

Towards a unification of gluon distributions at small and moderate x

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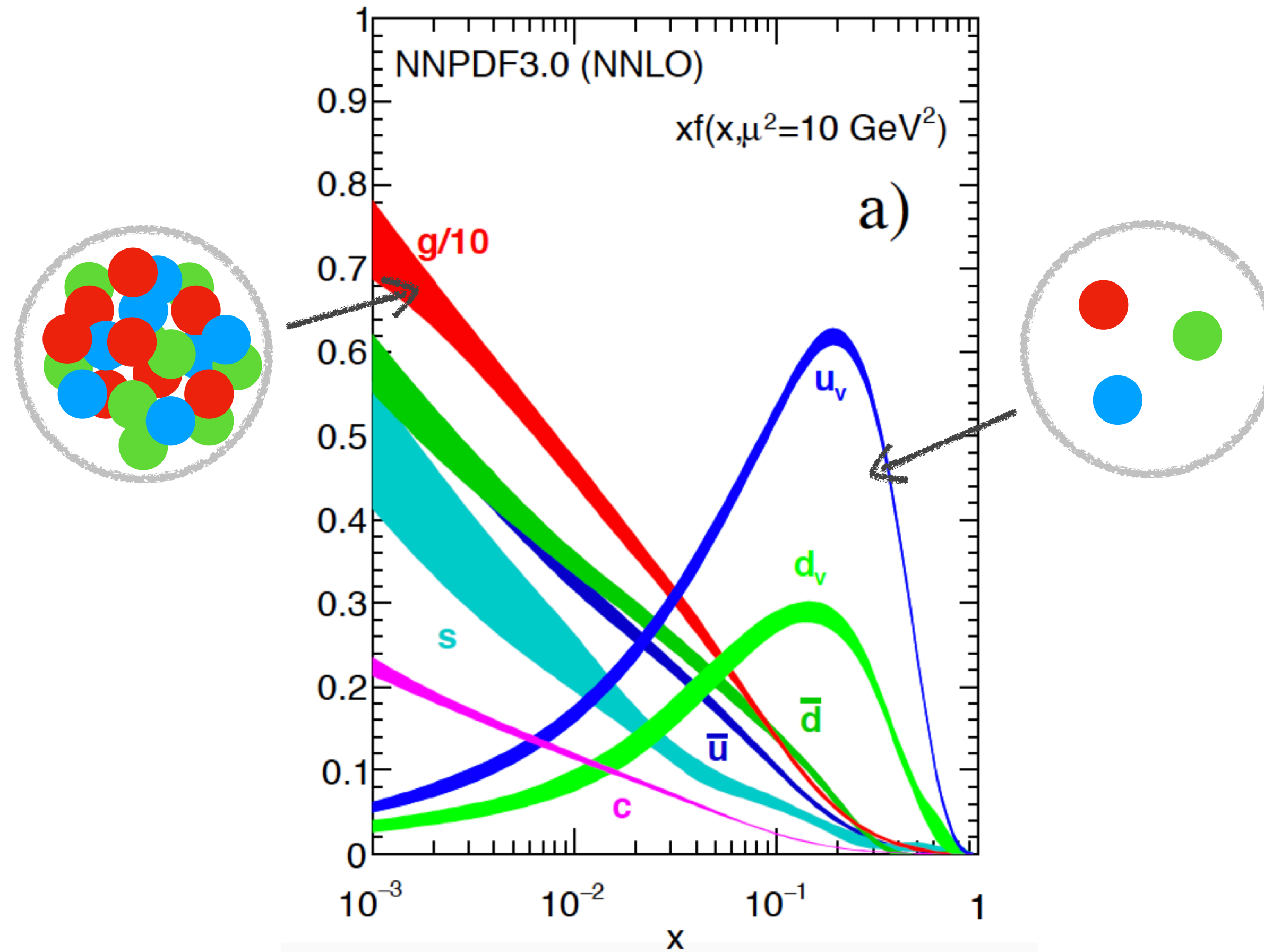
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In collaboration with Renaud Boussarie

2001.06449 [hep-ph]

2006.14569 [hep-ph]

Gluon saturation at small x



Gluon density rises rapidly at small x : large occupation numbers $\rightarrow$
Regime of strong classical fields: breakdown of the parton picture

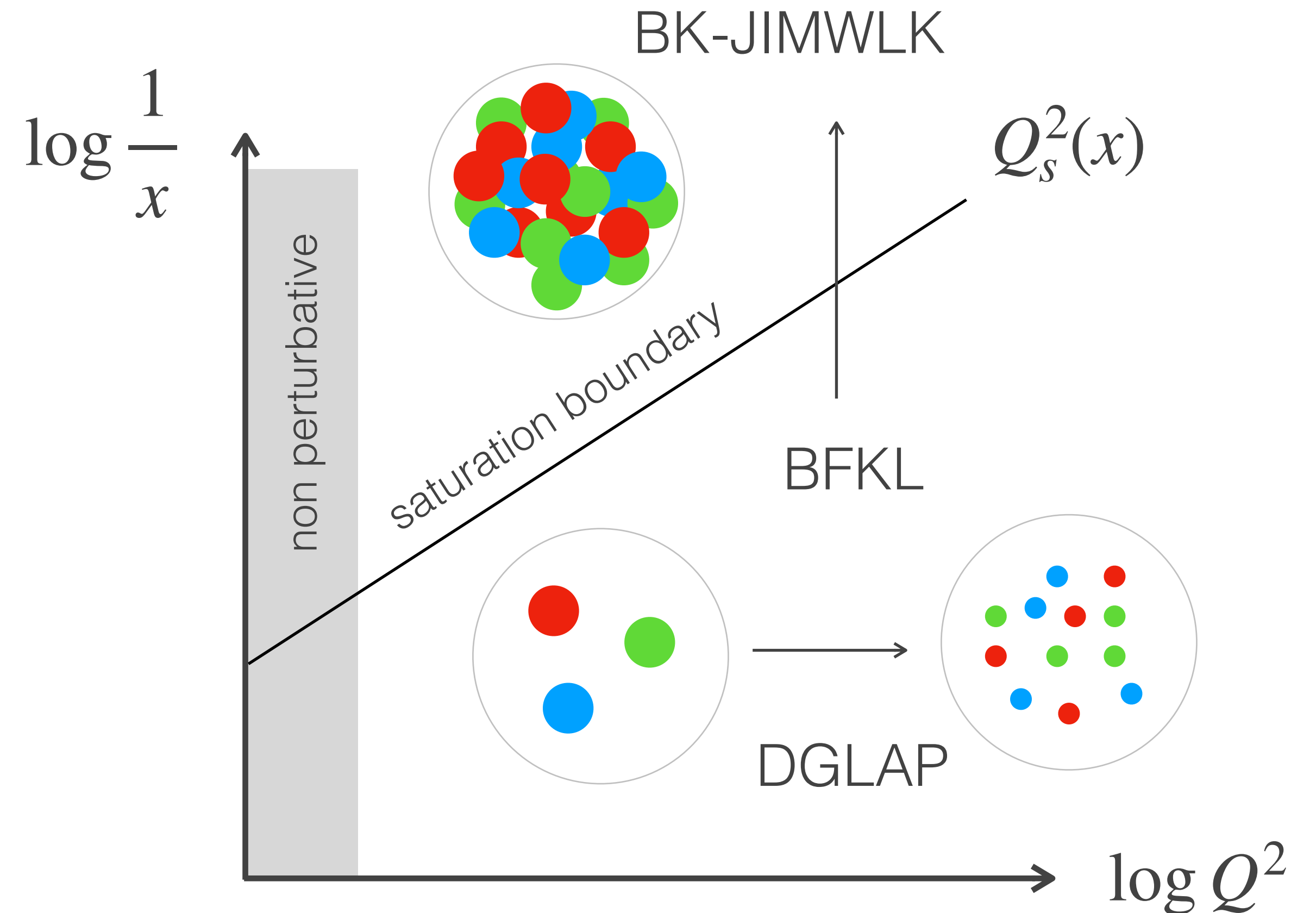
Gluon saturation at small x

- **Gluon saturation criterion:** large number of gluons populate the transverse extend of the proton leading to saturation when

$$S_{\perp} \sim \frac{\alpha_s}{Q^2} \times xg(x, Q^2)$$

- Defining the **saturation scale**

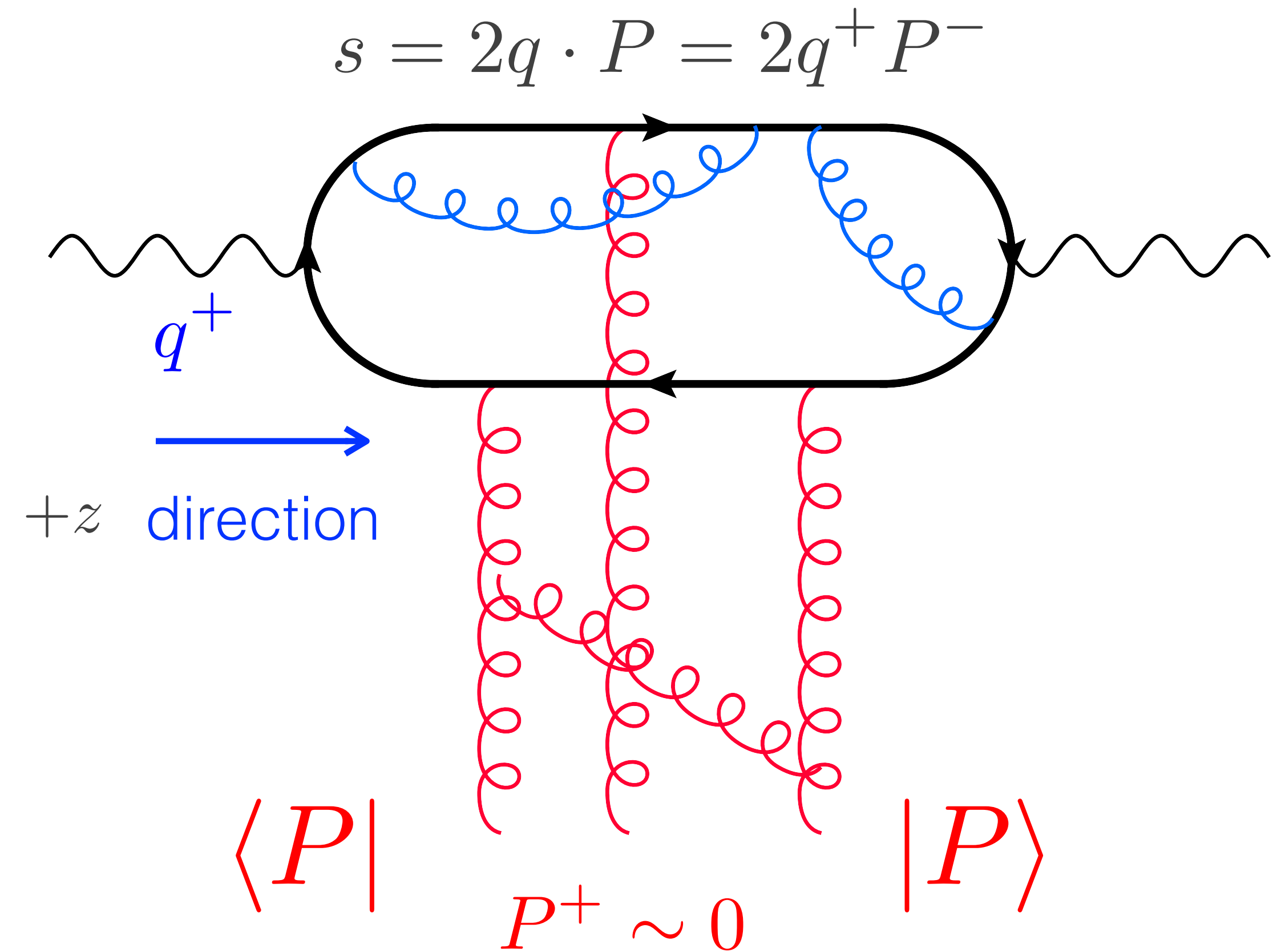
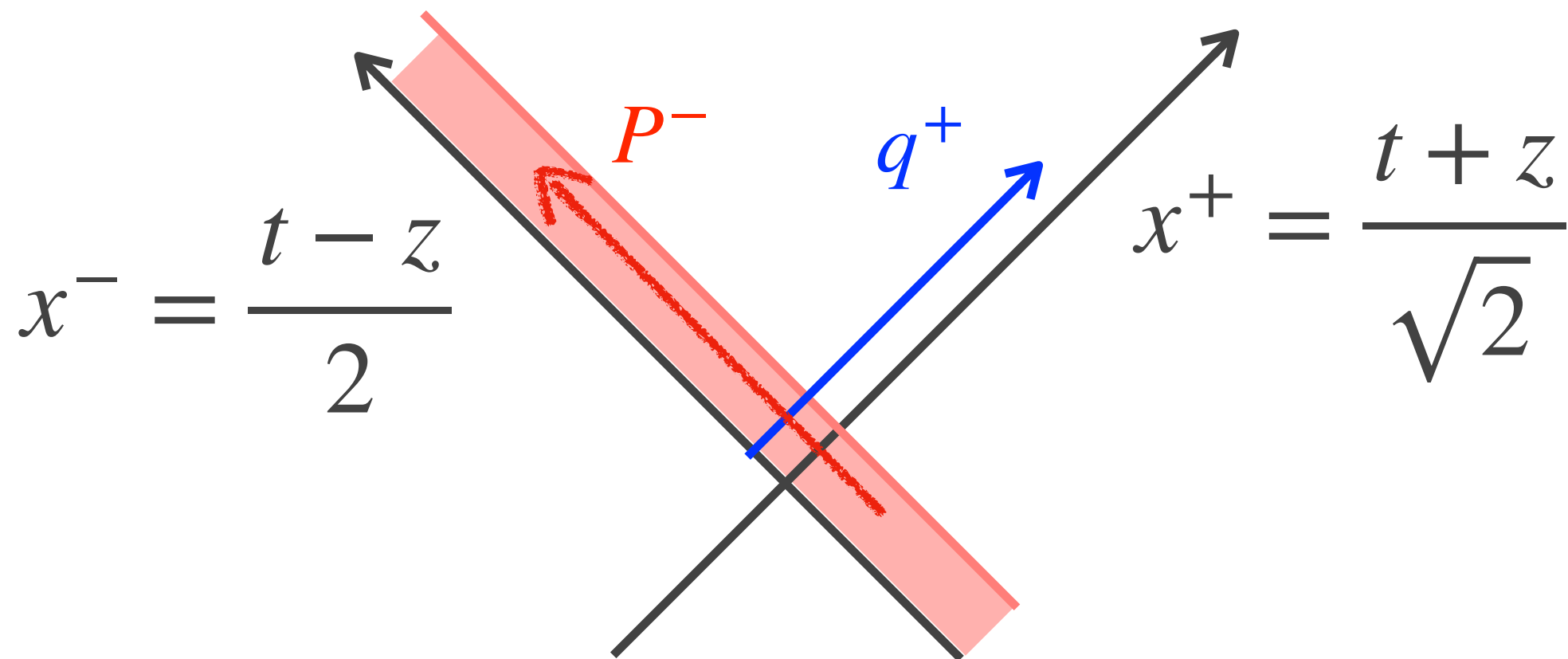
$$Q^2 \sim Q_s^2(x) \sim x^{-\lambda}$$



[Gribov, Levin, Ryskin, 1983- Mueller, Qiu, 1986, Venugopalan, McLerran (MV), Balitsky, Kovchegov (BK) Jallilian-Marian, Iancu, McLerran, Weigert, Leonidov, Kovner (JIMWLK) (1993-2001)]

Rapidity factorization at small x - coherent scattering

- **Regge limit:** $s \rightarrow \infty$ with Q^2 fixed
- Although gluons contribute only at NLO they dominate the cross section at small x. Dominant diagram in DIS: scattering of **quark dipole moving in the +z direction** off **longitudinally polarized gluons in the target**



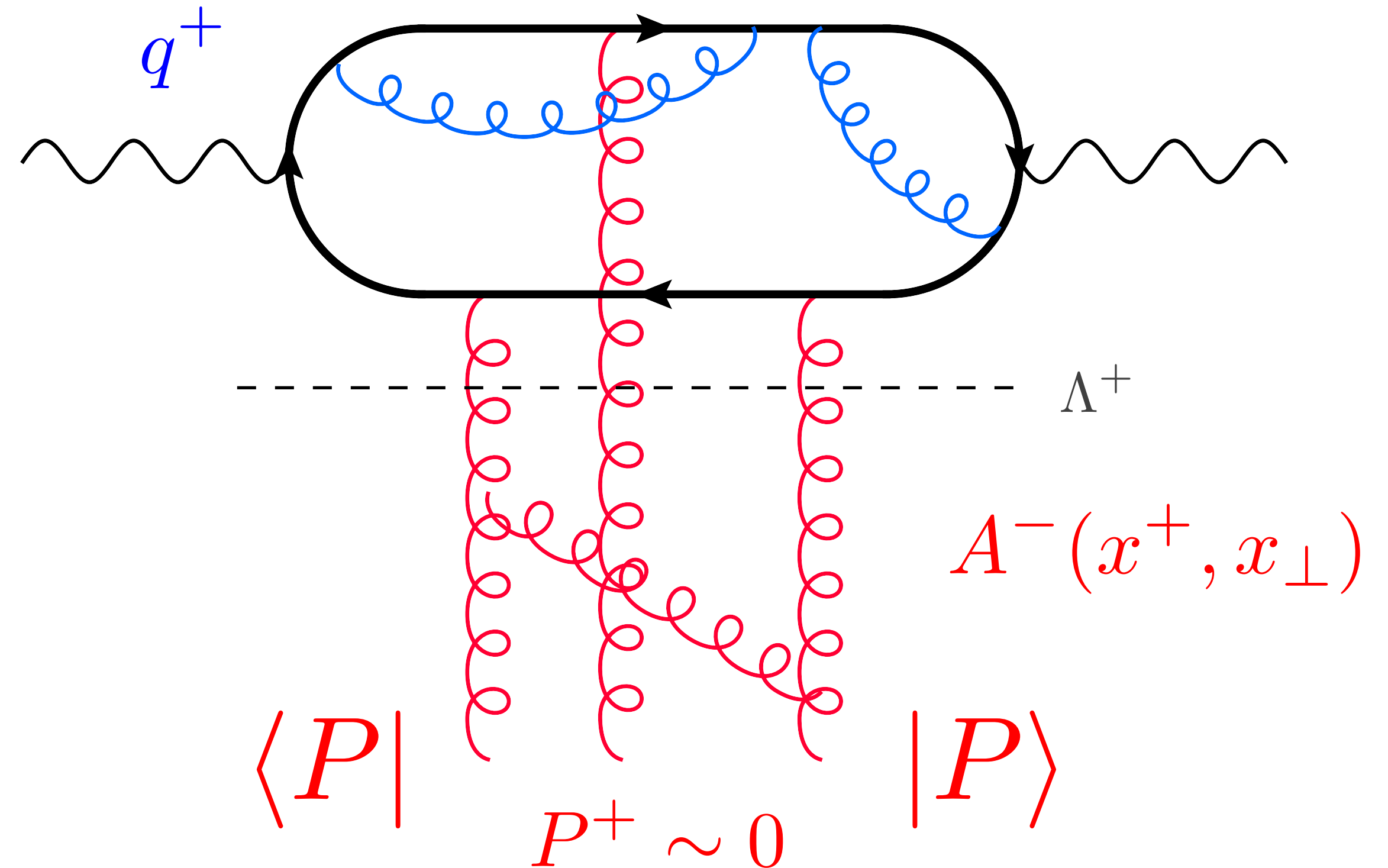
$$x_{Bj} = Q^2/s \rightarrow 0$$

Rapidity factorization at small x - coherent scattering

Step 1: Split the gluon fields into **fast** and **slow** gluons

$$A^\mu(k) \equiv A^\mu(k^+ < \Lambda^+) + a^\mu(k^+ > \Lambda^+)$$

- The relevant d.o.f. in the saturation regime are strong classical fields
 $g A^- \sim 1$: boosted target field dominated by its - component



$$A^\mu(x) \rightarrow \gamma A^-\left(\gamma x^+, \frac{x^-}{\gamma}, \mathbf{x}\right) \quad A^+ \sim O(1/\gamma) \quad A_\perp \sim O(1)$$

Rapidity factorization at small x - coherent scattering

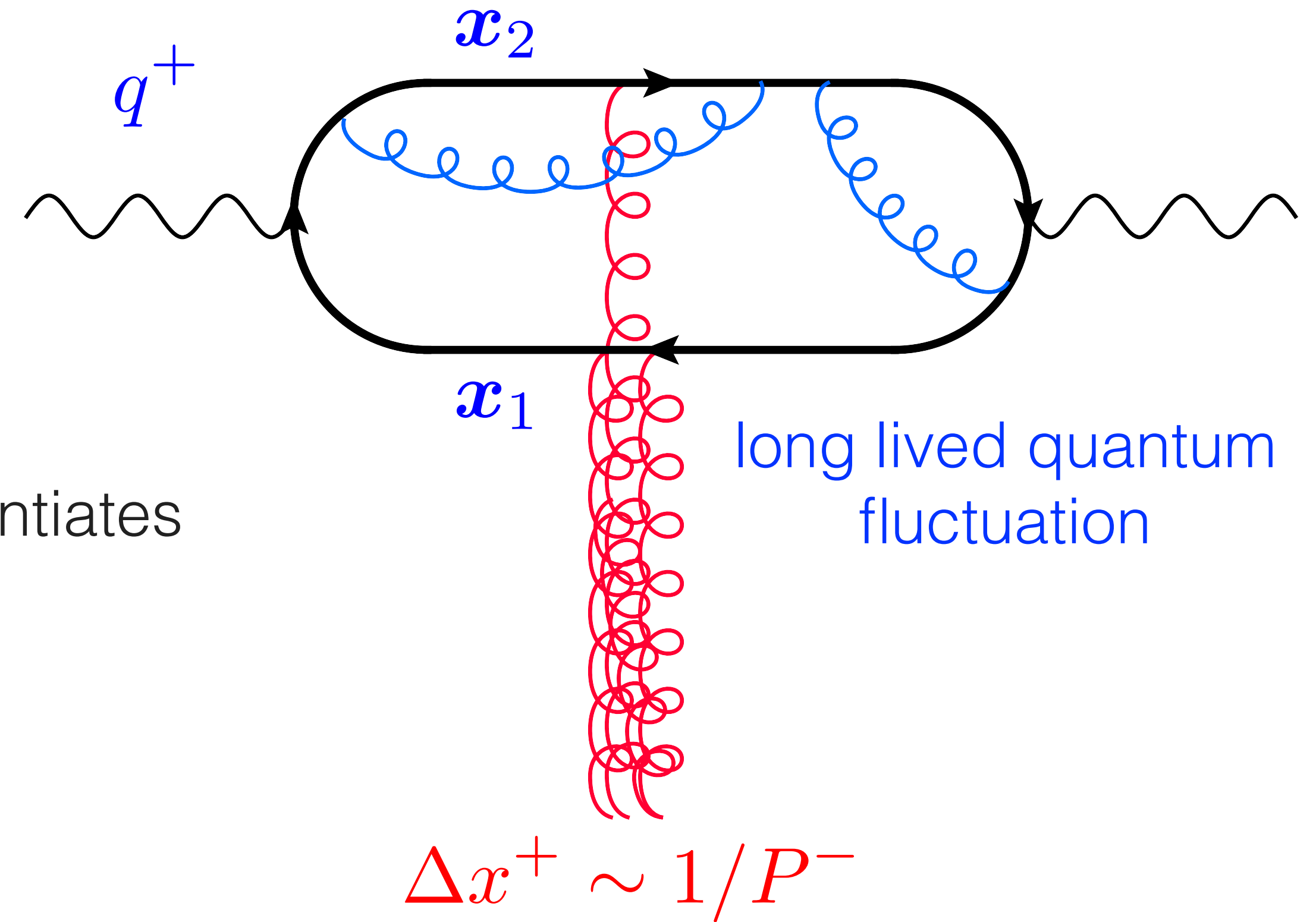
Step 2: shock wave approximation at small x
by symmetry the $-$ component of the momenta must also be strongly separated

$$q^- \equiv x_{\text{Bj}} P^- \ll P^-$$

The interaction with the background field exponentiates and **light cone time integrations decouple**

$$U_{\mathbf{x}} \equiv \mathcal{P}_+ \exp \left[ig \int_{-\infty}^{+\infty} dz^+ A^-(z^+, \mathbf{x}_\perp) \right]$$

$$\Delta x^+ \sim 1/P^-$$



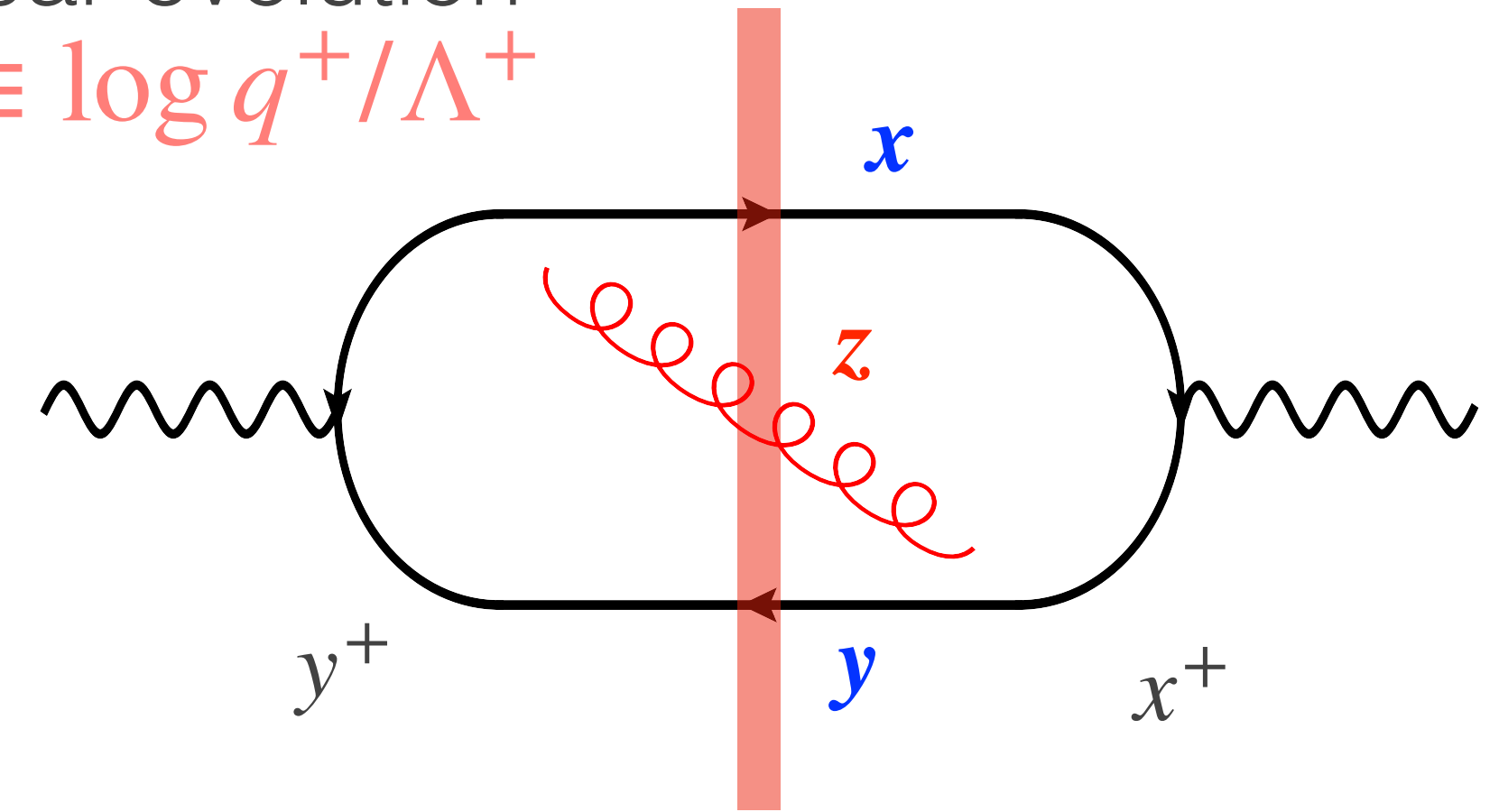
Factorization at small x :

$$\sigma \sim \int_{\mathbf{x}_1, \mathbf{x}_2} \mathcal{H}(\mathbf{x}_2 - \mathbf{x}_1) \otimes \langle P | \text{Tr } U_{\mathbf{x}_1} U_{\mathbf{x}_2}^\dagger | P \rangle$$

BK and the NLO crisis

- Balitsky-Kovchegov (1996-1999) equation describes the non-linear evolution of the dipole scattering amplitude as function of the rapidity $Y \equiv \log q^+/\Lambda^+$

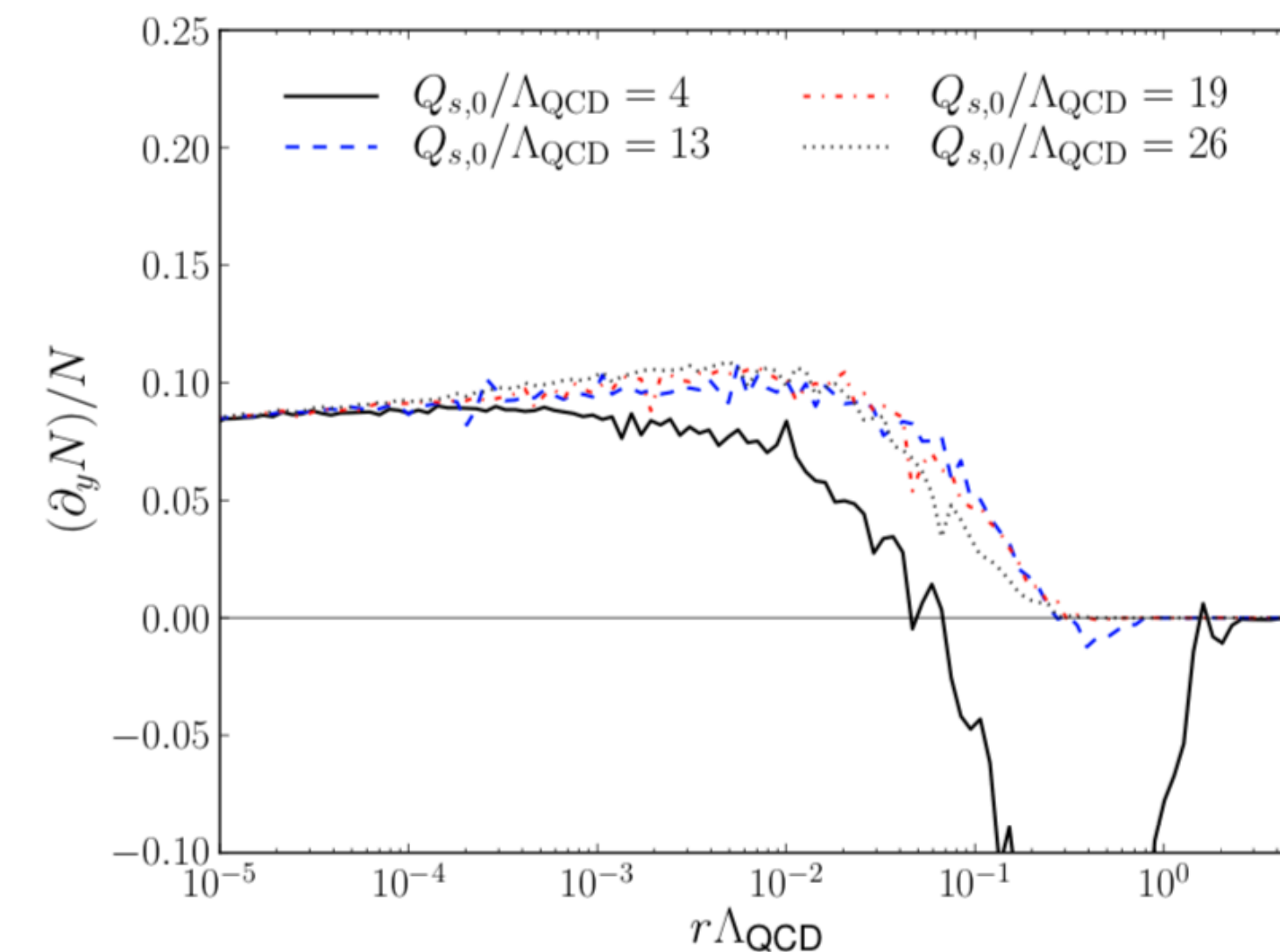
$$S_Y(\boldsymbol{x} - \boldsymbol{y}) \equiv \frac{1}{N_c} \langle \text{Tr } U(\boldsymbol{x}) U^\dagger(\boldsymbol{y}) \rangle_Y$$



$$\frac{\partial}{\partial Y} S_Y(\boldsymbol{x} - \boldsymbol{y}) = \bar{\alpha} \mathcal{K}_{NLO} \otimes [S_Y(\boldsymbol{x} - \boldsymbol{z}) S_Y(\boldsymbol{z} - \boldsymbol{y}) - S_Y(\boldsymbol{x} - \boldsymbol{y})] + \text{non-dipole}$$

- At NLO BK equation [Balitsky and Chirilli (2008)] was found to be numerically unstable

[Lappi and Mäntysaari (2015)]



BK and the NLO crisis

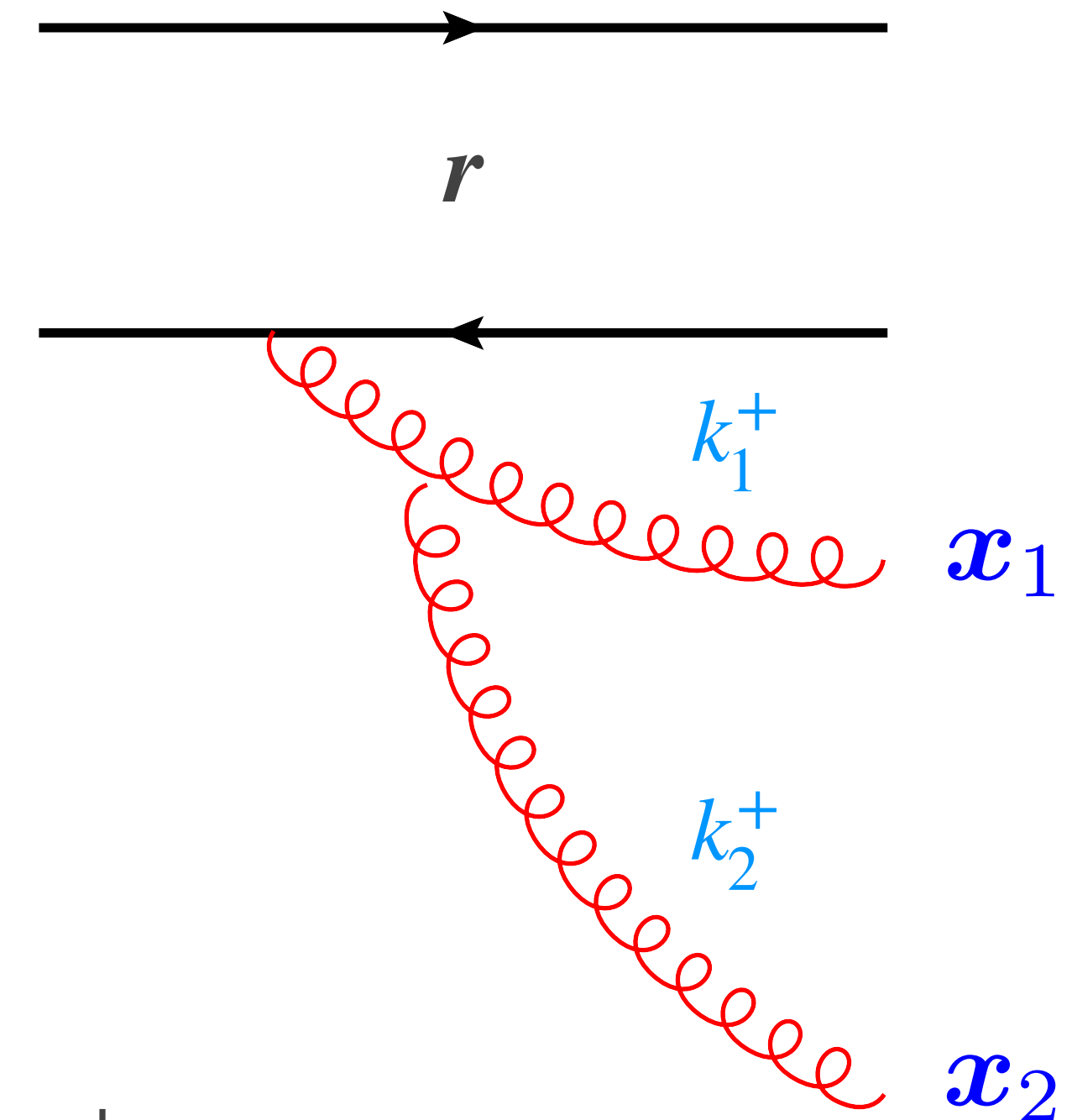
- This is due to the fact that the rapidity variable $Y \equiv \log \frac{q^+}{\Lambda^+}$ evolves independently from $x_\perp$ violating $k^- = xP^-$ ordering

formation time ordering implies

$$\begin{aligned} k_1^- = x_1^2 k_1^+ < k_2^- = x_2^2 k_2^+ &\Rightarrow \frac{x_1^2}{x_2^2} < \frac{k_2^+}{k_1^+} \ll 1 \\ k_1^+ \gg k_2^+ & \end{aligned}$$

- Small dipoles that radiate larger dipoles $x_1 \ll x_2$ generate large collinear logarithms when $k_1^- \sim k_2^-$

$$\Rightarrow \text{potentially large double logs} : \text{NLO/LO} \sim \alpha_s \log^2 \frac{1}{r_\perp^2} > 1$$



BK and the NLO crisis

[Beuf (2014) Ducloué, Iancu, Mueller, Soyez, Triantafyllopoulos (2015-2019)]

- Several solutions have been proposed: implementing kinematic bound, modification of the evolution kernel, better choice of the evolution variable, etc. However,
 - ▶ they spoil the renormalization picture established at LO
 - ▶ exhibit scheme dependence
 - ▶ no operator definition for systematic order by order calculations
 - ▶ not discussed at the level of observables: inclusive DIS

The elephant in the room: x dependence

- In the **Regge limit** distributions evaluated in the strict $x=0$ limit

$$f(k_{\perp}, x = 0)$$

- No x dependence at LO: quantum evolution generates rapidity dependence. **Ambiguous connection to x .**
- The dipole model (with locality in transverse space) is inconsistent with x dependence

Bialas, Navelet and Peschanski (2000)

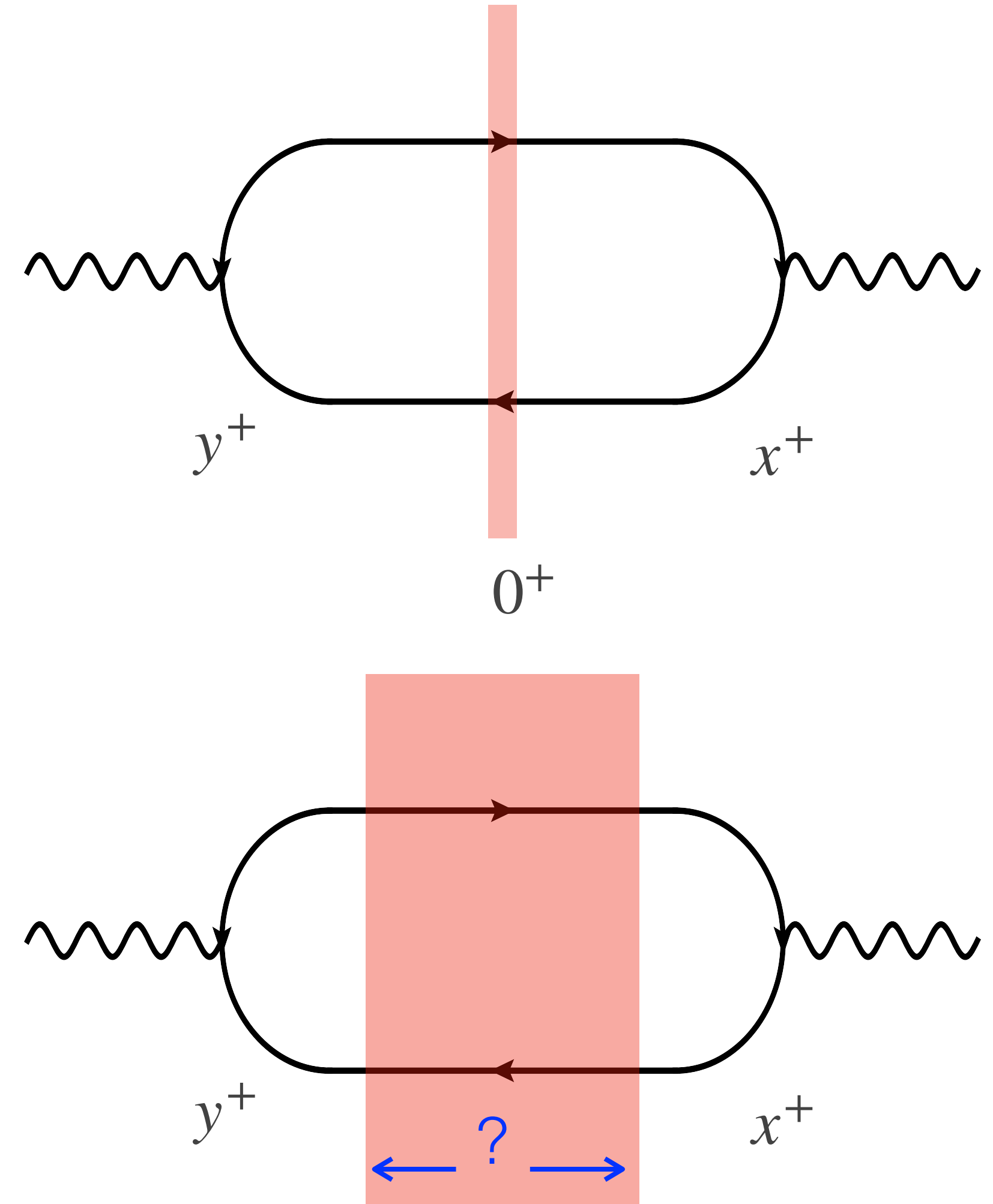
Neglecting x in the distribution is the origin of the improper account of the collinear logs at NLO

What (else) can be done to solve these issues?

- ▶ revisit the shock wave factorization scheme
- ▶ restore the x dependence of the gluon distribution to connect Regge and Bjorken limits for proper and systematic account of collinear logs

Inclusive DIS beyond shock wave

- **Decoupling of time integrals:** In the shock wave approximation the times of the photon splitting into quark antiquark pair are integrated from $0 < x^+ < +\infty$ and $-\infty < y^+ < 0$ which yields the photon wave functions
- What are the integration limits of the vertices if one relaxes the shock wave approximation?
- What is the longitudinal extent of the shock wave?



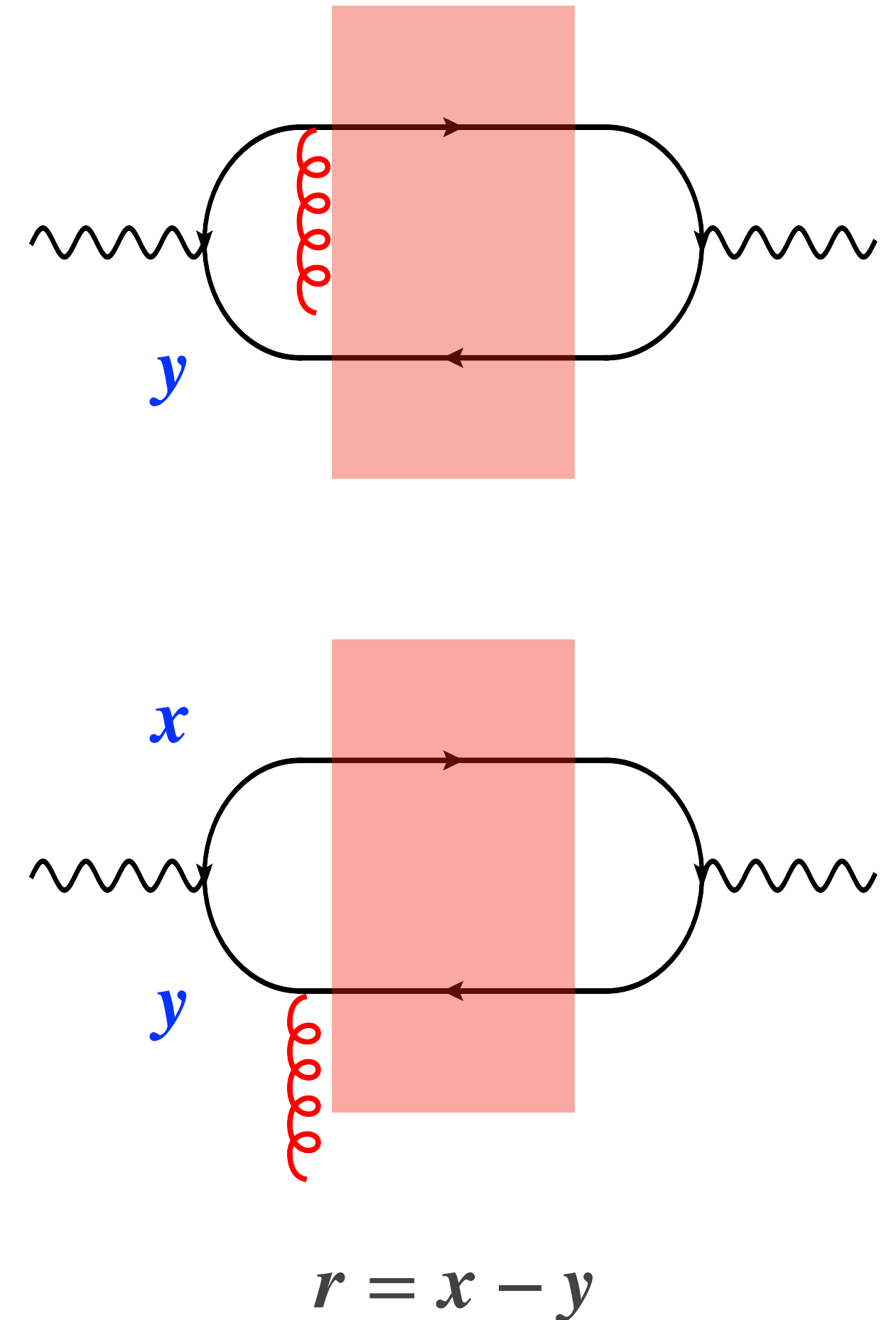
Inclusive DIS beyond shock wave

- **Solution:** extracting the first and last interactions provides a physical boundary to the shock wave
- 4 contributions that combines into to one
- Consider the left side of the diagram first
- The two gluon fields combine to generate field strength tensor

$$A^-(\mathbf{x}) - A^-(\mathbf{y}) = \int_0^1 ds \, r^i \partial^i A^-(\mathbf{y} + s\mathbf{r}) = \int_0^1 dz^i F^{i-}(\mathbf{z})$$

- Where $\mathbf{z}(s) \equiv s\mathbf{x} + (1-s)\mathbf{y}$ is a straight line trajectory in the transverse plane

- More generally: $\frac{\partial^+}{\partial x^+} [x^+, y^+]_x [\mathbf{x}, \mathbf{y}]_{x^+} [x^+, y^+]_y$



Inclusive DIS beyond shock wave

- DIS cross-section takes a similar form to that of the shock wave: **identical wave functions**

Shock wave factorization:

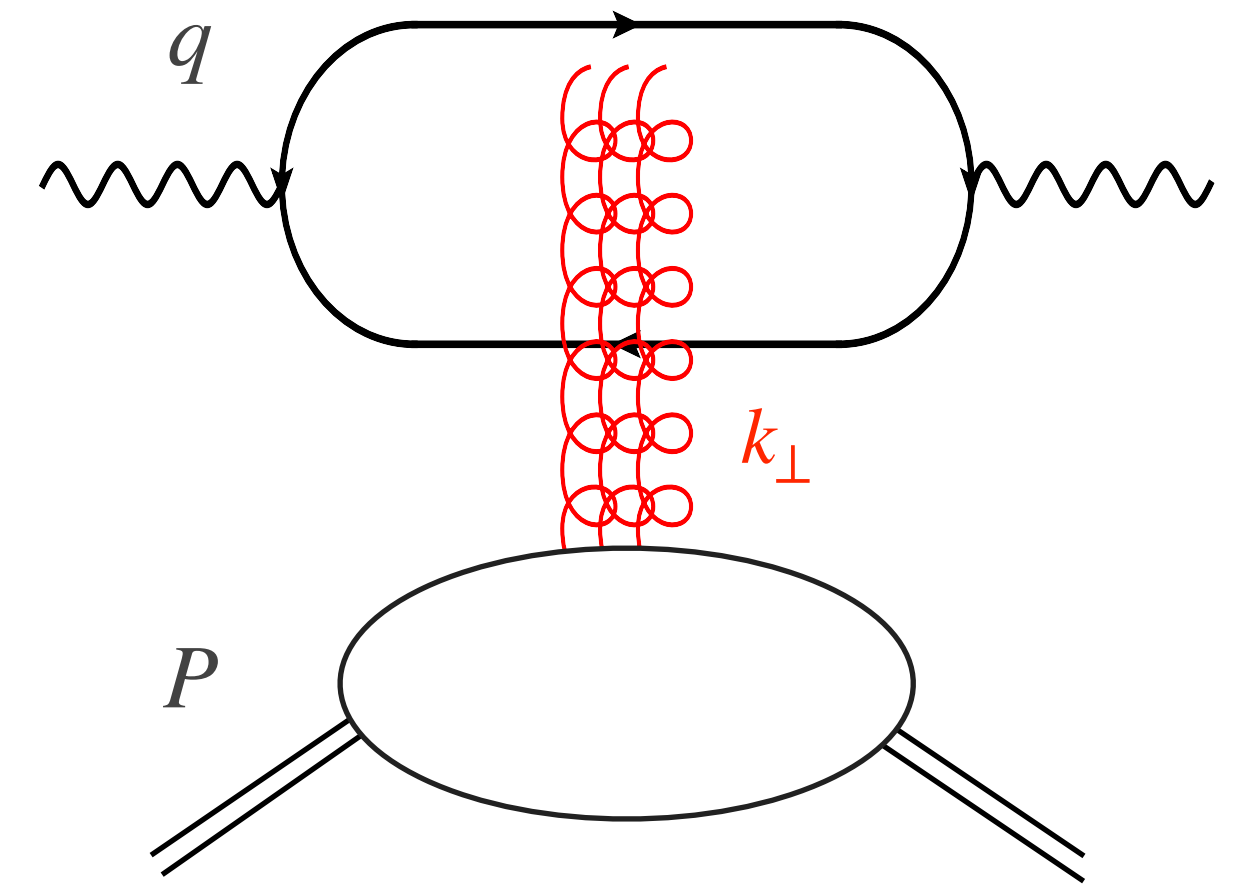
$$\sigma(x_{Bj}, Q^2) \sim e^2 \int_0^1 dz P(z) \int_{r,b} d\mathbf{r} |\varphi(z(1-z)|\mathbf{r}|^2 Q^2)|^2 \langle \text{Tr} U(\mathbf{r}) U^\dagger(0) \rangle_Y$$

Beyond shock wave:

$$\sigma(x_{Bj}, Q^2) \sim e^2 \int_0^1 dz \int_{r_1, r_2, b_1, b_2} \varphi(z(1-z)|\mathbf{r}_1|^2 Q^2) \varphi^*(z(1-z)|\mathbf{r}_2|^2 Q^2) \\ \times e^{i x_{Bj} P^-(x_2^+ - x_1^+)} \langle P | O_{\text{dipole}}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{b}_1, \mathbf{b}_2, x_1^+, x_2^+ | z) | P \rangle$$

x-dependent Fourier phase

gauge invariant dipole operator



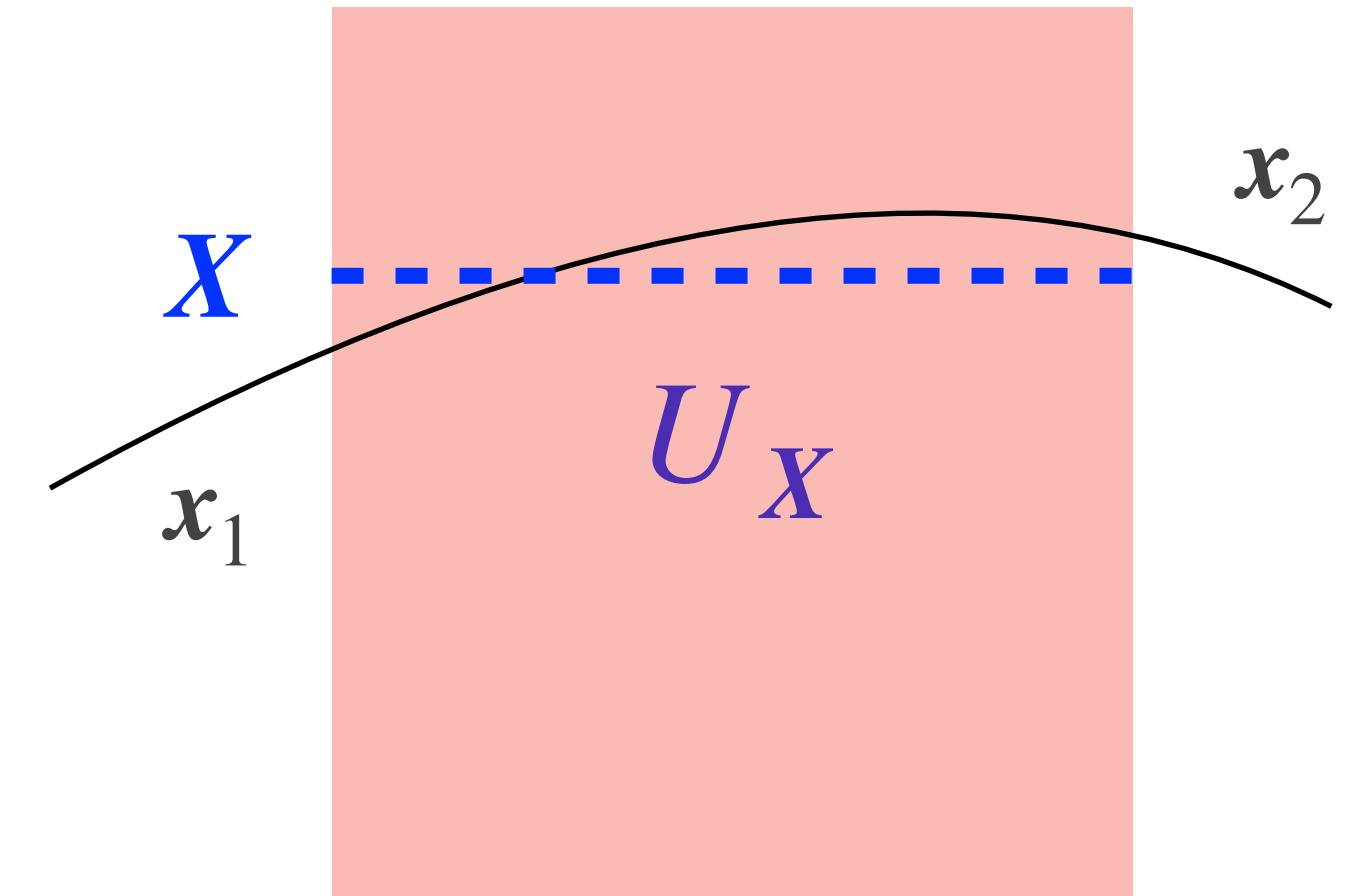
Partial twist expansion

- Leading power in s and Q^2 can be obtained by neglecting transverse recoil of fast partons

$$\Delta \mathbf{x}^2 \sim x_{\text{Bj}}/Q^2$$

x_{Bj} -suppressed in the Regge limit

Q^2 -suppressed in the Bjorken limit



$$\mathcal{G}_{p^+}(x^+, \mathbf{x}_2; y^+, \mathbf{x}_1) = \mathcal{G}_0(\mathbf{x}_2 - \mathbf{x}_1, x_2^+ - y_1^+) U_X(x_2^+, x_1^+) + \dots$$

Altinoluk, Armesto, Beuf, Martinez, Salgado (2015)

shock wave (eikonal limit)

$$\lim_{p^+ \rightarrow +\infty} \mathcal{G}_{p^+}(x^+, \mathbf{x}; y^+, \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y}) U_x(x^+, y^+)$$

- $\mathbf{x}$ encoded in quantum diffusion: FT w.r.t. $\mathbf{u} = \mathbf{x}_2 - \mathbf{x}_1$

$$\mathcal{G}_{p^+}(x_2, x_1^+, X; \ell) = e^{i \frac{\ell^2}{2zq^+} \Delta x^+} U_X(x_2^+, x_1^+) + \dots$$

$$x_F \equiv \frac{\ell^2 + z\bar{z}Q^2}{2z\bar{z}s}$$

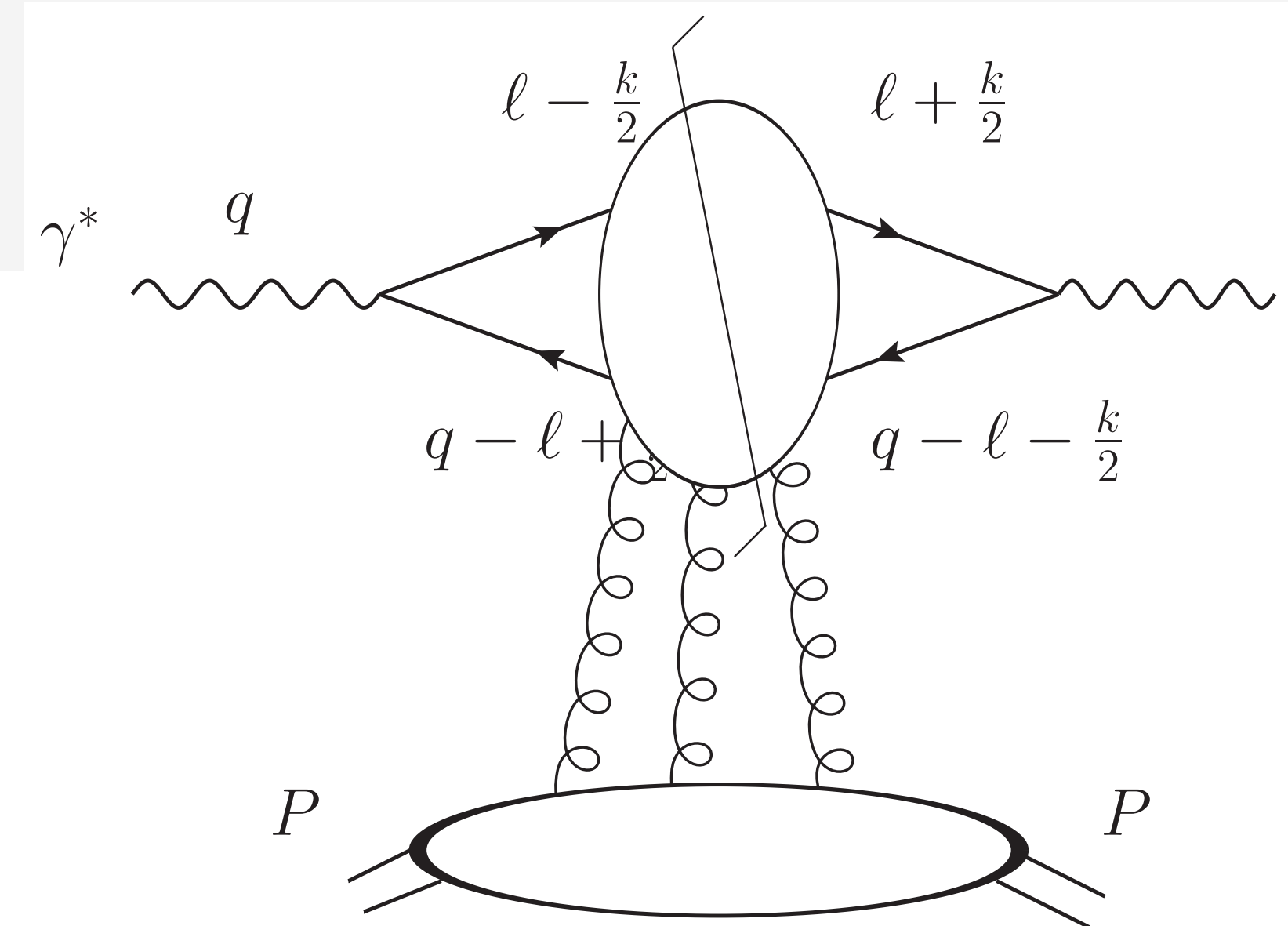
A Novel Unintegrated Gluon Distribution

2006.14569 [hep-ph]

Factorization formula for DIS at arbitrary x

- After integration over the factorized free propagators that lead to the Feynman x phase, we obtain the factorization formula (for the transverse photon cross-section), in momentum space,

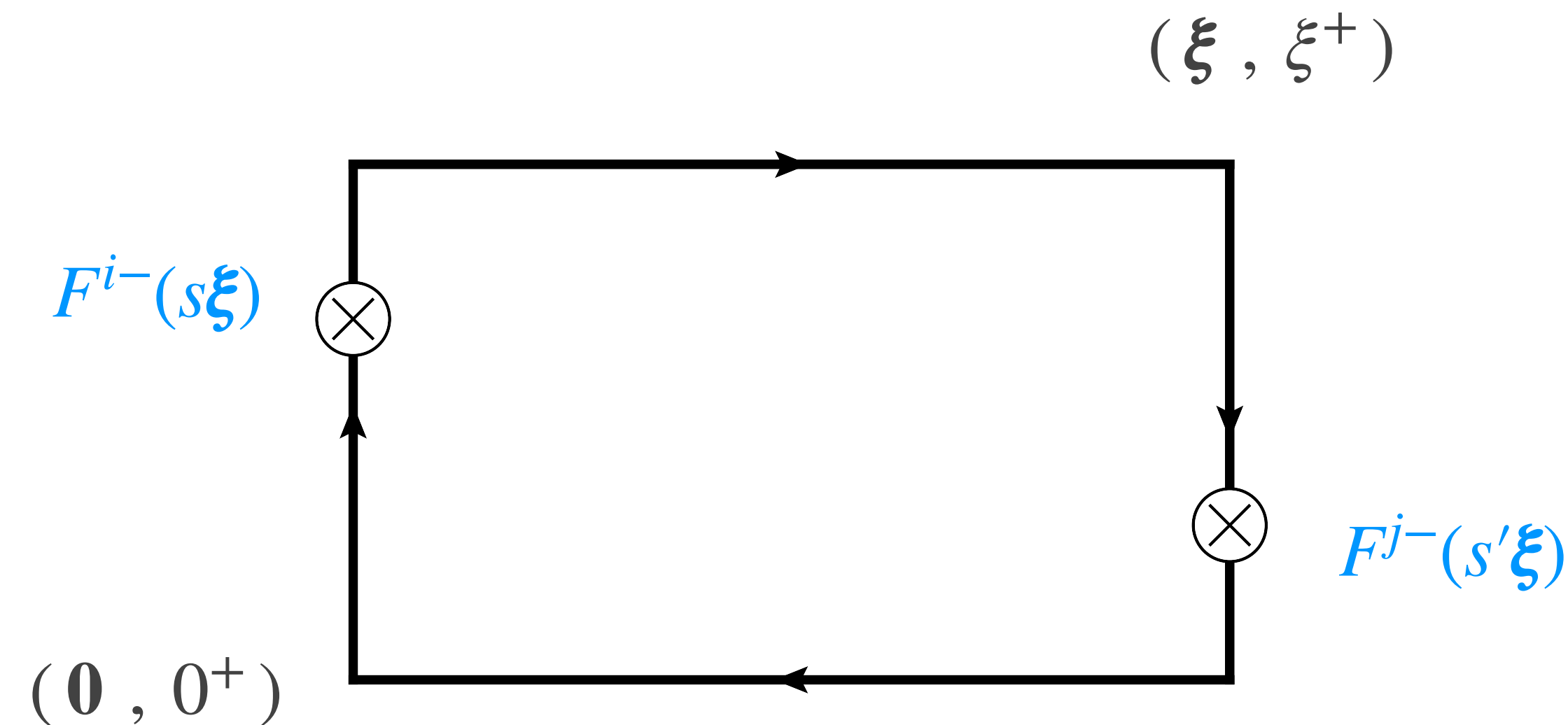
$$\sigma(x_{Bj}, Q^2) \sim e^2 \int_0^1 dz \int_0^1 dx \int_{\ell, k} \partial^i \varphi \left(\ell - \frac{k}{2} \right) \partial^j \varphi^* \left(\ell + \frac{k}{2} \right) \delta \left(x - x_{Bj} - \frac{\ell^2}{2z\bar{z}q^+} \right) \\ \times x G^{ij}(x, k) + O(k_{\perp}^2/s)$$



- Same wave functions as in small x
- The delta function relates x in the gluon distribution to x_{Bj}
- Gluon distribution different from small x

x-dependent and gauge invariant unintegrated gluon distribution

$$xG^{ij}(x, k_{\perp}) \equiv 2 \int \frac{d\xi^+ d\xi}{(2\pi)^3 P^-} e^{i x P^- \xi^+ - i k \cdot \xi} \langle P | \text{Tr} [0, \xi^+]_{\xi} F^{j-}(\xi^+, s' \xi) [\xi^+, 0]_0 F^{i-}(0, s \xi) | P \rangle$$

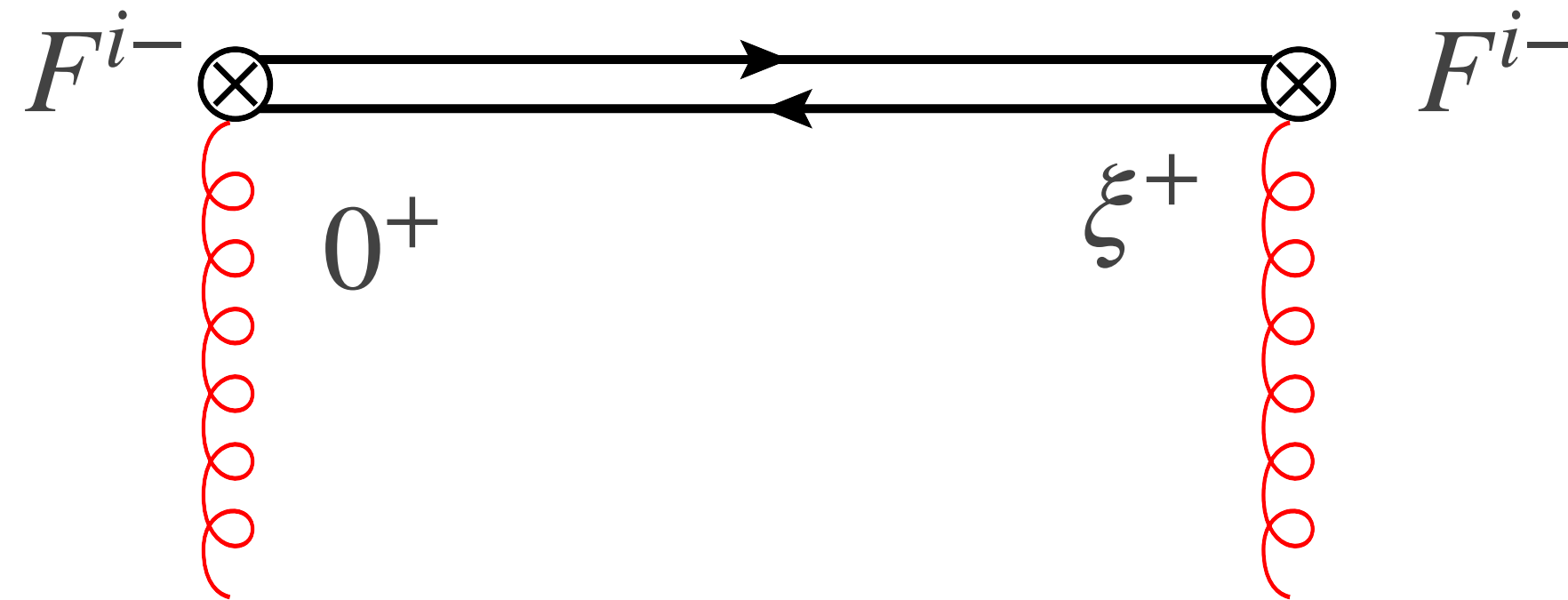


- Note that this uPDF involves finite Wilson lines in contrast to gluon TMDs such as Weizsacker-Williams

Bjorken and Regge limits of the uPDF

- **Collinear limit:** Integrating over $k_{\perp}$ yields $\xi_{\perp} = 0$ and we recover the gluon PDF

$$xg(x, \mu^2) = 2 \int \frac{d\xi^+}{(2\pi)P^-} e^{ixP^-\xi^+} \langle P | \text{Tr} [0, \xi^+] F^{i-}(\xi^+) [\xi^+, 0] F^{i-}(0) | P \rangle$$



- **Small-x limit:** At $x = 0$ we recover the small x dipole operator

$$\xi^i \xi^j G^{ij}(x=0, \xi) \rightarrow \langle P | \text{Tr} U_{\xi} U_0^{\dagger} | P \rangle$$

Provides the interpolation between the **leading twist term** in the **Bjorken limit** and the **eikonal term** in the **Regge limit**

Complete DIS at one-loop order

- In the Bjorken limit $Q^2 \rightarrow \infty$, we reproduce the 1-loop contribution to the DIS structure function

$$F_T(x_{Bj}, Q^2) = \frac{\alpha_s}{\pi} \sum_f q_f^2 \int_{x_{Bj}}^1 dy \, xg(x_{Bj}/y, \mu^2) \\ \times \left[\frac{1}{\epsilon} \left(\frac{e^{\gamma_E}}{4\pi} \right)^\epsilon P_{qg}(y) + [(1-y)^2 + y^2] \log \left[\frac{Q^2(1-y)}{\mu^2 y} \right] - 1 + 4y(1-y) \right]$$

J. Collins, Foundations of pQCD 2011

Summary

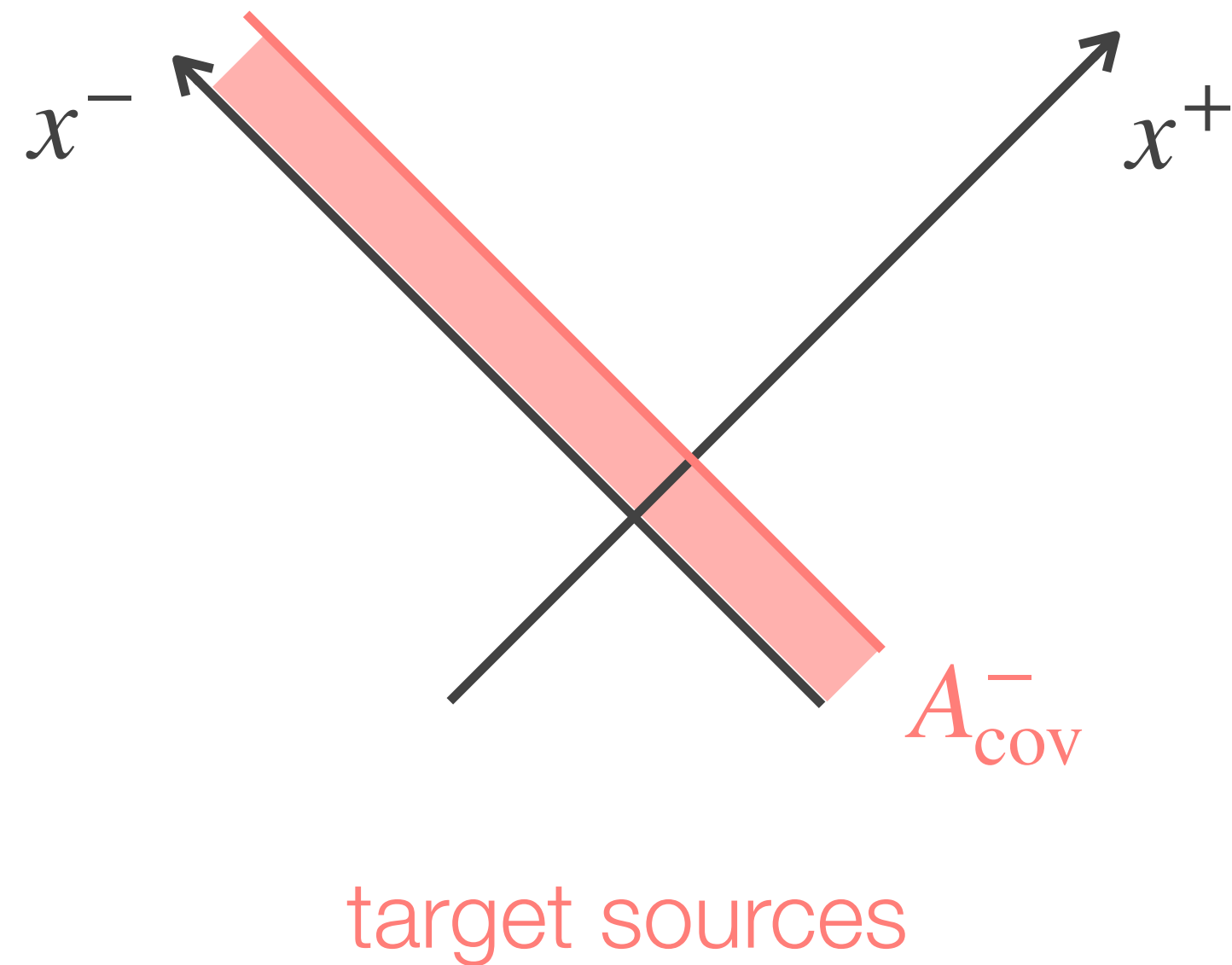
- There are issues with **collinear logs at small x** that can be traced back to the shock wave prescription
- **Minimal correction** of the semi-classical approach to small x solves the problem from first principles
- In the case of inclusive DIS: while the hard part is unchanged a **new (gauge invariant) unintegrated gluon distribution** compatible with x dependence emerges that interpolates between the dipole operator at small x and the gluon PDF at leading twist

Outlook

- Compute the evolution equation for the x -dependent uPDF
- Investigate other processes (eg. DVCS)
- Potential probe of gluon saturation on the lattice?

Backup

Background field and transverse gauge links



- consider a target boosted along the $-z$ direction close to the light cone. Due to time dilation the target color sources are “frozen” in the $-$ direction
- Yang-Mills equations $[D_\mu, F^{\mu\nu}] = J^\nu$ can be solved exactly (together with the continuity equation $[D_\mu, J^\mu] = 0$) in covariant gauge $\partial \cdot A = 0$ (or light-cone gauge $A^+ = 0$)

$$J^\nu(x) \rightarrow J^-(x^+, x_\perp)$$

and $J^+ = J_\perp = 0$

$$A_{\text{cov}}^- = -\frac{1}{\partial_\perp^2} J^- \quad \text{and} \quad A^+ = A_\perp = 0$$

Background field and transverse gauge links

- under an arbitrary gauge rotation $\Omega(x^+, x_\perp)$ the target field transforms as

$$A^- \rightarrow \Omega_x(x^+) A_{\text{cov}}^-(x^+, \mathbf{x}) \Omega_x^{-1}(x^+) - \frac{1}{ig} \Omega_x(x^+) \partial^- \Omega_x^{-1}(x^+)$$

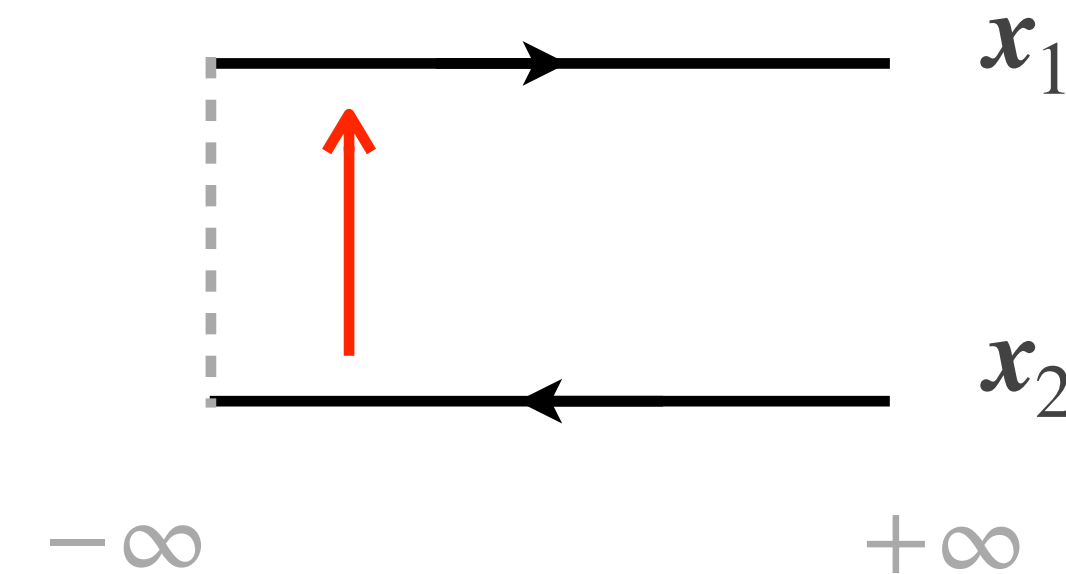
$$A^i \rightarrow -\frac{1}{ig} \Omega_x(x^+) \partial^i \Omega_x^{-1}(x^+)$$

- exploiting the residual gauge freedom we can generate a transverse pure gauge
- N.B.: the partonic picture is manifest in the LC-gauge $A^- = 0$ (with $A_\perp \neq 0$)
- small x observables are (in the dilute/dense limit) more naturally expressed in the wrong LC-gauge $A^- \neq 0$ (with $A_\perp = 0$).
- in order to connect to the partonic interpretation one needs to deal with transverse fields

Background field and transverse gauge links

- geometric interpretation of the all twist resummation

$$U_{x_1} = U_{x_2} - r^i \int_0^1 ds (\partial^i U_{x_2+sr})$$



- and noticing that $\frac{1}{ig}(\partial^i U_x)U_x \equiv A^i(x)$, one can express the **dipole operator** (in the background field A^-) as a transverse gauge link:

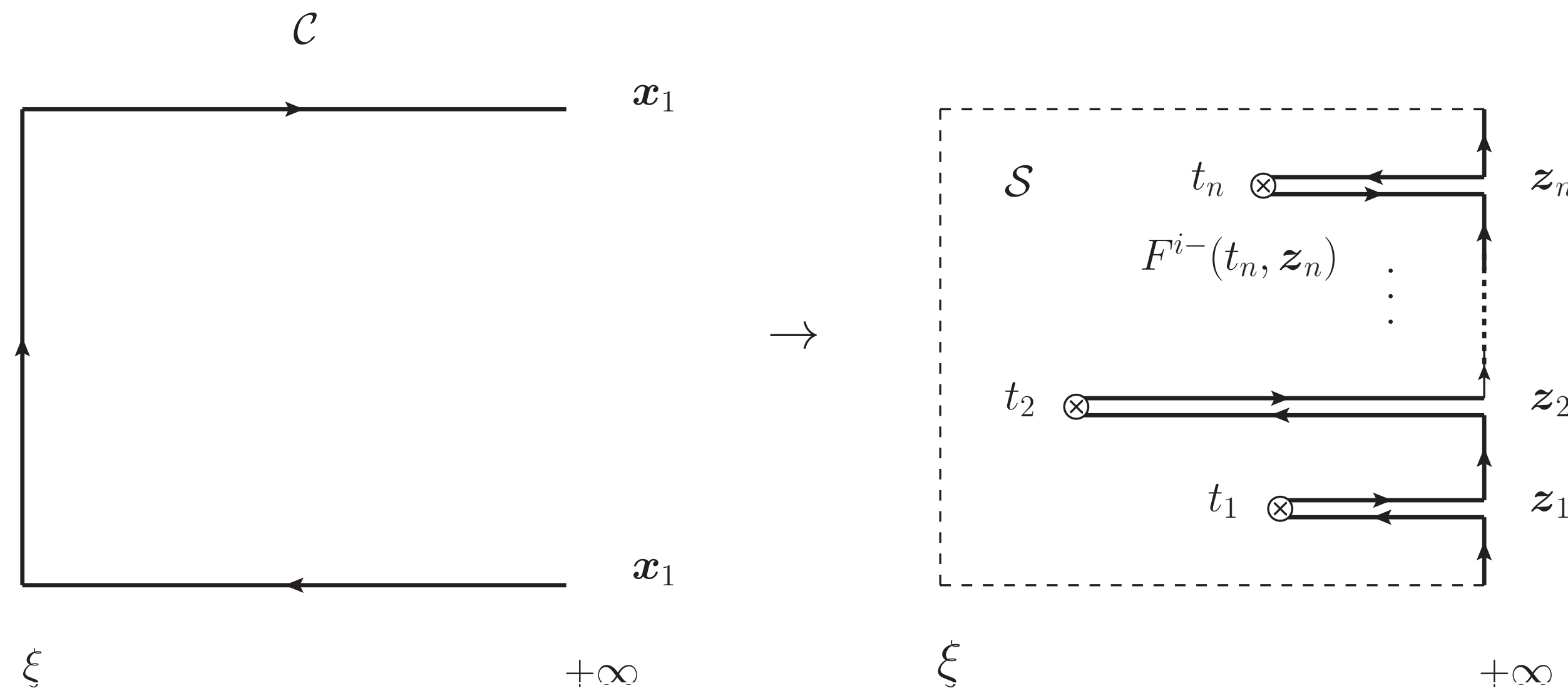
$$U_{x_1} U_{x_2}^\dagger = [x_1, x_2] = 1 - ig \int_{x_2}^{x_1} dz A^i(z) [z, x_2]$$



- dipole operator can be expressed in terms of transverse link operators

Background field and transverse gauge links

- **non-Abelian Stokes' theorem:** more generally, the dipole operator can be written as a path ordered tower of “twisted” field strength tensor (i.e. dressed with future pointing Wilson lines)



[Fishbane, Gasiorowicz, Kaus (1981) Wiedemann (2000)
YMT, Boussarie (2020)]

$$U_{x_2} U_{x_1}^\dagger \equiv P \exp \left[-ig \int_S dt dz \left[+\infty, x^+ \right]_x F^{i-}(x^+, \mathbf{x}) \left[x^+, +\infty \right]_x \right]$$

PDF's and gauge invariance

- Number density interpretation possible in light-cone gauge $A^- = A^0 - A^3 = 0$

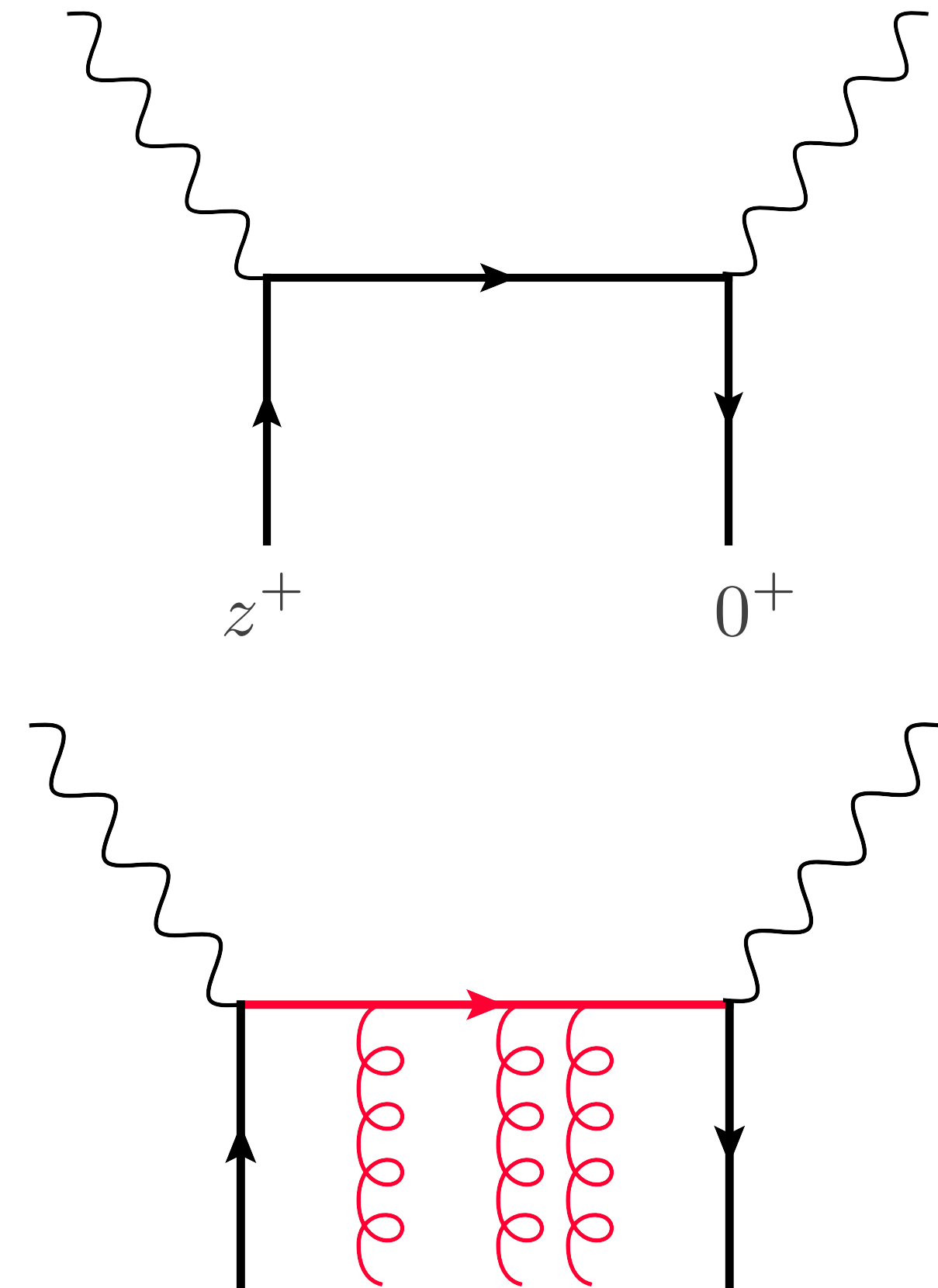
$$xq(x) \propto \int dz^+ e^{i x P^- z^+} \langle P | \bar{\psi}(z^+) \gamma^- \psi(0) | P \rangle$$

$$P^- = \frac{E - P_z}{\sqrt{2}} \quad x^+ = \frac{t + z}{\sqrt{2}}$$

- In an arbitrary gauge a path ordered Wilson line (gauge link) is required. Resums interactions with longitudinally polarized gluons in the target

$$xq(x) \propto \int dz^+ e^{i x P^- z^+} \langle P | \bar{\psi}(z^+) [z^+, 0^+] \psi(0) | P \rangle$$

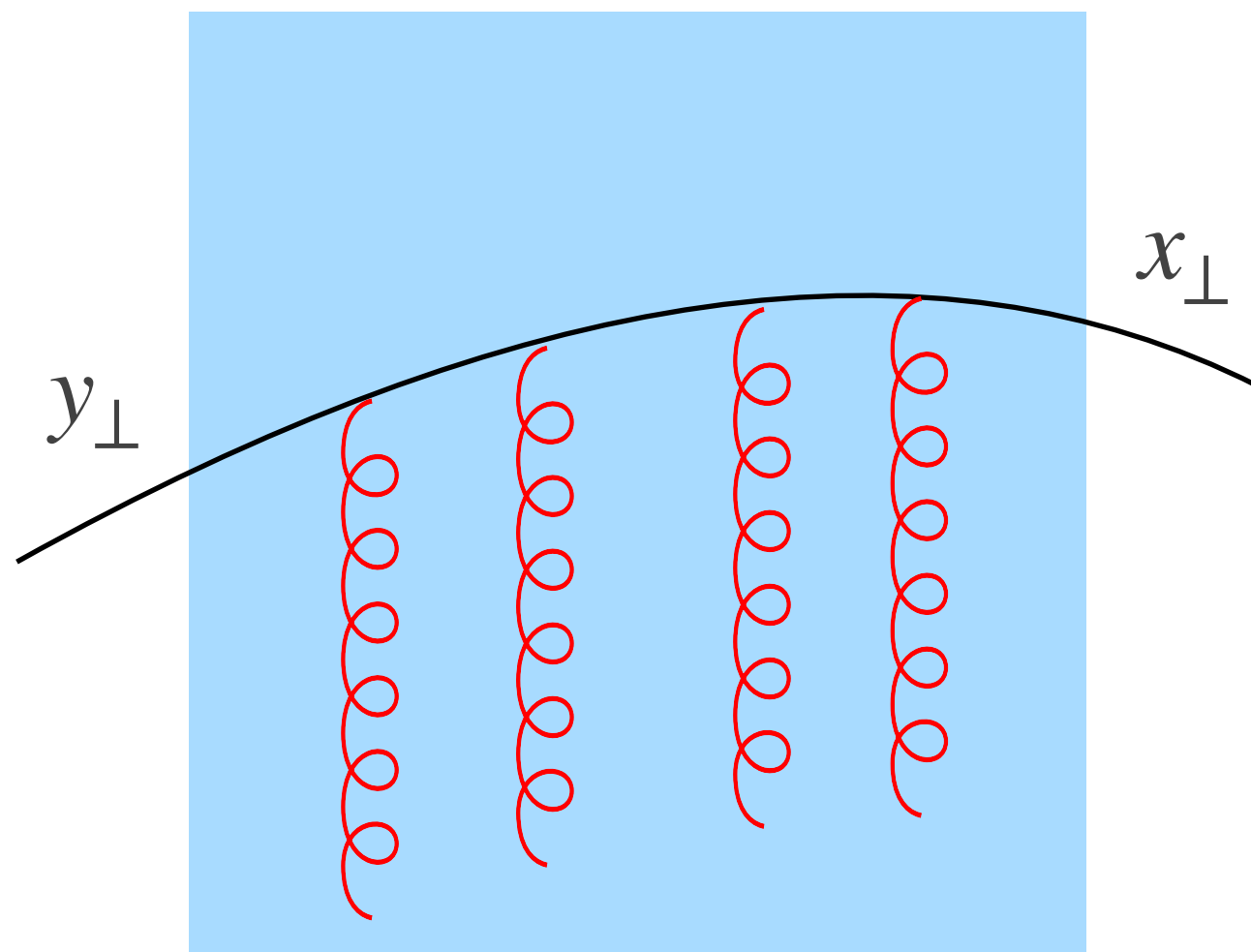
$$[z^+, 0^+] \equiv \mathcal{P}_+ \exp \left[i g \int_0^{z^+} dz'^+ A^-(z'^+, z_\perp = 0) \right]$$



$$A^- \equiv A_a^- t^a$$

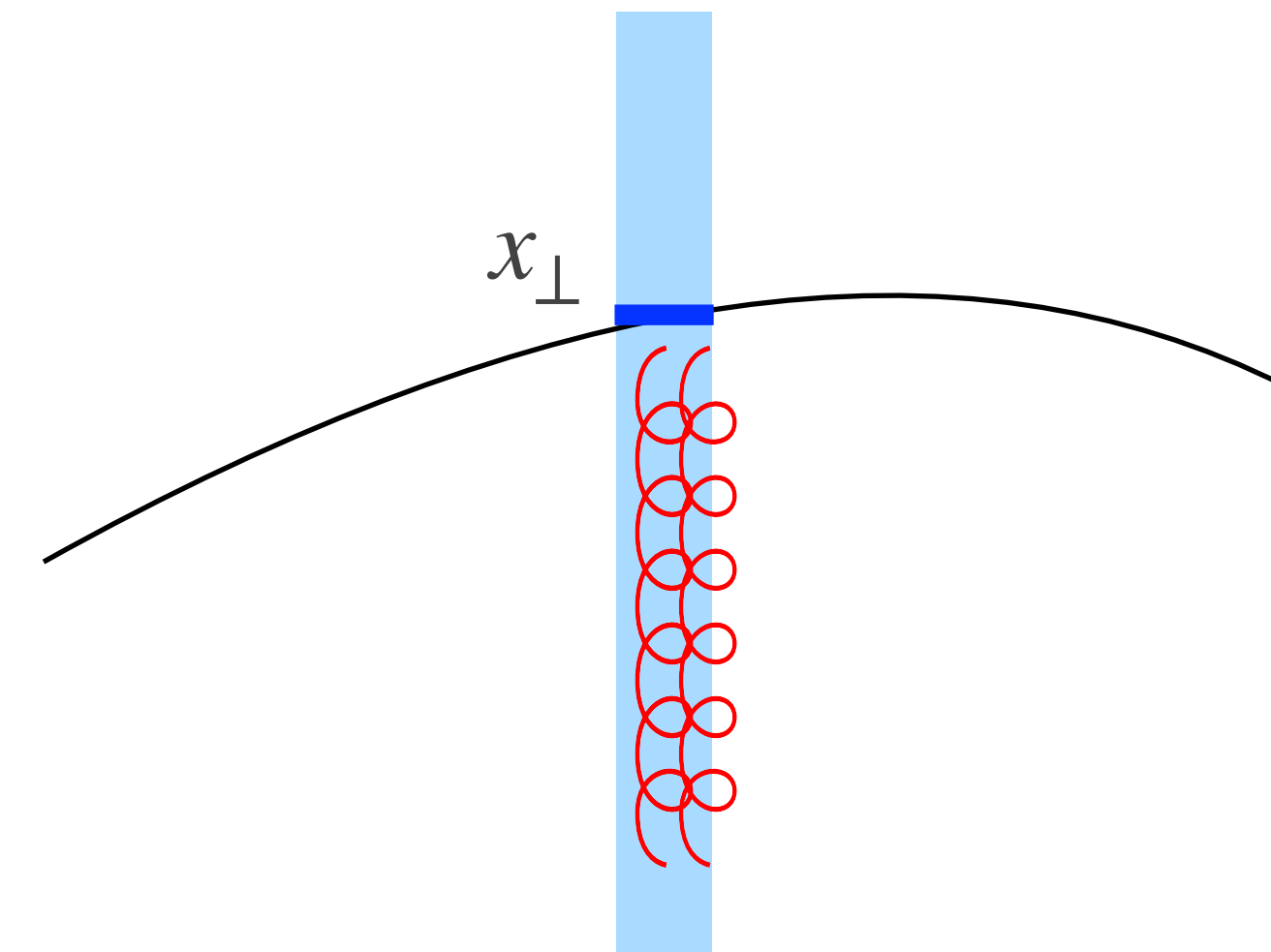
Revisiting the Shock Wave Approximation

- Under the assumption that all transverse momenta are of same order along the ladder, the leading power $1/s$ is obtained letting $p^+ \rightarrow \infty$ for any particle propagating inside the shock wave



Non-eikonal propagator

$$\mathcal{G}_{p^+}(x^+, y^+) = \left[i \frac{\partial}{\partial x^+} - \frac{\hat{p}_\perp^2}{2p^+} - gA^- \right]^{-1}$$



shock wave (eikonal limit)

$$\lim_{p^+ \rightarrow +\infty} \mathcal{G}_{p^+}(x^+, \mathbf{x}; y^+, \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y}) U_x(x^+, y^+)$$

Revisiting the Shock Wave Approximation

- This limit neglects quantum diffusion

$$\mathcal{G}_{p^+}^0(x^+, \mathbf{x}; y^+, \mathbf{y}) = \frac{p^+}{2i\pi \Delta x^+} e^{i \frac{(x-y)^2 p^+}{\Delta x^+}}$$

- it is important when $(\Delta \mathbf{x})^2 \sim \Delta x^+ / p^+ \sim s^{-1}$
- In effect, the phase relates the transverse dynamics to longitudinal dynamics, this is the phase that appears in the definition of PDF's
- It encodes the information about k^- 's in the target. It is expected to be non-negligible away from the strongly ordered region in k^- .

Factorization formula for DIS at arbitrary x

- Combining all phases we obtain

$$ik^- \Delta x^+ \equiv i \frac{\ell^2 + z\bar{z}Q^2}{2z\bar{z}q^+} \Delta x^+$$

- This is nothing but Feynman x that we encounter when deriving the DGLAP limit

$$x_F \equiv \frac{\ell^2 + z\bar{z}Q^2}{2z\bar{z}s}$$

Inclusive DIS beyond shock wave

- applying the same trick to the r.h.s. we obtain the following hadronic operator

$$O_{\text{dipole}}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_1, \mathbf{y}_2, x_1^+, x_2^+) \equiv \int_{\mathbf{y}_1}^{\mathbf{x}_1} dz_1^i \int_{\mathbf{y}_2}^{\mathbf{x}_2} dz_2^j$$

$$\langle P | \text{Tr } \mathcal{G}(\mathbf{y}_2, x_2^+; \mathbf{y}_1, x_1^+ | (1-z)q^+) F^{j-}(x_1^+, z_1) \mathcal{G}(\mathbf{x}_1, x_1^+; \mathbf{x}_2, x_2^+ | zq^+) F^{i-}(x_2^+, z_2) | P \rangle$$

- performing a gauge rotation leads to transverse gauge links: **explicit gauge invariance**
- dependence on + momenta of the dipole zq^+ and $(1-z)q^+$ can be factorized with further approximations

