Evolution with transverse momentum dependent splitting functions

based on ongoing work with
- Francesco Hautmann (Antwerp, Oxford)
- Hannes Jung (Hamburg)
- Lissa Keersmaekers (Antwerp)
- Aleksander Kusina (Cracow)
- Krzysztof Kutak (Cracow)
- Aleksandra Lelek (Antwerp)

Martin Hentschinski
Universidad de las Americas Puebla
Ex-Hacienda Santa Catarina Martir S/N
San Andrés Cholula
72820 Puebla, Mexico
martin.hentschinski@gmail.com

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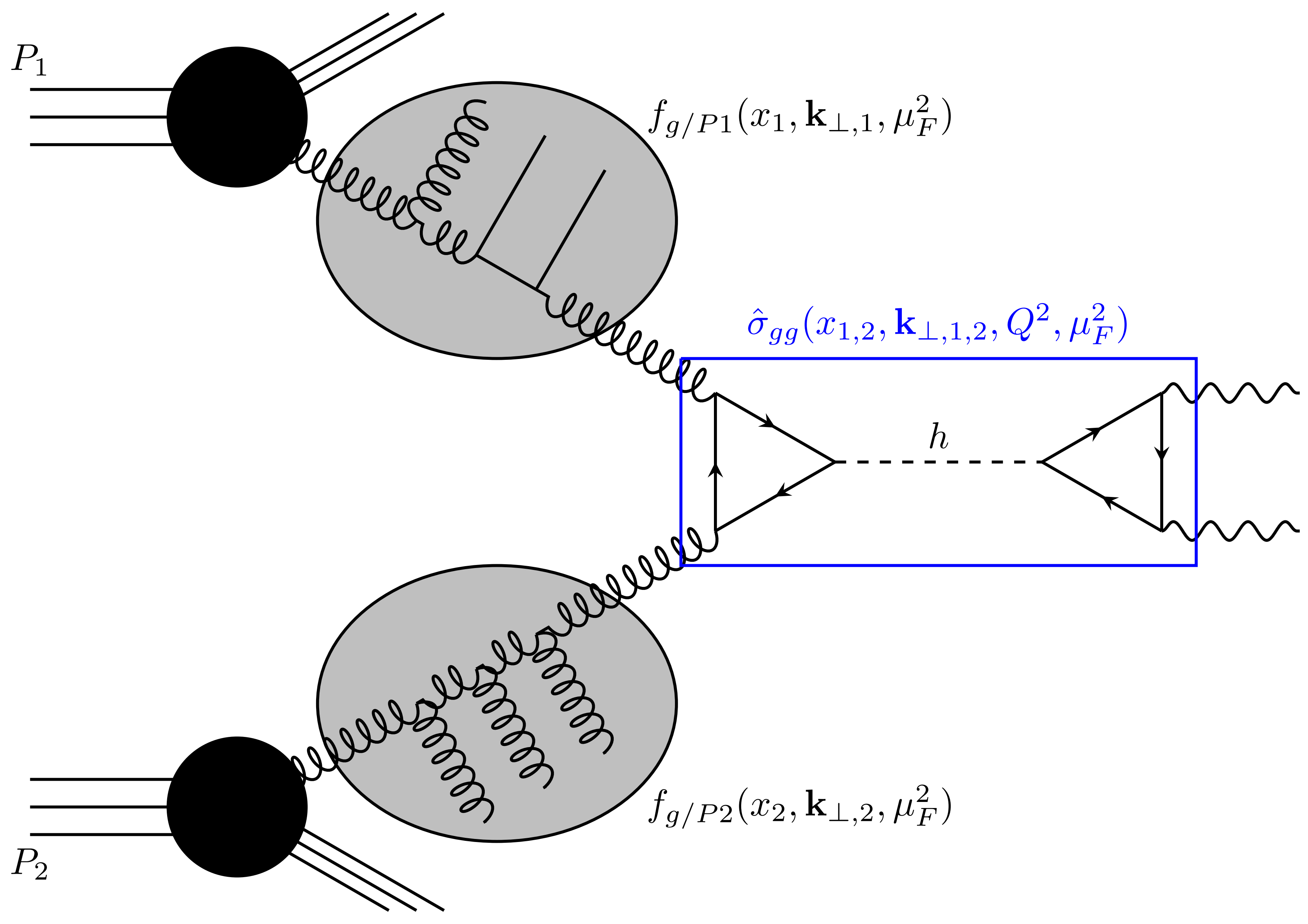
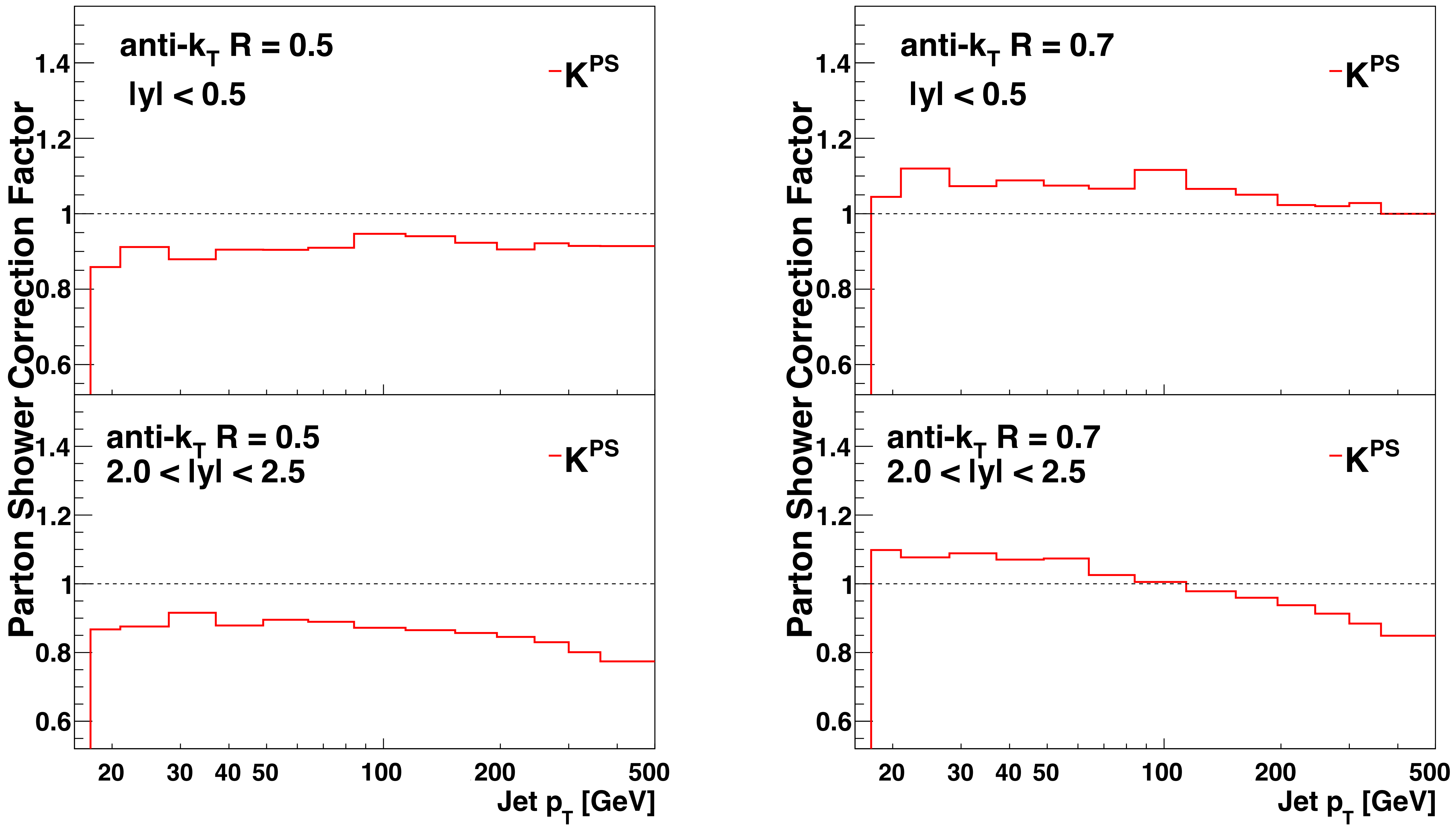
Transverse Momentum Dependent pdfs at LHC



conventional collinear factorization: TM in final state from higher order perturbative corrections
→ slows down convergence of perturbative series



NLO-MC for jet p_T

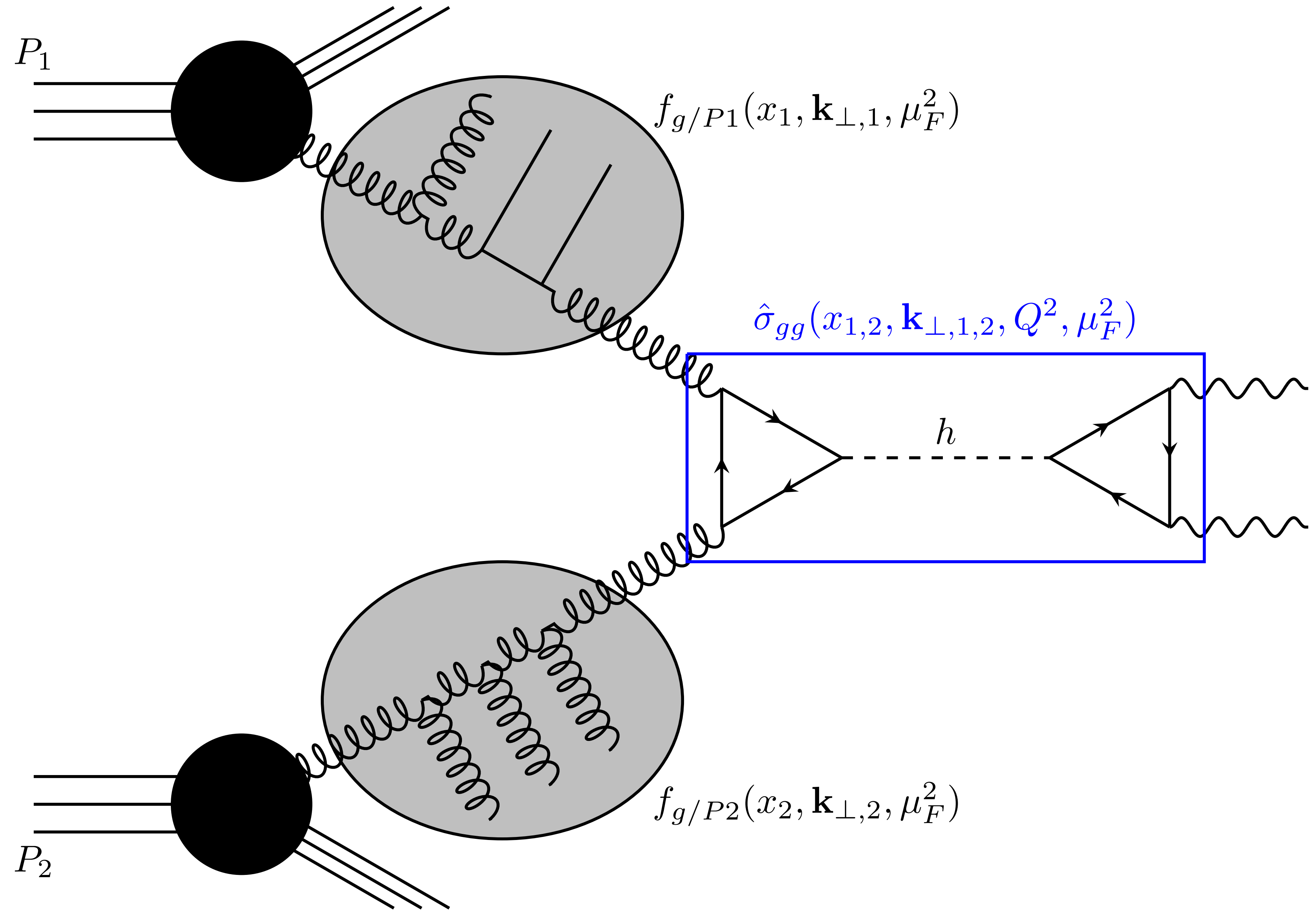
$$K^{PS} = \frac{N^{(ps)}_{NLO-MC}}{N^{(0)}_{NLO-MC}}$$

Dooling, Gunnellini, Hautmann, Jung; 1304.7180

Parton Branching method

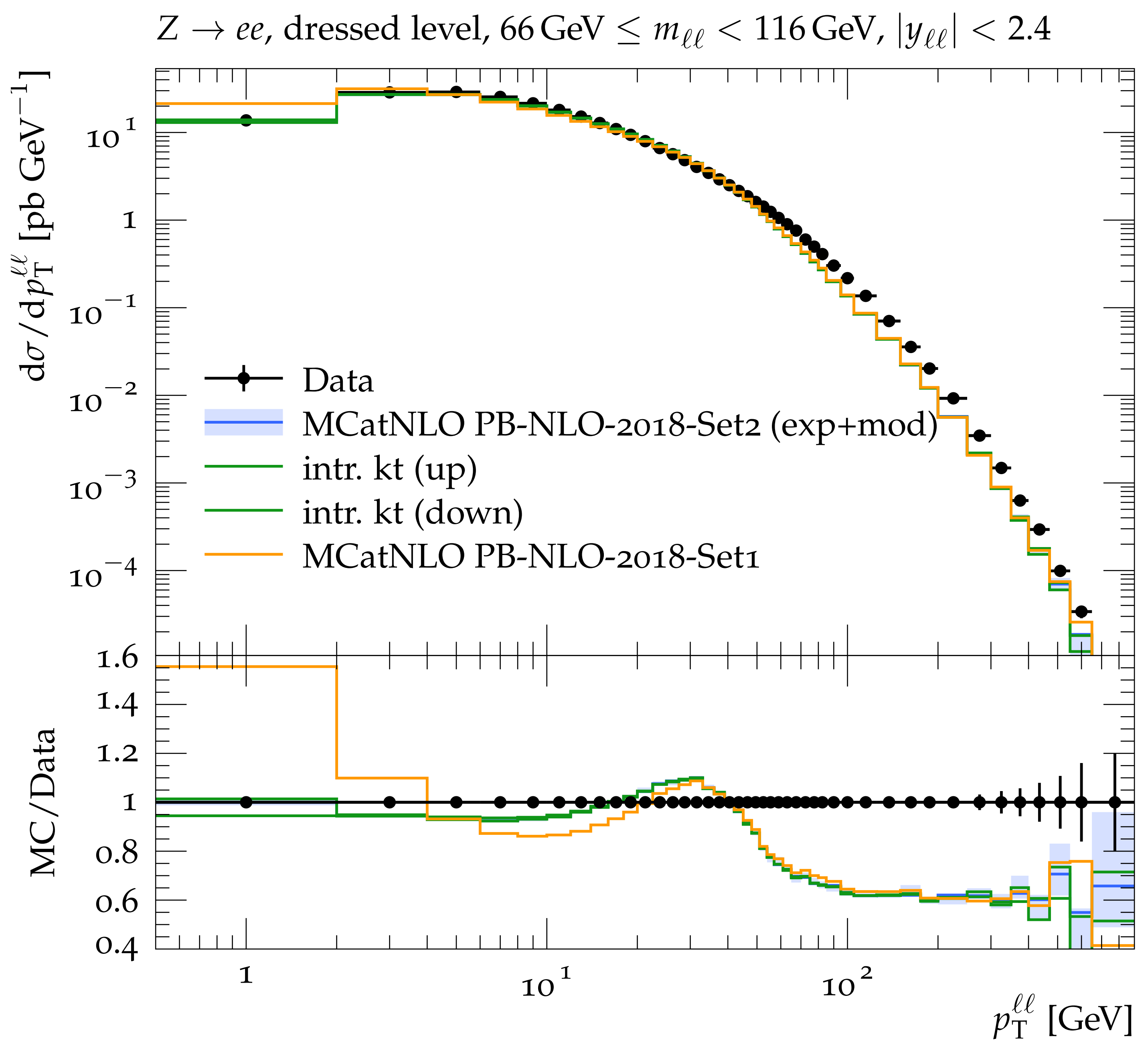
underlying idea:

- keep track of transverse momenta along the DGLAP evolution chain → TMD PDF (or unintegrated pdf) → MC formulation of DGLAP evolution
- $P^R_{ab}(z)$ real splitting kernels: prob. that parton splits
- Sudakov form factor: prob. that parton does not split
- both closely related through momentum sum rules



- can be formulate at LO, NLO, NNLO
- implemented in Xfitter framework
- particularly useful for MC studies
- available through https://tmdlib.hepforge.org/, see also [2103.0974]
- also implemented in Cascade MC [Baranov et. al.; 2101.10221]

phenomenology: Z-boson production [Bermudez Martinez et. al. 1906.00919]



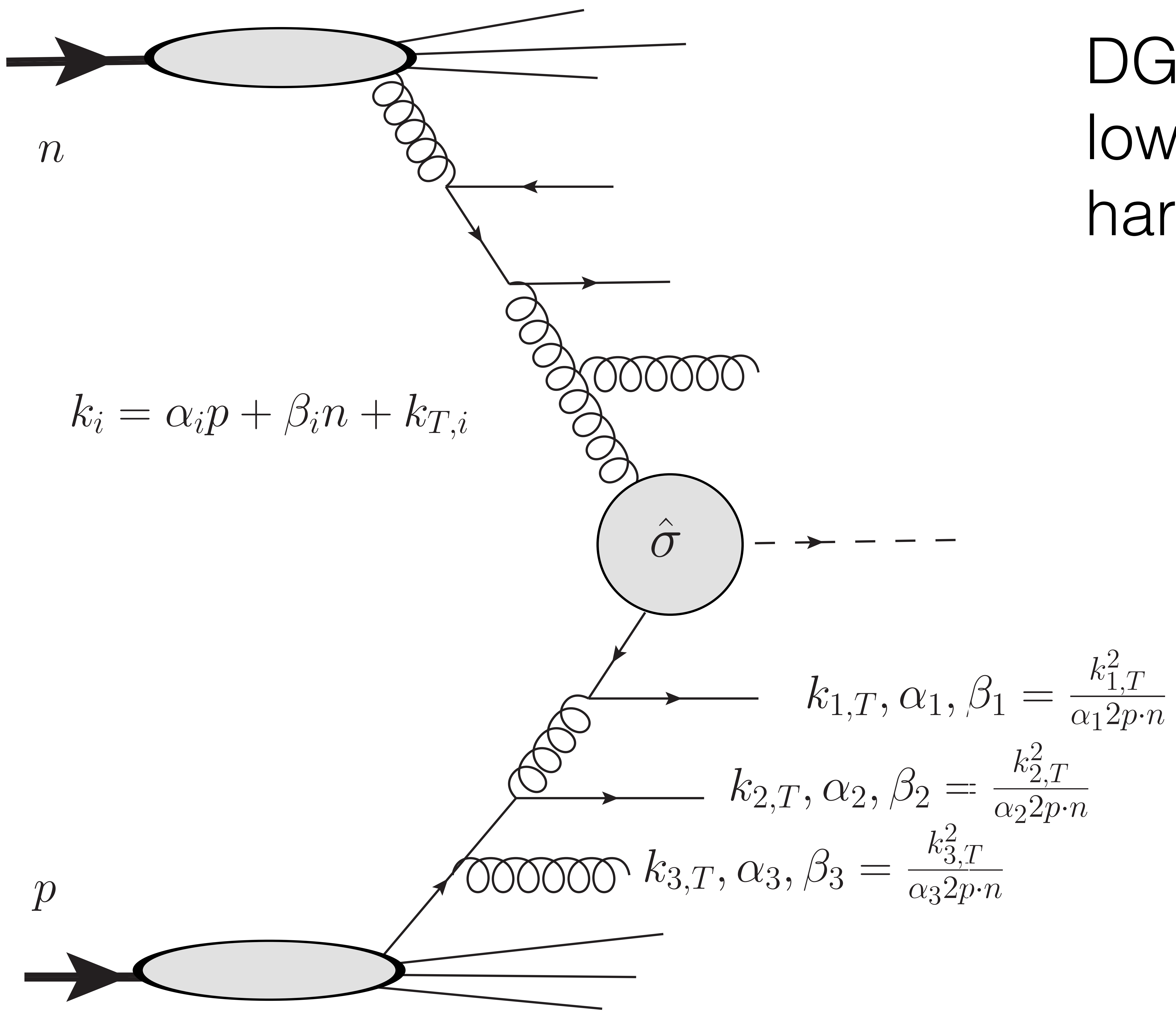
Open questions (personal list)

- precise relation to TMD QCD operator definition (→underway, but not topic of this talk)
- high energy/low x limit: QCD amplitudes are naturally factorized into TMD unintegrated gluon distribution
  - high energy factorization in the dilute limit (no high density effects)
  - BFKL evolution [Kuraev, Lipatov, Fadin; SPJ (1977)], [Balitsky, Lipatov; SJNP (1978)] and unintegrated gluon density
  - k_T-factorization: matching to collinear factorization → analytic continuation of partonic cross-section

$$4M^2\sigma(x, M^2) = \int d^2\boldsymbol{k} \int_0^1 dz_1 \int_0^1 dz_2 \hat{\sigma}\left(z_1, \frac{\boldsymbol{k}^2}{M^2}\right) \mathcal{F}(z_2, \boldsymbol{k}) \delta(z_1 z_2 - x)$$

[Catani, Ciafaloni, Hautmann; NPB366]

BFKL is an exact QCD result→necessary to re-obtain it within the parton branching method

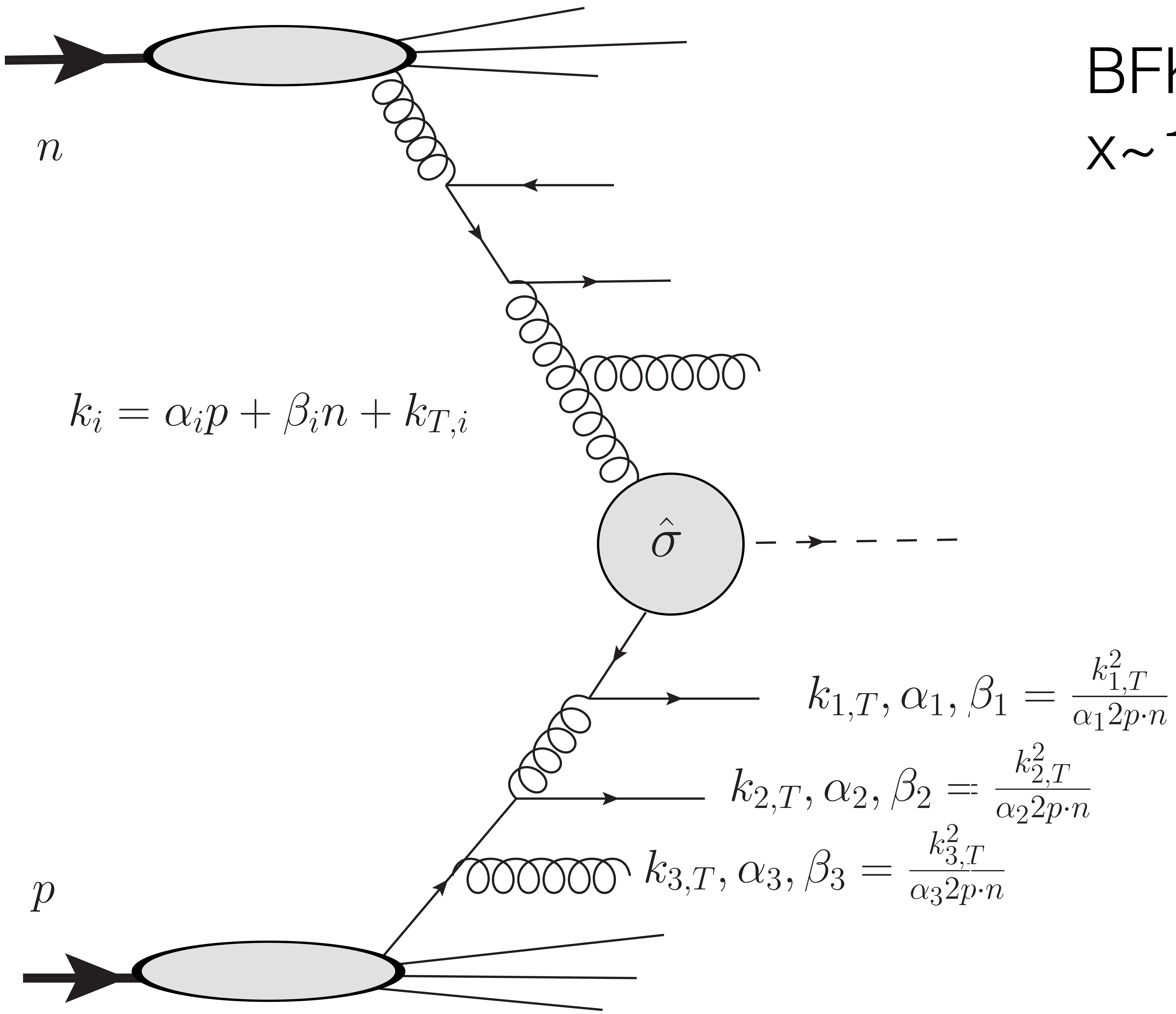


DGLAP= evolution from low scale (hadron) to hard scale (process)

transverse momenta strongly ordered $\mathbf{k}_{T,i} \gg \mathbf{k}_{T,i+1}$(=neglect information on kT⟷ isolate logarithmic enhanced term ~ collinear factorization)

proton momentum fraction α treated exactly (no approximation), but implicitly $\alpha_i \sim \alpha_j$

BFKL= evolution from intermediate x~10^{-3} to low x~10^{-6}

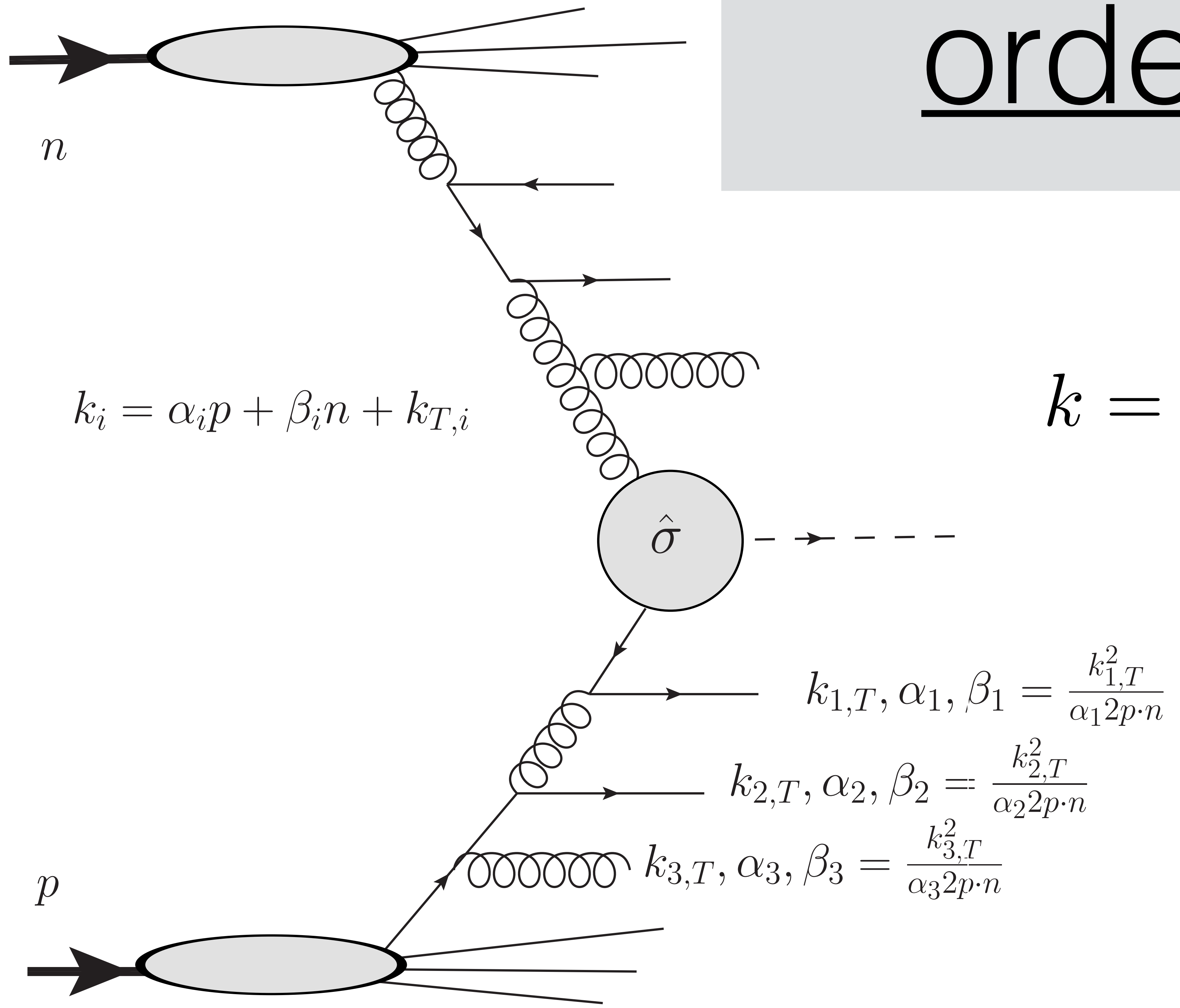


proton momentum fraction α strongly ordered $\alpha_i \gg \alpha_j$

(=neglect information on α ⟷ isolate logarithmic enhanced term ~ high energy factorization)

transverse momentum treated exactly (no approximation), but implicitly $\mathbf{k}_{T,i} \sim \mathbf{k}_{T,i+1}$

ordering in β (momentum fraction w.r.t. collision partner)



$$k = \alpha p + \beta n + k_T$$

goal:

- combine DGLAP & BFKL
- extend validity of TMD evolution to the region $x = 10^{-3} - 0.7$

$\beta_1 \gg \beta_2 \gg \beta_3 \gg \dots$ means $\frac{k_{T,1}^2}{\alpha_1 2n\cdot p} \gg \frac{k_{T,1}^2}{\alpha_1 2n\cdot p} \gg \frac{k_{T,1}^2}{\alpha_1 2n\cdot p} \gg \dots$

implies: $\alpha_1 \ll \alpha_2 \ll ....$

$k_{T,1} \sim k_{T,2} \sim ...$

BFKL/Multi-Regge Kinematics

AND

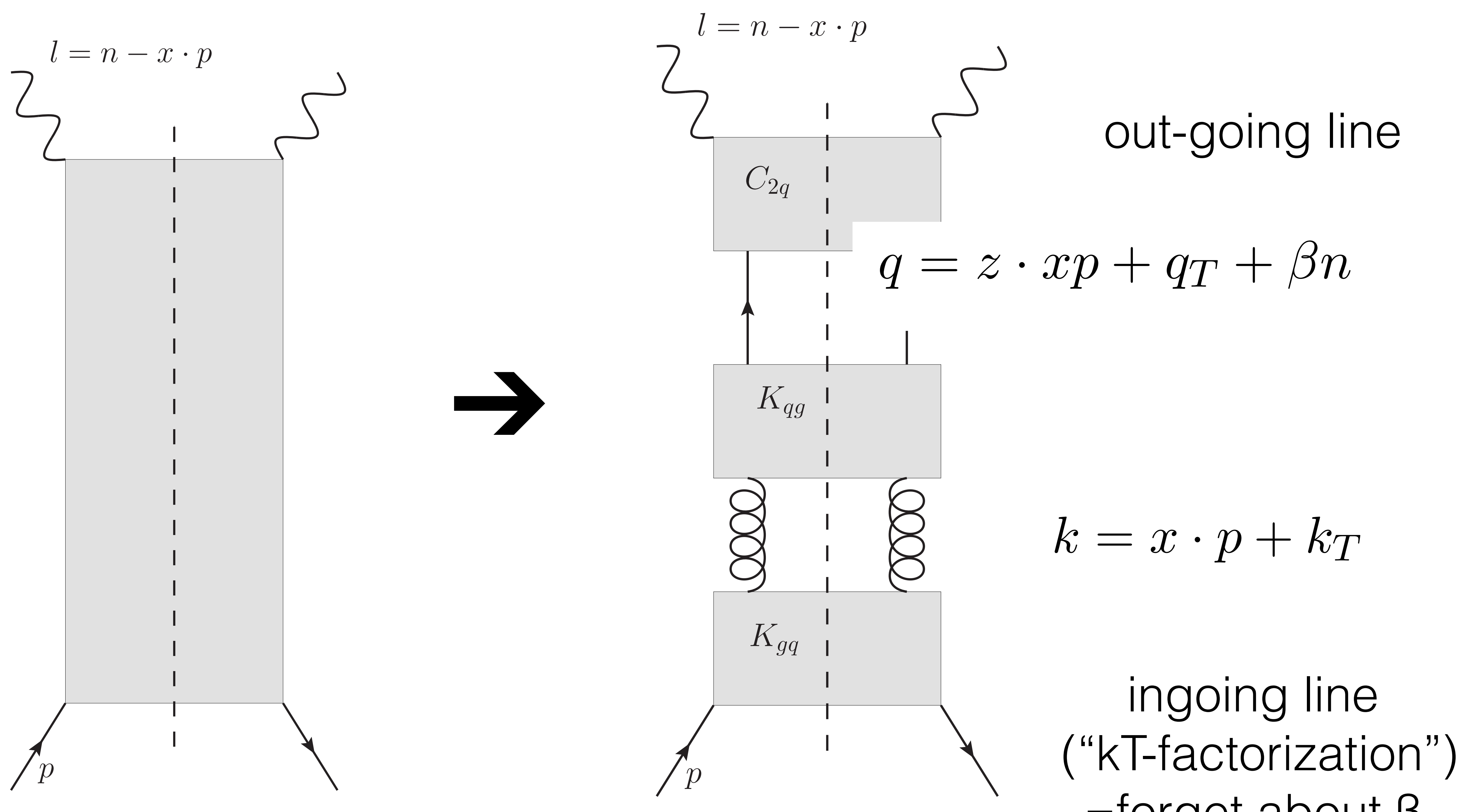
$\alpha_1 \sim \alpha_2 \sim ....$

$k_{T,1} \gg k_{T,2} \gg ...$

DGLAP/collinear kinematics

Real TMD splitting kernels

task: search for factorization of correlators which is only ordered in β



real part:

- start from diagrammatic definition of collinear factorization in axial gauge [Curci, Furmanski, Petronzio; NPB 1908]
- incoming off-shell legs: high energy factorization as formulated in high energy effective action [Lipatov, hep-ph/9502308]
- gauge invariant kernel + correct high energy & collinear limit
- allows to derive <u>real</u> splitting kernels [Gituliar, MH, Kutak; 1511.08439]; [MH, Kusina, Kutak, Serino; 1711.04587]
- fails for virtual corrections → TMD distribution in light cone gauge requires transverse gauge link → work in progress

this talk:

- explore real TMD splitting within parton branching method
- fix missing virtual contributions through probabilistic interpretation

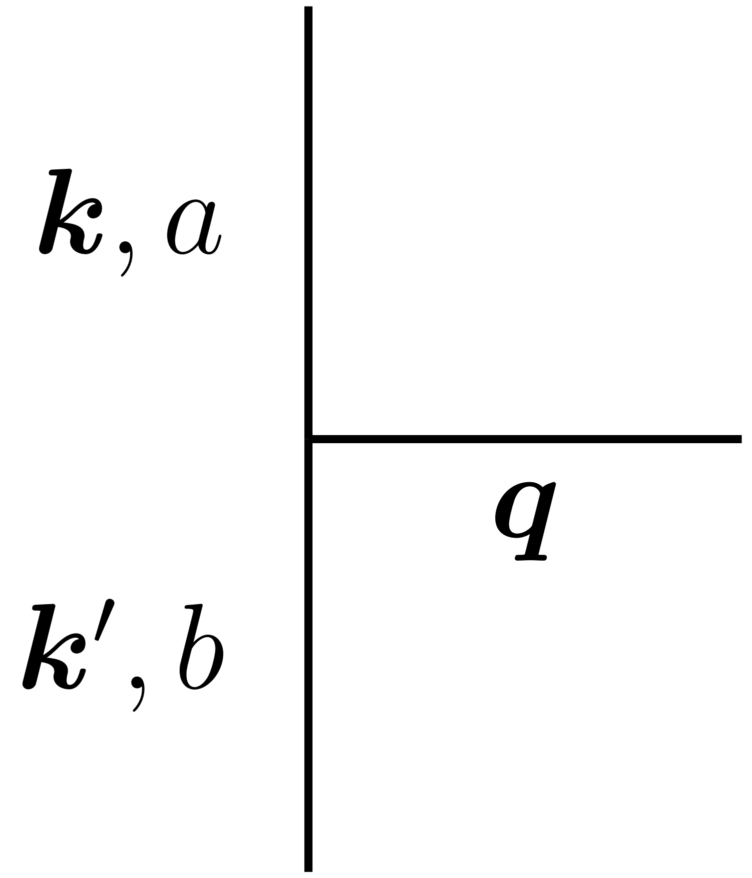
Heuristic generalization of the BFKL equation

$$\mathcal{F}(x,\boldsymbol{k}^2) = \mathcal{F}^{(0)}(x,\boldsymbol{k}^2) + \int_x^1 dz \int \frac{d^2\boldsymbol{q}}{\pi \boldsymbol{q}^2} P_{gg}^{\mathrm{TM}}(z,\boldsymbol{q}+\boldsymbol{k},\boldsymbol{q}) \mathcal{F}\left(\frac{x}{z},(\boldsymbol{q}+\boldsymbol{k})^2\right)$$

leading order BFKL: $\boldsymbol{P}_{gg}^{TM} \to \frac{\alpha_s C_A}{\pi z}\left(\frac{1}{\mathbf{q}^2}+\text{virtual}\right)$

$$\frac{1}{\boldsymbol{q}^2} P_{gg}^{\mathrm{TM},R}(z,\boldsymbol{k}',\boldsymbol{q}) = \frac{\alpha_s C_A}{2\pi}\frac{e^{-\gamma_E \epsilon}}{\mu^{2\epsilon}}\left[\frac{2}{z(1-z)\boldsymbol{q}^2} + \frac{1}{\boldsymbol{q}^2}\frac{\boldsymbol{k}'^2 - 3\boldsymbol{q}^2 - \boldsymbol{k}^2}{z\boldsymbol{q}^2 + (1-z)\boldsymbol{k}^2} + \frac{(1+\epsilon)z(1-z)}{(z\boldsymbol{q}^2+(1-z)\boldsymbol{k}^2)^2}\frac{(2\boldsymbol{k}'\cdot\boldsymbol{q} - \boldsymbol{k}'^2)^2}{\boldsymbol{k}'^2}\right]$$

real splitting kernel



Properties:

$$\lim_{\boldsymbol{k}'^2\to 0} P_{gg}^{\mathrm{TM},R}(z,\boldsymbol{k}',\boldsymbol{q}) = \frac{\alpha_s 2C_A}{2\pi}\frac{e^{-\gamma_E\epsilon}}{\mu^{2\epsilon}}\left[\frac{1}{z(1-z)} - 2 + (1+\epsilon)\frac{(\boldsymbol{q}\cdot\boldsymbol{k}')^2}{\boldsymbol{k}'^2}\right];$$

- DGLAP slitting function in the collinear limit
- real part of BFKL kernel for $z \to 0$

double Mellin space:

$$\hat{P}_{gg}^{R}(\omega,\gamma;\epsilon) = \int_0^1 dz z^{\omega}\int\frac{d^{2+2\epsilon}\boldsymbol{q}}{\pi}\left(\frac{(\boldsymbol{k}+\boldsymbol{q})^2}{\boldsymbol{k}^2}\right)^{\gamma-1}\frac{1}{\boldsymbol{q}^2}P_{gg}^{\mathrm{TM},R}(z,\boldsymbol{k}+\boldsymbol{q},\boldsymbol{q})$$
$$= \frac{\alpha_s C_A}{\pi\omega}\left[\frac{1}{\epsilon} + \ln\frac{\boldsymbol{k}^2}{\mu^2} + \chi_0(\gamma) + \mathcal{O}(\epsilon)\right] + \mathcal{O}(\omega^0),$$

anticipating virtual correction (= gluon Regge trajectory)

within high energy effective action: [MH, Sabio Vera; 1110.6741]

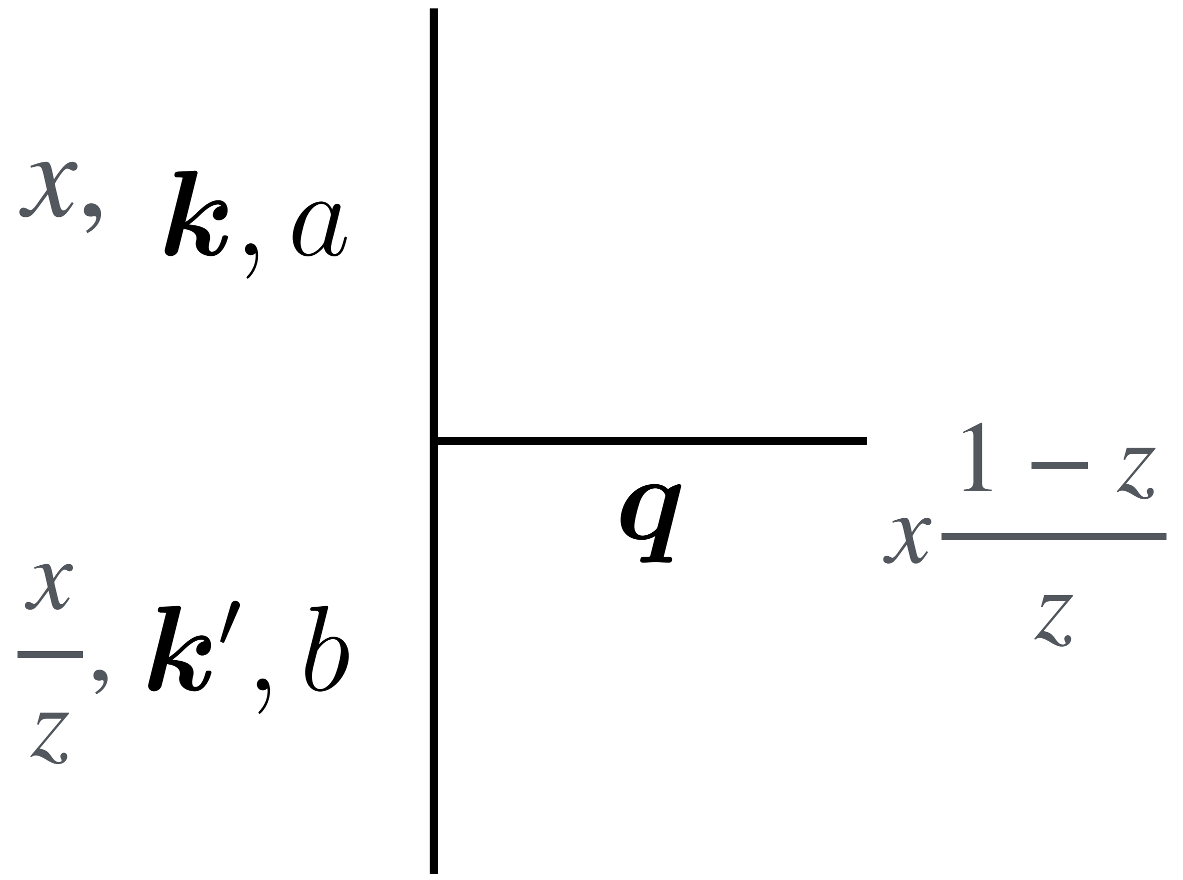
BFKL unintegrated gluon density after resumming $(\alpha_s/\omega)^n$ to all orders

$$\mathcal{F}(x,\boldsymbol{k}^2) = \frac{1}{\boldsymbol{k}^2}\int\frac{d\omega}{2\pi i}x^{-\omega}\int\frac{d\gamma}{2\pi i}\left(\frac{\boldsymbol{k}^2}{Q_0^2}\right)^{\gamma}\frac{h^{(0)}(\gamma)}{\omega-\bar{\alpha}_s\chi_0(\gamma)}.$$

collinear limit: BFKL anomalous dimension

$$\gamma^{\mathrm{BFKL}}(\alpha_s,\omega) = \frac{\bar{\alpha}_s}{\omega} + 2\zeta(3)\left(\frac{\bar{\alpha}_s}{\omega}\right)^4 + \ldots$$

Quark splittings



$$\frac{1}{\boldsymbol{q}^2} P_{gq}^{\mathrm{TM}}(z, \boldsymbol{k}', \boldsymbol{q}) = \frac{\alpha_s C_F}{2\pi} \frac{e^{-\gamma_E \epsilon}}{\mu^{2\epsilon}} \left[\frac{2}{z\boldsymbol{q}^2} - \frac{2}{z\boldsymbol{q}^2 + (1-z)\boldsymbol{k}^2} + \frac{(1+\epsilon) z \boldsymbol{q}^2}{[z\boldsymbol{q}^2 + (1-z)\boldsymbol{k}^2]^2} \right]$$

$$\frac{1}{\boldsymbol{q}^2} P_{qg}^{\mathrm{TM},R}(z, \boldsymbol{k}', \boldsymbol{q}) = \frac{\alpha_s T_R}{2\pi} \frac{e^{-\gamma_E \epsilon}}{\mu^{2\epsilon}} \left[\frac{1}{z\boldsymbol{q}^2 + (1-z)\boldsymbol{k}^2} + \frac{z(1-z)\boldsymbol{q}^2 (2\boldsymbol{q}\cdot\boldsymbol{k}' - \boldsymbol{k}'^2)^2}{\boldsymbol{k}^2 [z\boldsymbol{q}^2 + (1-z)\boldsymbol{k}^2]^2} \right]$$

$$\frac{1}{\boldsymbol{q}^2} P_{qq}^{\mathrm{TM},R}(z, \boldsymbol{k}', \boldsymbol{q}) = \frac{\alpha_s C_F}{2\pi} \frac{e^{-\gamma_E \epsilon}}{\mu^{2\epsilon}} \left[\frac{2}{(1-z)\boldsymbol{q}^2} + \frac{\boldsymbol{k}'^2 - \boldsymbol{k}^2}{\boldsymbol{k}^2 [z\boldsymbol{q}^2 + (1-z)\boldsymbol{k}^2]} - \frac{z\boldsymbol{q}^2 + \epsilon(1-z)\boldsymbol{k}^2}{[z\boldsymbol{q}^2 + (1-z)\boldsymbol{k}^2]^2} \right]$$

- correct high energy and collinear limits easily verified
- generalization to arbitrary flavors …

$$\mathcal{F}_a(x, \boldsymbol{k}^2) = \mathcal{F}_a^{(0)}(x, \boldsymbol{k}^2) + \sum_b \int_x^1 dz \int \frac{d^2\boldsymbol{q}}{\pi \boldsymbol{q}^2} P_{ab}^{\mathrm{TM}}(z, \boldsymbol{q}, \boldsymbol{q}+\boldsymbol{k}) \mathcal{F}_b\left(\frac{x}{z}, (\boldsymbol{q}+\boldsymbol{k})^2\right)$$

but: still lacks virtual contribution + requires “renormalization”

From ‘bare' to ‘physical' TMD PDF

$$xG^{(1)}(x,\boldsymbol{k}) = \int \frac{d\xi^- d^2\boldsymbol{\xi}}{(2\pi)^3 P^+} e^{ixP^+\xi^- - i\boldsymbol{k}\cdot\boldsymbol{\xi}} \langle P|F_a^{+i}(\xi^-,\boldsymbol{\xi})\mathcal{L}_\xi^\dagger \mathcal{L}_0 F_a^{+i}(0)|P\rangle$$

$$\mathcal{L}_\xi = \mathrm{P}\exp\{-ig\int_{\xi^-}^{\infty} d\zeta^- A^+(\zeta,\boldsymbol{\xi})\}$$

these operator definitions require

- soft factor
- UV renormalization
- virtual corrections

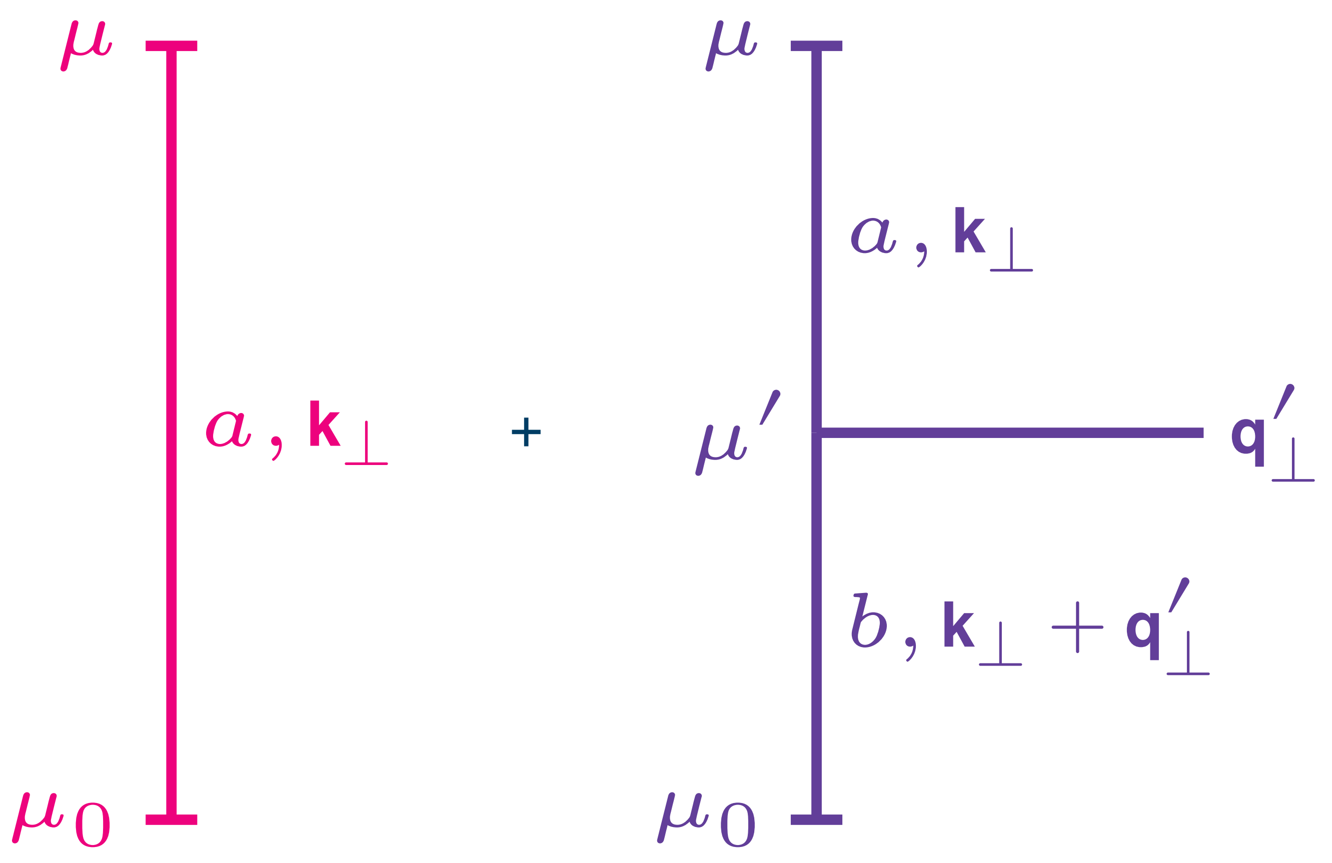
- real splitting kernels can be re-obtained from operator definitions of unpolarized gluon and quark TMD (real 1-loop)
- initial state: reggeized gluon and quark field as defined through high energy effective action

our treatment:

- determine/constraint these conditions through consistency requirement of the parton branching method
- independent formal calculation is under way ….

Parton branching method: collinear case

prob. for evolution



with no branchings

with branching

$$\widetilde{A}_a\left(x, k_\perp, \mu^2\right) = \Delta_a\left(\mu^2, \mu_0^2\right) \widetilde{A}_a\left(x, k_\perp, \mu_0^2\right) + \sum_b \int_{\mu_0^2}^{\mu^2} \frac{\mathrm{d}\mu'^2}{\mu'^2} \int_0^{2\pi} \frac{d\phi}{2\pi} \int_x^{z_M(\mu')} \mathrm{d}z K_{ab}(z, \mu') \widetilde{A}_b\left(\frac{x}{z}, |\boldsymbol{k} + a(z)\boldsymbol{\mu}'|, \mu'^2\right)$$

$$\Delta_a\left(\mu^2, \mu_0^2\right) = \exp\left[-\sum_b \int_{\mu_0^2}^{\mu^2} \frac{\mathrm{d}\mu'^2}{\mu'^2} \int_0^{z_M(\mu')} \mathrm{d}z z P_{ba}^R\left(z, \alpha_s\left(b(z)^2 \mu'^2\right)\right)\right]$$

Sudakov form factor

$$\mathbf{q}'_\perp = (1 - z')\mu'$$ collinear angular ordering

$$K_{ab}(z, \mu') = \frac{\Delta_a\left(\mu^2, \mu_0^2\right)}{\Delta_a\left(\mu'^2, \mu_0^2\right)} P_{ab}^R\left(z, \alpha_s(b(z)^2 \mu'^2)\right)$$ collinear splitting kernel

Naive replacement: $P_{ab}(z) \to P_{ab}^{TM}(z, \mathbf{k}', \mu')$

- technically possible: since we already integrate over transverse momenta through angular ordering prescription
- Sudakov form factors etc. remain with the collinear prescription

Consequences: rather strong violation of momentum sum rules

$$xf_a(x,\mu) = \int \frac{d^2\mathbf{k}}{\pi} \mathscr{A}(x, \mathbf{k}, \mu)$$

relation to conventional integrated pdfs (“DGLAP up to phase space”)

$$\int_0^1 dx \sum_a xf_a(x,\mu) = 1$$

need to obey momentum sum rule

$$\int_0^1 dx \sum_a x f_a(x,\mu) = 1$$ for collinear splitting kernels + collinear Sudakov

→ works ✔

μ^2 [GeV^2]	$\alpha_s(\mu^2)$, fix. z_M	$\alpha_s(q_\perp^2)$, fix. z_M	$\alpha_s(q_\perp^2)$, dyn. z_M
3	1.000	1.000	1.000
10	0.999	0.999	0.999
10^2	0.997	0.997	0.997
10^3	0.995	0.993	0.995
10^4	0.992	0.989	0.992
10^5	0.986	0.981	0.984

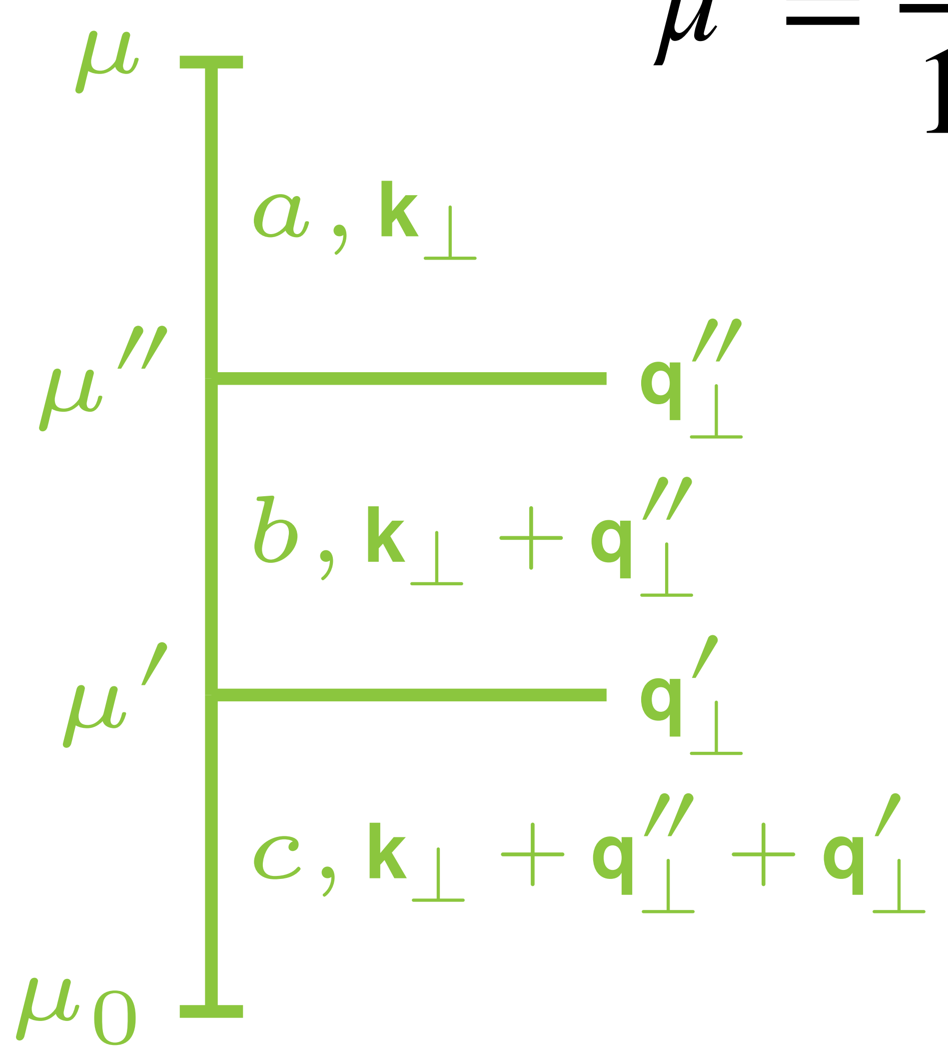
$$\int_0^1 dx \sum_a x f_a(x,\mu) = 1$$ for TMD splitting kernels + collinear Sudakov

→ does not work ✘

μ^2 [GeV^2]	$\alpha_s(\mu^2)$, fix. z_M	$\alpha_s(q_\perp^2)$, fix. z_M	$\alpha_s(q_\perp^2)$, dyn. z_M
3	1.029	1.038	1.000
10	1.087	1.139	1.007
10^2	1.156	1.304	1.045
10^3	1.195	1.413	1.091
10^4	1.219	1.478	1.129
10^5	1.229	1.507	1.148

From the 'bare' TMD

$$\mathcal{F}_a(x,\boldsymbol{k}^2) = \mathcal{F}_a^{(0)}(x,\boldsymbol{k}^2) + \sum_b \int_x^1 dz \int \frac{d^2\boldsymbol{q}}{\pi \boldsymbol{q}^2} P_{ab}^{\rm TM}(z,\boldsymbol{q},\boldsymbol{q}+\boldsymbol{k})\mathcal{F}_b\left(\frac{x}{z},(\boldsymbol{q}+\boldsymbol{k})^2\right)$$



$$\mu' = \frac{\mathbf{q}_T}{1-z'} \text{ etc}$$

- formulation based on angular ordering
- exact angular/rapidity ordering $\mu > z''\mu'', \quad \mu'' > z'\mu'$ etc,

- Current Parton branching code is based on angular ordering, but ignores the $'z'$

simplified ordering prescription: $\mu > \mu'' > \mu'$

- misses complete TM range for $z \to 0$: incomplete BKFL ladder in low x region ☹️
- but control well the infra-red region $z \to 1$ 🙂

focus on these aspects in this work

towards parton branching

Note: this is the region which we need to control to make sense of our 'bare' equation

Define now a TMD PDF which depends on the scale μ

$$\widetilde{A}_a(x, k_\perp, \mu) = \mathcal{F}_a^{(0)}(x, \boldsymbol{k}^2) + \sum_b \int_x^1 dz \int_0^{2\pi} \frac{d\phi}{2\pi} \int_0^{\mu^2} \frac{d\boldsymbol{\mu}'^2}{\boldsymbol{\mu}'^2} \tilde{P}_{ab}(z, \boldsymbol{k}', \boldsymbol{q}) \widetilde{A}_b\left(\frac{x}{z}, |\boldsymbol{k}'|, \mu'\right)$$

- separate off modes with $\mu' > \mu_0$

$$\widetilde{A}_a(x, k_\perp, \mu_0) = \mathcal{F}_a^{(0)}(x, \boldsymbol{k}^2) + \sum_b \int_x^1 dz \int_0^{2\pi} \frac{d\phi}{2\pi} \int_0^{\mu_0^2} \frac{d\boldsymbol{\mu}'^2}{\boldsymbol{\mu}'^2} \tilde{P}_{ab}(z, \boldsymbol{k}', \boldsymbol{q}) \widetilde{A}_b\left(\frac{x}{z}, |\boldsymbol{k}'|, \mu\right)$$

- separate off modes with $z > z_M, \quad z_M \sim 1 - 10^{-5}$
  defines the "no emission probability" $F_a(\mu'^2, \mathbf{k}')$
- contains both (unknown) virtual and unresolved real contributions

note: z_M is a regulator; $z_M = 1 - 10^{-5}$ is the implementation used in the MC solution intermediate result:

probability of no emission: gathers unresolved real contributions + unknown virtual corrections

$$\tilde{\mathcal{A}}_a(x,|\boldsymbol{k}|,\mu^2) = \tilde{\mathcal{A}}_a(x,|\boldsymbol{k}|,\mu_0^2) - \int_{\mu_0^2}^{\mu^2} \frac{d\mu'^2}{\mu'^2} F_a(\mu'^2,\boldsymbol{k}^2)\tilde{\mathcal{A}}_a(x,|\boldsymbol{k}|,\mu'^2) +$$

$$+ \sum_b \int_{\mu_0^2}^{\mu^2} \frac{d\mu'^2}{\mu'^2} \int_0^{2\pi} \frac{d\phi}{2\pi} \int_x^{z_M} dz \tilde{P}_{ab}^R(z,\boldsymbol{k}',\boldsymbol{\mu}')\tilde{\mathcal{A}}_b\left(\frac{x}{z},|\boldsymbol{k}'|,\mu'^2\right)$$

real TMD splitting kernels (known): probability of emission

next step: impose momentum sum rule to fix the unknown no emission probability $\boldsymbol{F_a}$

$$\sum_a \int_0^1 dx \int dk_\perp^2 \tilde{\mathcal{A}}_a(x,k_\perp,\mu^2) = 1.$$

technically: extend usual MC arguments to k_T dependent case, obtain from sum-rule:

$$F_a(\mu'^2, k_\perp) = \sum_b \int_0^{2\pi} \frac{d\phi}{2\pi} \int_0^{z_M} dz\ z\tilde{P}^R_{ba}(z, \boldsymbol{k}, \boldsymbol{\mu}')$$

- sufficient to satisfy the sum-rule
- not necessarily the most general expression

Finally reformulate evolution equation using a TMD Sudakov form factor

$$\Delta_a(\mu^2, k_\perp) = \exp\left(-\sum_b \int_{\mu_0^2}^{\mu^2} \frac{d\mu'^2}{\mu'^2} \int_0^{z_M} dz\ z\bar{P}^R_{ba}(z, k_\perp, \mu')\right).$$

depends now on TM!

eventually obtain the parton branching evolution equation:

$$\widetilde{A}_a\left(x, k_\perp, \mu^2\right) = \Delta_a\left(\mu^2, k_\perp\right) \widetilde{A}_a\left(x, k_\perp, \mu_0^2\right) + \\ + \sum_b \int_{\mu_0^2}^{\mu^2} \frac{\mathrm{d}\mu'^2}{\mu'^2} \int_0^{2\pi} \frac{d\phi}{2\pi} \int_x^{z_M(\mu')} \mathrm{d}z\, K_{ab}\left(z, \boldsymbol{k} + a(z)\boldsymbol{\mu}', \boldsymbol{\mu}'\right) \widetilde{A}_b\left(\frac{x}{z}, |\boldsymbol{k} + a(z)\boldsymbol{\mu}'|, \mu'^2\right)$$

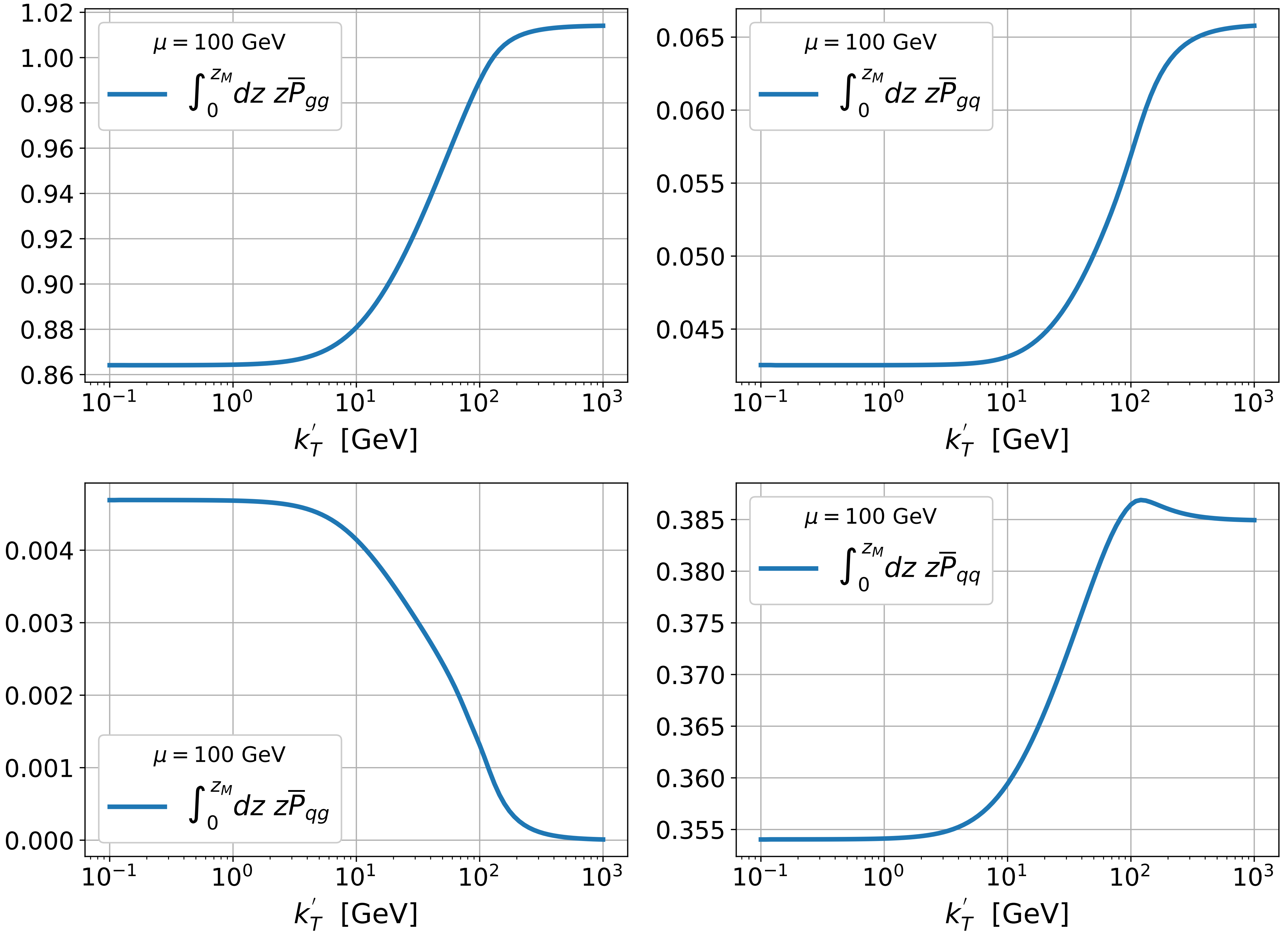
$$K_{ab}(z, \boldsymbol{k}', \boldsymbol{\mu}') = \frac{\Delta_a\left(\mu^2, k_\perp\right)}{\Delta_a\left(\mu'^2, k_\perp\right)} \tilde{P}^R_{ab}\left(z, \boldsymbol{k}', \boldsymbol{\mu}', \alpha_s\right)$$

TMD evolution equation with clear probabilistic interpretation

Properties of splittings & the TMD Sudakov

$$\Delta_a(\mu^2, k_\perp) = \exp\left(-\sum_b \int_{\mu_0^2}^{\mu^2} \frac{d\mu'^2}{\mu'^2} \int_0^{z_M} dz\ z\bar{P}^R_{ba}(z, k_\perp, \mu')\right).$$

angular average TMD kernels

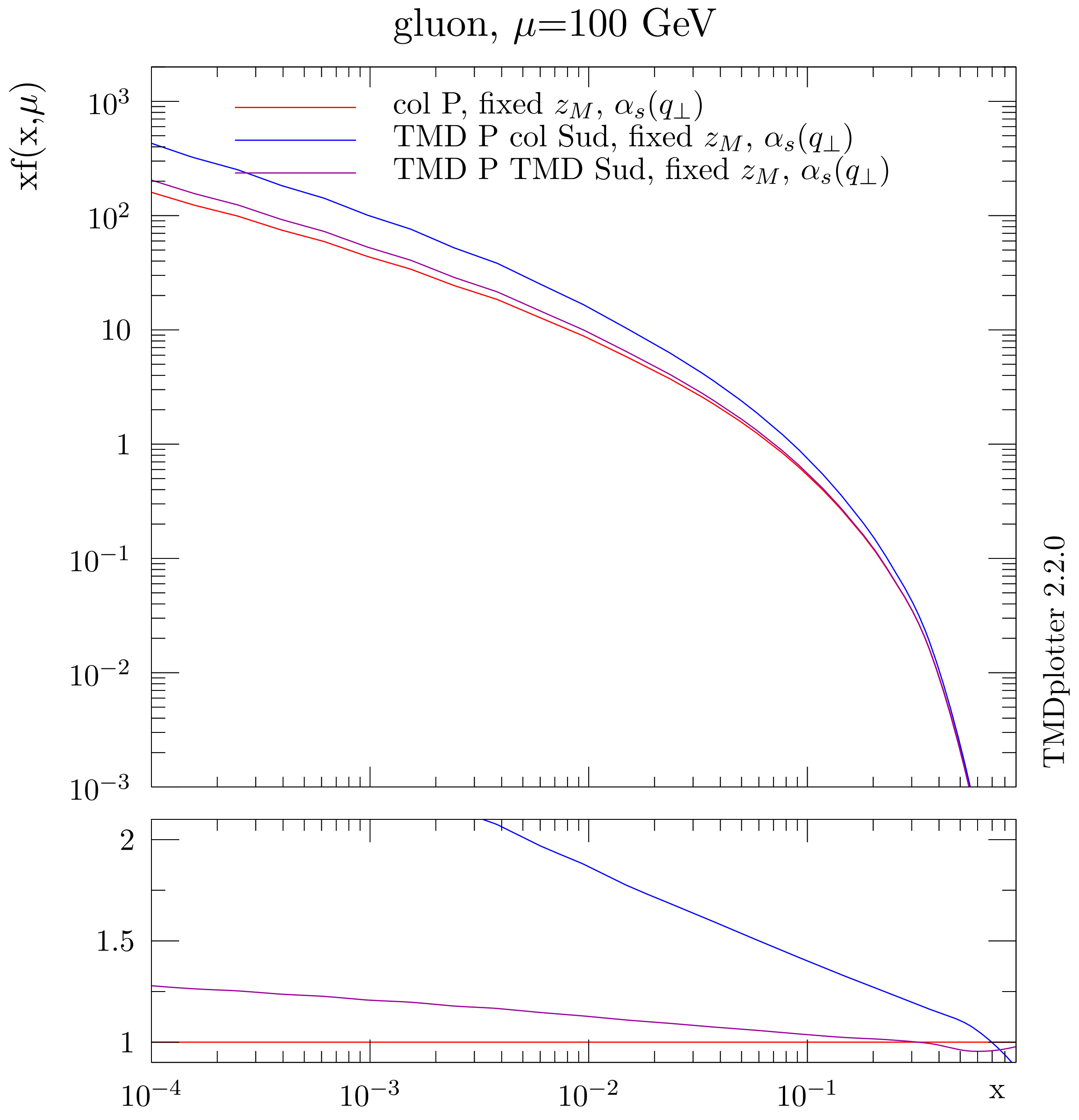


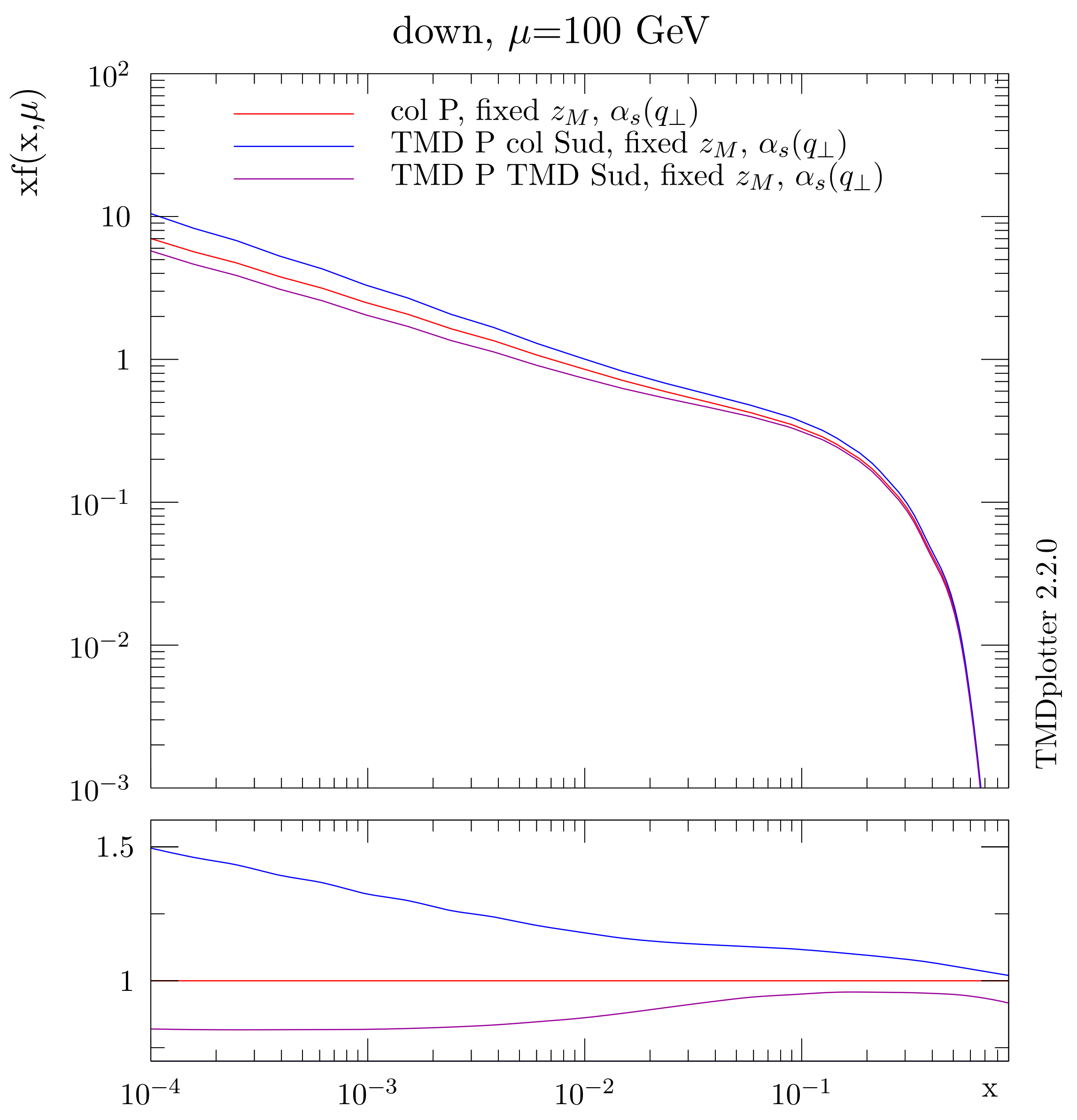
- angular average splitting kernels grow with t-channel transverse momentum → increased splitting probabilities
- TMD Sudakov (no splitting probability): drops off for large TM
- collinear limit by construction
- low x resummation only complete for small transverse momenta $k_T \ll \mu$

$\int_0^1 dx \sum_a x f_a(x, \mu) = 1$ for TMD splitting kernels + TMD Sudakov

μ^2 (GeV^2)	$\alpha_s(\mu^2)$ fix. z_M	$\alpha_s(q_\perp^2)$, fix. z_M	$\alpha_s(q_\perp^2)$, dyn. z_M
3	1.000	1.000	1.000
10	0.999	0.999	0.999
10^2	0.997	0.996	0.997
10^3	0.994	0.992	0.994
10^4	0.991	0.987	0.991
10^5	0.984	0.978	0.983

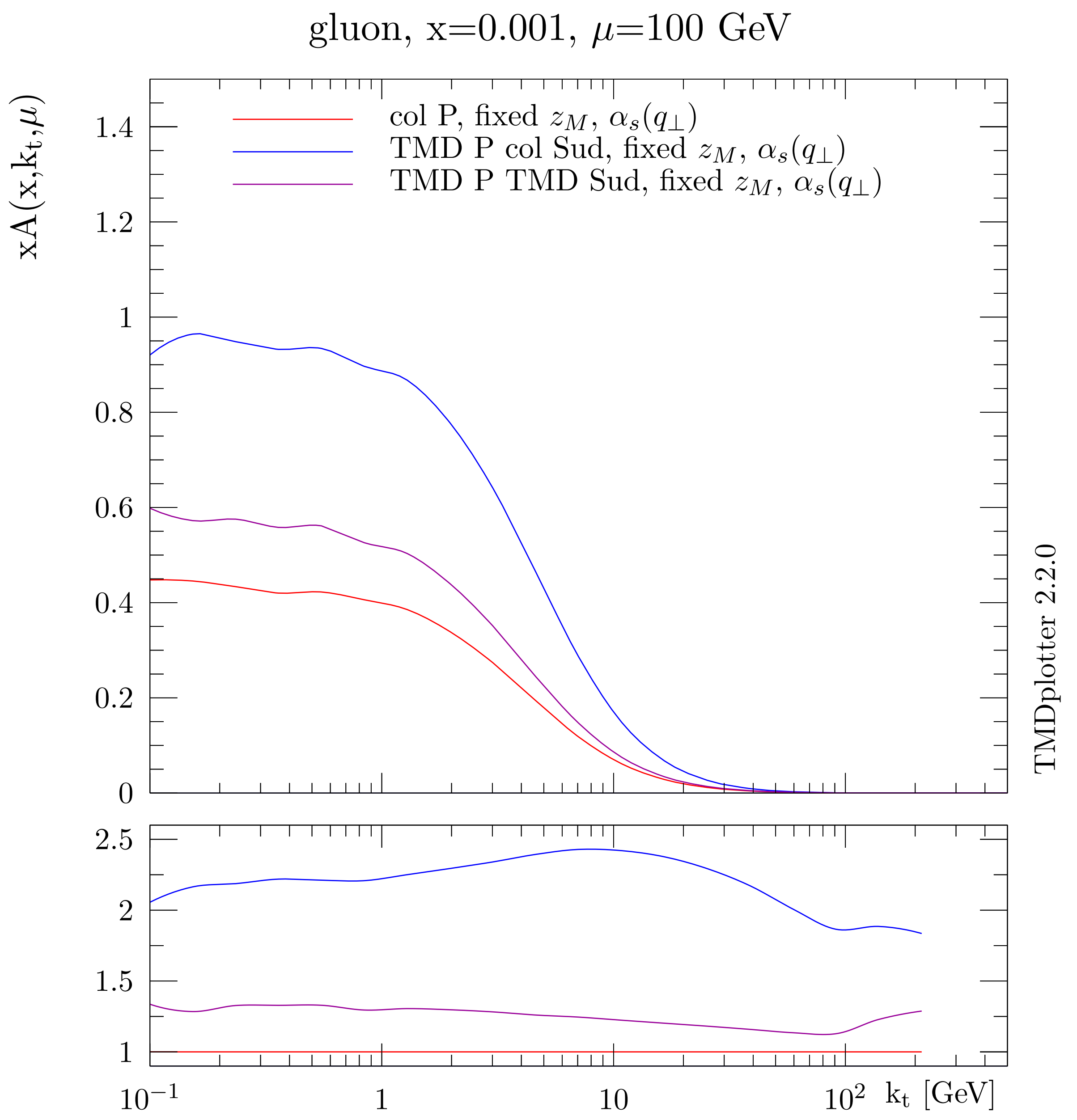
Distributions:

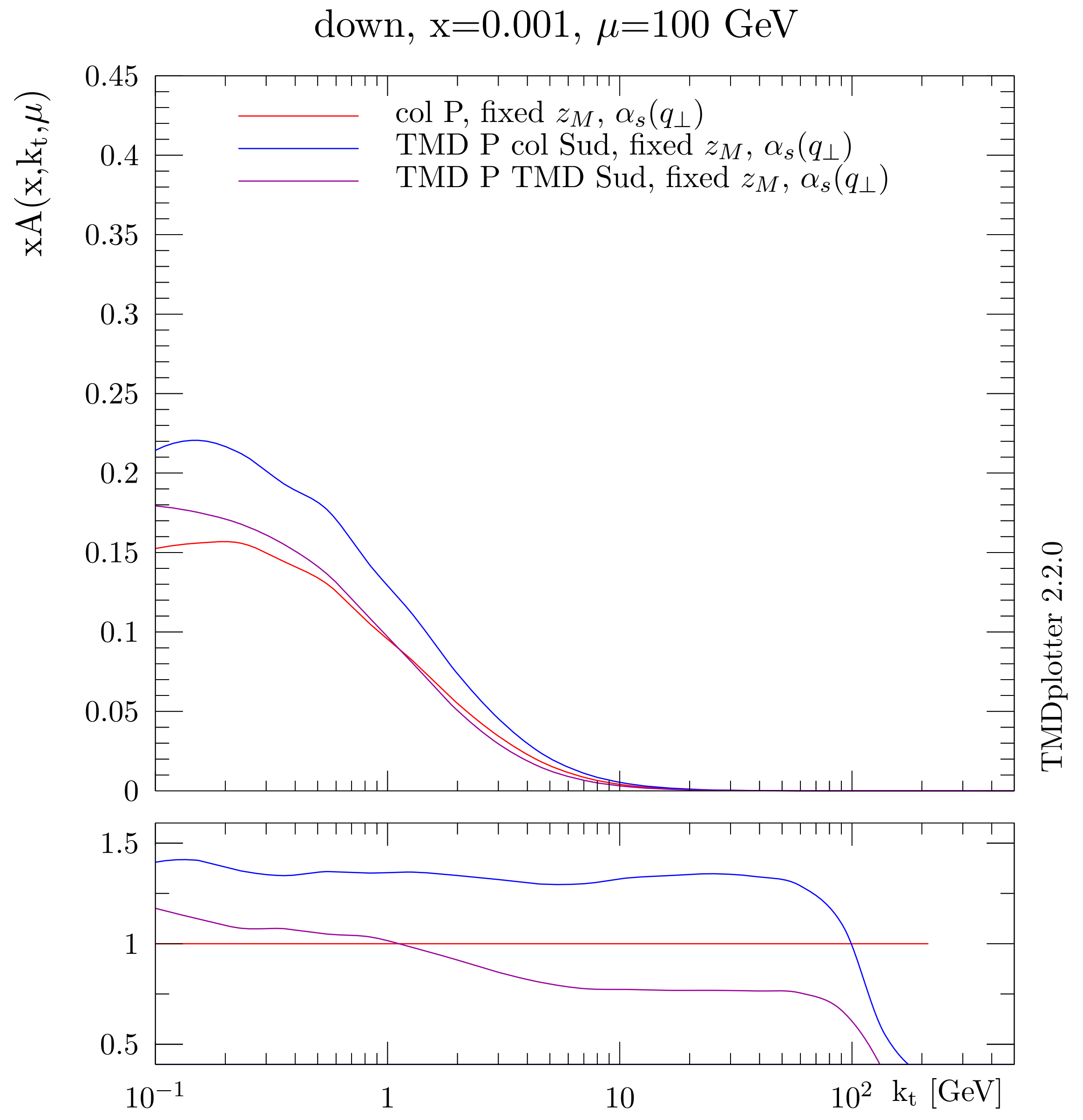




x-dependence

k_T-dependence





- so far: identical collinear initial condition
- separate fits left as task for the future

$$\tilde{\mathcal{A}}_a(x, k_{\perp,0}, \mu_0^2) = x f_a(x, \mu_0^2) \cdot \frac{1}{q_s\sqrt{\pi}} \exp\left(-\frac{k_{\perp,0}^2}{q_s^2}\right)$$

- large differences for TMD splitting + collinear Sudakov
- reason: violation of sum rules

Conclusion:

- First implementation of TMD splitting function within the Parton Branching method
- Region which needs to be controlled: soft/infra-red region $z \to 1$
  - implies a rapidity divergence
  - requires 'renormalization'; here achieved through the MC based Parton Branching method (require probabilistic picture)
- In parallel: formal study in progress; seems to lead to similar results
- Future plans:
  - can we include in this way the complete BFKL limit (arbitrary emitted parton q_T at $z \to 1$)?
  - explore relation to MC implementation of CCFM equation (old Cascade [Jung *et. al.*; 1008.0152]) and NLO BFKL MC (BFKLex [Chachamis, Sabio Vera *et.al.*;1606.07349 and others])
  - fits + phenomenology