Lorentz Invariance and Equation of Motion Relations for polarized GTMDs and Angular Momentum

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An old story – longitudinal quark angular momentum sum rule (Ji decomposition)

$$
\begin{aligned}
J_L \quad &= \quad L_L \quad &+ \quad S_L \\
\frac{1}{2}\int dx\, x(H+E) &= \int dx\, x(\widetilde{E}_{2T}+H+E) &+ \frac{1}{2}\int dx\, \widetilde{H} \\
&= -\int dx F_{14}^{(1)} &+ \frac{1}{2}\int dx\, \widetilde{H}
\end{aligned}
$$

All distributions taken in the forward limit, $\Delta = 0$.

Using nomenclature of Meißner, Metz and Schlegel
Notation for GTMD moment:

$$X^{(1)} = 2\int d^2k_T \frac{k_T^2\Delta_T^2 - (k_T\cdot\Delta_T)^2}{M^2\Delta_T^2} X$$

These relations can be understood as Lorentz Invariance and Equation of Motion relations

Can work at zero skewness, $\xi = 0$, for the following.

Lorentz Invariance Relations

Completely unintegrated correlator

$$W^{\Gamma}_{\Lambda'\Lambda}(P,k,\Delta) = \frac{1}{2}\int \frac{d^4z}{(2\pi)^4} e^{ik\cdot z} \langle P+\Delta/2,\Lambda'|\bar{\psi}\left(-\frac{z}{2}\right)\Gamma\mathcal{U}\psi\left(\frac{z}{2}\right)|P-\Delta/2,\Lambda\rangle$$

can be parametrized in terms of Lorentz invariant amplitudes $A(k^2, k\cdot P, k\cdot\Delta, \Delta^2, P\cdot\Delta)$.

Its k^- integral,

$$W^{\Gamma}_{\Lambda'\Lambda}(P,x,k_T,\xi,\Delta_T) = \int dk^- \, W^{\Gamma}_{\Lambda'\Lambda}(P,k,\Delta)$$

defines the GTMDs .

There are more GTMDs than A-amplitudes $\longrightarrow$ relations between GTMDs induced

$$\frac{d}{dx} F^{(1)}_{14} = \widetilde{E}_{2T} + H + E$$

Equation of Motion Relations

Introduce $(i\not{D} - m)\psi = 0$ into the GTMD correlator W, integrate by parts

Result: Relations between GTMD correlators, e.g.,

$$-\frac{\Delta^+}{2} W^{\gamma^i \gamma^5}_{\Lambda'\Lambda} + ik^+ \epsilon^{ij} W^{\gamma^j}_{\Lambda'\Lambda} + \frac{\Delta^i}{2} W^{\gamma^+ \gamma^5}_{\Lambda'\Lambda} - i\epsilon^{ij} k^j W^{\gamma^+}_{\Lambda'\Lambda} + \mathcal{M}^{i,S}_{\Lambda'\Lambda} = 0$$

with quark-gluon-quark term $\mathcal{M}^{i,S}_{\Lambda'\Lambda}$, $\int dx \int d^2k_T \mathcal{M}^{i,S}_{\Lambda'\Lambda} = 0$ (but non-vanishing for staple link)

Consider combination $(W^{\Gamma}_{++} - W^{\Gamma}_{--})$, contract with Δ^i:

$$F^{(1)}_{14} = x\widetilde{E}_{2T} + \widetilde{H} + \mathcal{M}_L$$

$$\mathcal{M}_L = \int d^2k_T \frac{\Delta^i}{\Delta_T^2} \left(\mathcal{M}^{i,S}_{++} - \mathcal{M}^{i,S}_{--} \right)$$

Synopsis

$$\frac{d}{dx}F_{14}^{(1)} = \widetilde{E}_{2T} + H + E$$

$$F_{14}^{(1)} = x\widetilde{E}_{2T} + \widetilde{H} + \mathcal{M}_L$$

Eliminate F_{14} terms, integrate over x, take $\Delta_T \to 0$
Quark-gluon-quark term integrates to zero

Recover

$$\frac{1}{2}\int dx\, x(H+E) = \int dx\, x(\widetilde{E}_{2T} + H + E) \quad + \quad \frac{1}{2}\int dx\, \widetilde{H}$$

Alternatively, eliminate $\widetilde{E}_{2T}$.

Constructing the transverse quark angular momentum sum rule

Transverse angular momenta are not boost-invariant, contrary to longitudinal ones

Construct leading two orders,

$$J_T = J_T^{(1)} P^+ + J_T^{(-1)} \frac{1}{P^+} + \dots$$

$$L_T = L_T^{(1)} P^+ + L_T^{(-1)} \frac{1}{P^+} + \dots$$

$$S_T = S_T^{(1)} P^+ + S_T^{(-1)} \frac{1}{P^+} + \dots$$

$$J_T^{(1)} = L_T^{(1)} + S_T^{(1)}$$

$$J_T^{(-1)} = L_T^{(-1)} + S_T^{(-1)}$$

What do we already know about these quantities?

Transverse Angular Momentum

Use Lorentz transformation properties of $(0\,i)$ components of rank-2 tensor for longitudinal/transverse i

$$J_T = \gamma J_0 = \gamma J_L \;=\; \left(P^+ \frac{1}{\sqrt{2}M} + \frac{1}{P^+}\frac{M}{2\sqrt{2}}\right) \frac{1}{2}\int dx\, x(H+E)$$

Note: Implicit choice of reference point for angular momentum!

Reference point: Proton rest frame center of mass

$$J_T^{(1)} = \frac{1}{\sqrt{2}M}\,\frac{1}{2}\int dx\, x(H+E)$$

$$J_T^{(-1)} = \frac{M}{2\sqrt{2}}\,\frac{1}{2}\int dx\, x(H+E)$$

All distributions in the forward limit, $\Delta = 0$.

Transverse Spin

$$S_T^{(1)} = 0$$

$$S_T^{(-1)} = \frac{M}{\sqrt{2}} \, \frac{1}{2} \int dx \, g_T = \frac{M}{\sqrt{2}} \, \frac{1}{2} \int dx \, H'_{2T} \Big|_{\Delta=0}$$

Transverse Orbital Angular Momentum

$$L_T^{(1)} = J_T^{(1)} = \frac{1}{\sqrt{2}M} \frac{1}{2} \int dx \, x(H+E)$$

Construct $L_T^{(-1)}$ via LIRs and EoMs!

In the transverse case, need to utilize the full range of kinematics, including the skewness variable.

Equation of Motion Relation

$$-\frac{\Delta^+}{2} W_{\Lambda'\Lambda}^{\gamma^i\gamma^5} + ik^+\epsilon^{ij} W_{\Lambda'\Lambda}^{\gamma^j} + \frac{\Delta^i}{2} W_{\Lambda'\Lambda}^{\gamma^+\gamma^5} - i\epsilon^{ij}k^j W_{\Lambda'\Lambda}^{\gamma^+} + \mathcal{M}_{\Lambda'\Lambda}^{i,S} = 0$$

Consider combination $(\Delta_1 + i\Delta_2)W^{\Gamma}_{+-} + (\Delta_1 - i\Delta_2)W^{\Gamma}_{-+}$, contracted with Δ^i:

$$\left[(1-\xi^2)H'_{2T} - \xi^2 E'_{2T} + \xi\widetilde{E}'_{2T}\right] \;+\; x\left[(1-\xi^2)\frac{1}{\xi}H_{2T} - \xi E_{2T} + \widetilde{E}_{2T}\right] \;+\; \frac{\Delta_T^2}{4M^2}\widetilde{E}$$

$$+\;\frac{1}{4}\left[\frac{\Delta_T^2}{M^2}F_{14}^{(1)} + \frac{2}{\xi}(1-\xi^2)F_{12}^{(1)}\right] \;+\; \mathcal{M}_T \;=\; 0$$

$$\mathcal{M}_T = \frac{\sqrt{1-\xi^2}\,\Delta^i}{4M\xi\Delta_T^2}\int d^2k_T\left((\Delta_1 + i\Delta_2)\mathcal{M}_{+-}^{i,S} + (\Delta_1 - i\Delta_2)\mathcal{M}_{-+}^{i,S}\right)$$

Lorentz Invariance Relations

$$\frac{d}{dx}F_{14}^{(1)} = \widetilde{E}_{2T} + H + E - \xi E_{2T}$$

$$\frac{d}{dx}F_{12}^{(1)} = -2H_{2T} - \frac{2\xi P^2}{M^2}\left(\widetilde{E}_{2T} + H + E - \xi E_{2T}\right)$$

Synopsis

Eliminate the GTMD moments $F_{12}^{(1)}$ and $F_{14}^{(1)}$

Integrate over x, take $\Delta \to 0$

$$\left[\frac{1}{2}\int dx\, x(H+E)\right] = \left[\int dx\, x\left(\widetilde{E}_{2T} + H + E + \lim_{\xi\to 0}\frac{H_{2T}}{\xi}\right)\right] + \left[\frac{1}{2}\int dx\, H_{2T}'\right]$$

$1/P^+$ order of the transverse spin sum rule for reference point at proton rest frame center of mass

Multiply by $M/\sqrt{2}$, balance kinematic effects:

$$J_T^{(-1)} \quad = \quad L_T^{(-1)} \quad + \quad S_T^{(-1)}$$

$$\frac{M}{2\sqrt{2}}\left[\frac{1}{2}\int dx\, x(H+E)\right] = \frac{M}{\sqrt{2}}\left[\int dx\, x\left(\widetilde{E}_{2T} + H + E + \lim_{\xi\to 0}\frac{H_{2T}}{\xi}\right)\right] - \frac{M}{2\sqrt{2}}\left[\frac{1}{2}\int dx\, x(H+E)\right] + \frac{M}{\sqrt{2}}\left[\frac{1}{2}\int dx\, H_{2T}'\right]$$

Transverse vs. longitudinal sum rule

$$\frac{1}{2}\int dx\, x(H+E) = \int dx\, x\left(\widetilde{E}_{2T} + H + E + \lim_{\xi\to 0}\frac{H_{2T}}{\xi}\right) + \frac{1}{2}\int dx\,(g_1+g_2)$$

$$= \frac{1}{2}\lim_{\xi\to 0}\int dx \frac{F_{12}^{(1)}}{\xi} + \frac{1}{2}\int dx\,(g_1+g_2)$$

$\frac{1}{2}\lim_{\xi\to 0}\frac{F_{12}^{(1)}}{\xi}$ $\longrightarrow$ has structure of $\langle r_L \times k_T \rangle$

$\longrightarrow$ replace straight link by staple link to obtain Jaffe-Manohar version

$$\frac{1}{2}\int dx\, x(H+E) = \int dx\, x(\widetilde{E}_{2T} + H + E) + \frac{1}{2}\int dx\, g_1$$

$$= -\int dx F_{14}^{(1)} + \frac{1}{2}\int dx\, g_1$$

All distributions taken in the forward limit, $\Delta = 0$.