}}{2M(p^0 + M)} \qquad \langle \vec{S}_M \rangle = \frac{p^0}{2M} \left(\vec{s} - \frac{\vec{p}(\vec{p} \cdot \vec{s})}{p^0(p^0 + M)} \right) \qquad \langle O \rangle \equiv \langle O \rangle_{\vec{\mathcal{R}},\vec{p}}$$

$$\langle \vec{R}_c \rangle = \vec{\mathcal{R}} \qquad \langle \vec{S}_c \rangle = \frac{\vec{s}}{2}$$

We define quark and gluon contributions to internal AM operator as

$$S^k_{X,a} \equiv \epsilon^{kij} \int \mathrm{d}^3 r \; (r^i - \langle R^i_X \rangle) \, T^{0j}_a(r) = J^k_a - \epsilon^{kij} \langle R^i_X \rangle P^j_a \qquad X = E, M, c$$

Relativistic spin sum rules [C.L., arXiv:2103.10100]

$$\langle \vec{S}_{X,a} \rangle = \langle \vec{S}_X \rangle \, A_a(0) + \langle \vec{S}_M \rangle \, B_a(0)$$

Leader: $X = c$ canonical spin sum rule

Ji-Yuan: $X = M$ covariant spin sum rule

(When « spin » is properly defined, there is no need to drop terms!)

Summary

- **« Spin » is understood as AM about the center of the nucleon**
- **There are at least 3 possible centers in relativity**
- **Phase-space approach allows one to compute for the first time the contribution from the motion of the center about the origin**
- **Transverse spin sum rules by Leader and Ji-Yuan are both correct and refer to different definitions of the center**