JIMWLK Evolution, Lindblad Equation and Quantum-Classical Correspondence

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Ming Li and Alex Kovner, JHEP 05 (2020) 036
Alex Kovner, Ming Li and Michael Lublinsky, in progress

Outline

- **Brief Reviews** (*JIMWLK High Energy Evolution Equation, Lindblad Equation for Open Quantum Systems, Hamiltonian Approach to the Color Glass Condensate Effective theory*).
- **Effective Density Matrix and Lindblad High Energy Evolution Equation.**
- **Quantum-Classical Correspondence: Reproducing JIMWLK Equation.**
- **Summary**

A re-formulation (interpretation) of the JIMWLK evolution from the perspective of Open Quantum Systems

The Color Glass Condensate Framework

- **Separating gluonic degrees of freedoms according to longitudinal momentum.**

Λ^+ $\quad p^+$

$A^\mu(x)$ $\qquad j^{\mu,a}(x^-,\mathbf{x}_\perp)=\delta^{\mu+}j^a(x^-,\mathbf{x}_\perp)$

- **The action of CGC effective theory**

$$S[A,j]=-\int d^4x\frac{1}{4}F^a_{\mu\nu}F^{a,\mu\nu}+\frac{i}{gN_c}\int dx^-d^2\mathbf{x}_\perp \mathrm{Tr}\left\{j(x^-,\mathbf{x}_\perp)W[A^-](x^-,\mathbf{x}_\perp)\right\}$$

$$W[A^-](x^-,\mathbf{x}_\perp)=\mathcal{P}\exp\left\{ig\int_{-\infty}^{+\infty}dz^+A^-(z^+,x^-,\mathbf{x}_\perp)\right\}$$

Gauge invariant Eikonal coupling j^+A^-

- **Physical observables are calculated by averaging over soft gluons and hard gluons.**

$$\langle\hat{\mathcal{O}}\rangle=\int\mathcal{D}A^\mu\mathcal{D}j^a\mathcal{O}[A^\mu,j^a]e^{iS_{\mathrm{CGC}}[A,j]}=\int\mathcal{D}j^a\mathcal{W}_{\Lambda^+}[j^a]\langle\mathcal{O}\rangle_A[j^a]$$

- **The weight functional evolves with the cut-off scale : JIMWLK equation.**
(*leading logarithmic approximation)*

$$\frac{\partial\mathcal{W}_y[j^a]}{\partial y}=\frac{1}{2}\frac{\delta}{\delta j^a(\mathbf{x}_\perp)}\eta^{ab}[\mathbf{x}_\perp,\mathbf{y}_\perp;j]\frac{\delta}{\delta j^b(\mathbf{y})}\mathcal{W}_y[j^a]$$

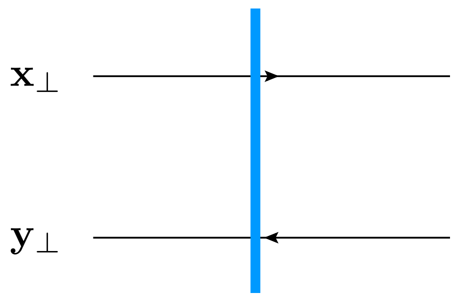
Fokker-Planck type evolution equation for a probability distribution functional.

η^{ab} : diffusion coefficient or width of Gaussian fluctuations.

$$\eta^{ab}[\mathbf{x}_\perp,\mathbf{y}_\perp;j]=\left\langle\delta j^{+a}(\mathbf{x}_\perp)\delta j^{+b}(\mathbf{y}_\perp)\right\rangle$$

JIMWLK Equation, Balistky Hierarchy, Balitsky-Kovchegov Equation

Evolution of Observables (scattering amplitudes) in Dilute-Dense Scatterings



Target

$$d(\mathbf{x}_\perp, \mathbf{y}_\perp) = \frac{1}{N_c}\mathrm{Tr}\left[V(\mathbf{x}_\perp)V^\dagger(\mathbf{y}_\perp)\right]$$

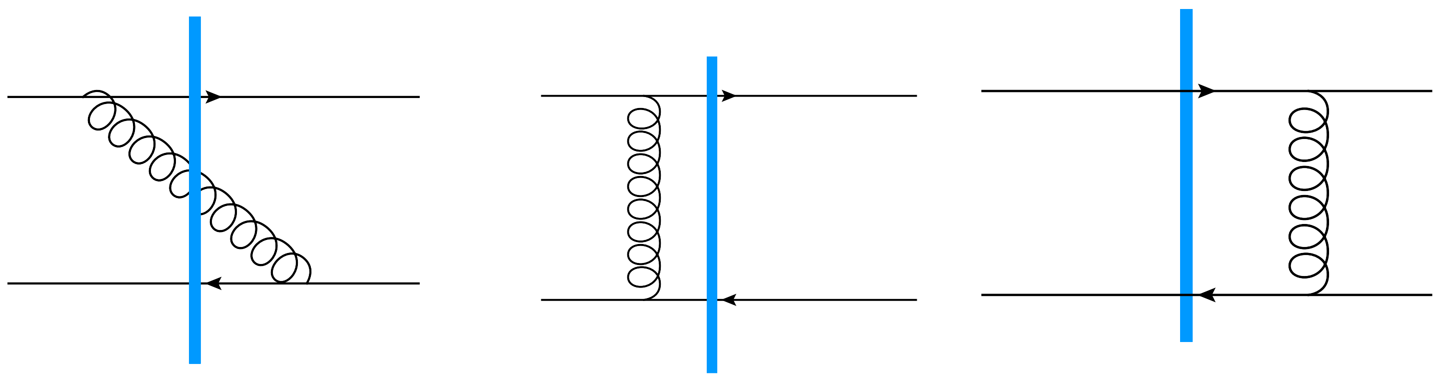
Balistky (1996), Kovchegov (1999), Mueller(2001), Kovner and Lublinsky (2005)

Lightlike Wilson Line

$$V^\dagger(\mathbf{x}_\perp) = \mathcal{P}\exp\left\{ig\int_{-\infty}^{+\infty} dx^- \alpha^a(x^-, \mathbf{x}_\perp)t^a\right\}$$

<u>Boosting the dipole by Δy</u>

Only one gluon emission needs to be considered in the leading logarithmic approximation



$$\frac{\partial \mathcal{W}_y[V]}{\partial y} = -\mathcal{H}_{\mathrm{JIMWLK}}\mathcal{W}_y[V]$$

$$\mathcal{H}_{\mathrm{JIMWLK}} = \frac{1}{2\pi}\int d^2\mathbf{z}_\perp\, Q_i^{a\dagger}(\mathbf{z}_\perp)Q_i^a(\mathbf{z}_\perp)$$

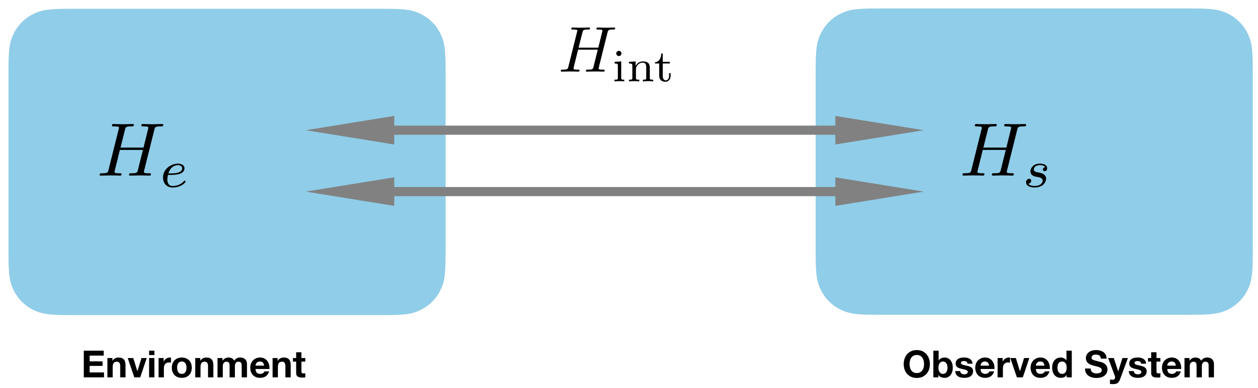
$$Q_i^a(\mathbf{z}_\perp) = \frac{g}{2\pi}\int d^2\mathbf{x}_\perp \frac{(\mathbf{z}_\perp - \mathbf{x}_\perp)_i}{(\mathbf{z}_\perp - \mathbf{x}_\perp)^2}\left[V(\mathbf{z}_\perp) - V(\mathbf{x}_\perp)\right]^{ab}\mathcal{J}_R^b(\mathbf{x}_\perp)$$

$$\mathcal{J}_R^a(\mathbf{x}_\perp) = -\mathrm{Tr}\left(V(\mathbf{x}_\perp)T^a\frac{\delta}{\delta V^\dagger(\mathbf{x}_\perp)}\right)$$

The Lindblad Equation for Open Quantum Systems

J. Preskill, Lecture notes on Quantum Computation

Lindblad equation is the most general *<u>Markovian</u>* and *<u>non-unitary</u>* time evolution equation for density matrix of an open quantum system. It preserves the properties of density matrix: hermiticity, unit trace and positivity.



- The total density matrix of the complete system evolves according to the quantum Liouville equation.

$$\frac{d\hat{\rho}}{dt} = -i[\hat{H}, \hat{\rho}]\,, \qquad \hat{\rho}(t) = \hat{U}(t)\hat{\rho}(0)\hat{U}^\dagger(t)\,, \quad \hat{U}(t) = e^{-i\hat{H}t}\,, \quad \hat{H} = \hat{H}_s + \hat{H}_e + \hat{H}_{\text{int}}$$

- Tracing out the Hilbert space of the environment to obtain the density matrix of the observed system.

$$\hat{\rho}_s(t) = \text{Tr}_e\hat{\rho}(t) = \sum_n \langle n|\hat{U}(t)|0_e\rangle\hat{\rho}_s(0)\langle 0_e|\hat{U}^\dagger(t)|n\rangle = \sum_n \hat{M}_n(t)\hat{\rho}_s(0)\hat{M}_n^\dagger(t)$$

Assume the initial total density matrix is separable $\hat{\rho}(0) = \hat{\rho}_s(0) \otimes \hat{\rho}_e(0) = \hat{\rho}_s(0) \otimes |0_e\rangle\langle 0_e|$

$\{|n\rangle\}$: a complete basis in the Hilbert space of the environment.

The Lindblad Equation for Open Quantum Systems

- Time evolution of density matrix expressed in the *operator-sum* representation (Kraus representation)

$$\hat{\rho}_s(t) = \sum_n \hat{M}_n(t)\hat{\rho}_s(0)\hat{M}_n^\dagger(t)\,, \quad \hat{M}_n(t) = \langle n|\hat{U}(t)|0_e\rangle\,, \qquad \sum_n \hat{M}_n^\dagger(t)\hat{M}_n(t) = 1$$

- To extract a differential equation from the Kraus representation, one needs the Markovian approximation:

Correlation time between environmental degrees of freedom

$$\boxed{t_{\rm corr} \ll \Delta t_s}$$

Typical inverse frequency of the observed system

For an infinitesimal period of time:

$\hat{L}_n$ Jump Operator (Lindblad Operator)

$$\hat{M}_n(dt) = \sqrt{dt}\hat{L}_n\,, \quad n > 0$$

$$\hat{M}_0(dt) = 1 + (-i\hat{H}_s + \hat{K})dt \qquad \hat{K} = -\frac{1}{2}\sum_{n>0}\hat{L}_n^\dagger\hat{L}_n$$

The Lindblad equation (also called Gorini-Kossakowski-Lindblad-Sudarshan master equation):

$$\frac{d\hat{\rho}_s}{dt} = -i[\hat{H}_s, \hat{\rho}_s] + \sum_{n>0}\left(\hat{L}_n\hat{\rho}_s\hat{L}_n^\dagger - \frac{1}{2}\hat{L}_n^\dagger\hat{L}_n\hat{\rho}_s - \frac{1}{2}\hat{\rho}_s\hat{L}_n^\dagger\hat{L}_n\right)$$

The Hamiltonian Formalism for CGC Effective Theory

Kovner, Lublinsky, Wiedemann (2007)

- **The Light-Cone Hamiltonian of the pure gluonic sector of QCD**

$$H_{CGC} = \int dx^- d\mathbf{x}_\perp \left(\frac{1}{2}(\Pi_a^-(x^-, \mathbf{x}_\perp) + \frac{1}{\partial_-} J_a^+)^2 + \frac{1}{4} F_{ij}^a(x^-, \mathbf{x}_\perp) F_{ij}^a(x^-, \mathbf{x}_\perp) \right)$$

Light-Cone Quantization

$$\left[\hat{a}_i^a(k^+, \mathbf{x}_\perp), \hat{a}_j^{\dagger b}(p^+, \mathbf{y}_\perp)\right] = (2\pi)\delta^{ab}\delta_{ij}\delta(k^+ - p^+)\delta(\mathbf{x}_\perp - \mathbf{y}_\perp)$$

$$\left[\hat{j}^a(\mathbf{x}_\perp), \hat{j}^b(\mathbf{y}_\perp)\right] = igf^{abc}\hat{j}^c(\mathbf{x}_\perp)\delta(\mathbf{x}_\perp - \mathbf{y}_\perp)$$

- **Finding the Ground State of Soft Gluons**

<u>*Dilute Regime*</u> $(j^a \sim g)$

$$|\psi_0\rangle = \hat{\mathcal{C}}|0\rangle$$

$$\hat{\mathcal{C}} = \mathrm{Exp}\left\{ i \int d\mathbf{x}_\perp b_i^a(\mathbf{x}_\perp) \int \frac{dk^+}{\pi |k^+|^{1/2}} \left(\hat{a}_i^{\dagger a}(k^+, \mathbf{x}_\perp) + \hat{a}_i^a(k^+, \mathbf{x}_\perp) \right) \right\}$$

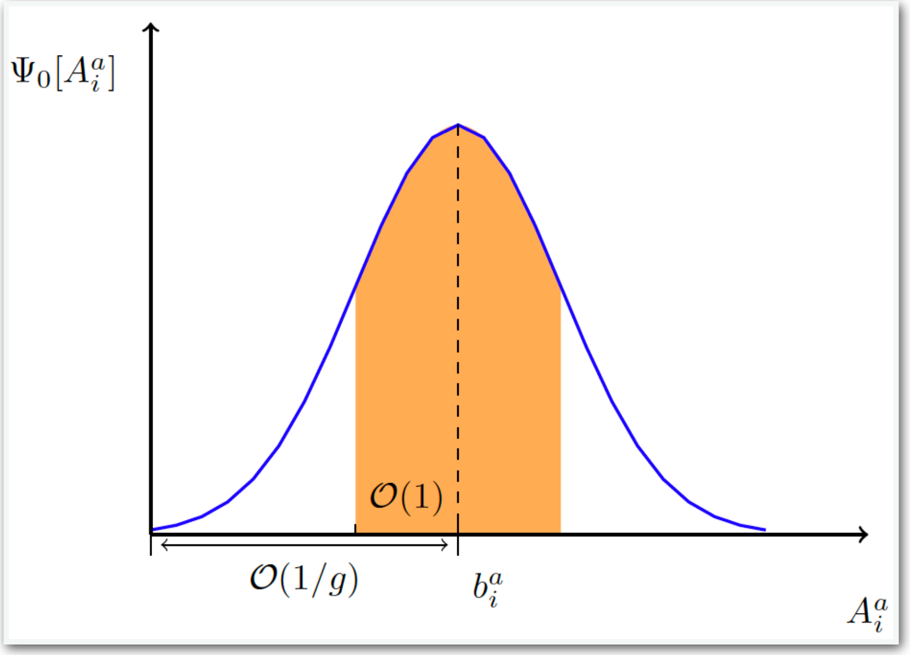
<u>*Dense Regime*</u> $(j^a \sim 1/g)$

$$|\psi_0\rangle = \hat{\mathcal{C}}\hat{\mathcal{B}}|0\rangle$$

$$\hat{\mathcal{B}} = \mathrm{Exp}\left\{ \hat{a}_\alpha^\dagger \Lambda_{\alpha\beta} \hat{a}_\beta^\dagger + \hat{a}_\alpha \Lambda_{\alpha\beta}^* \hat{a}_\beta \right\}$$

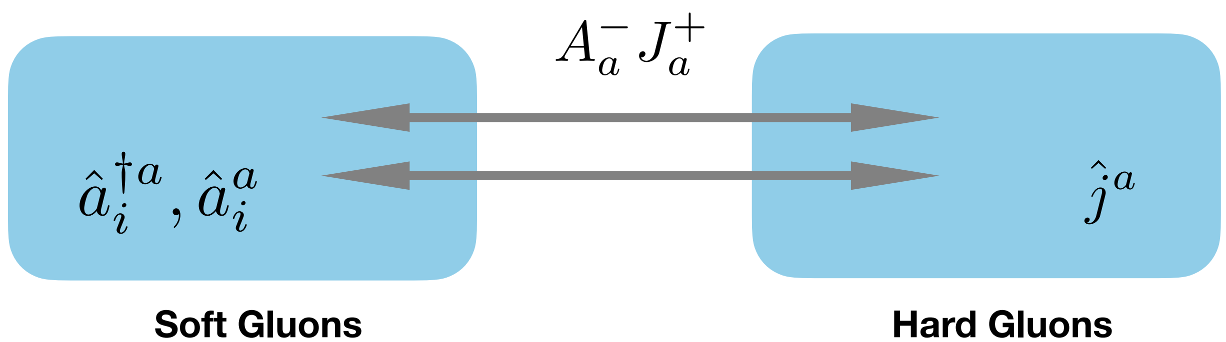
<u>*Weizsäcker-Williams gluon field*</u>

$$\boxed{\begin{aligned} &\partial_i b_i^a(\mathbf{x}_\perp) = j^a(\mathbf{x}_\perp)\,, \\ &\partial_i b_j^a(\mathbf{x}_\perp) - \partial_j b_i^a(\mathbf{x}_\perp) - g f^{abc} b_i^b(\mathbf{x}_\perp) b_j^c(\mathbf{x}_\perp) = 0\,. \end{aligned}}$$



High Energy Evolution for Density Matrix

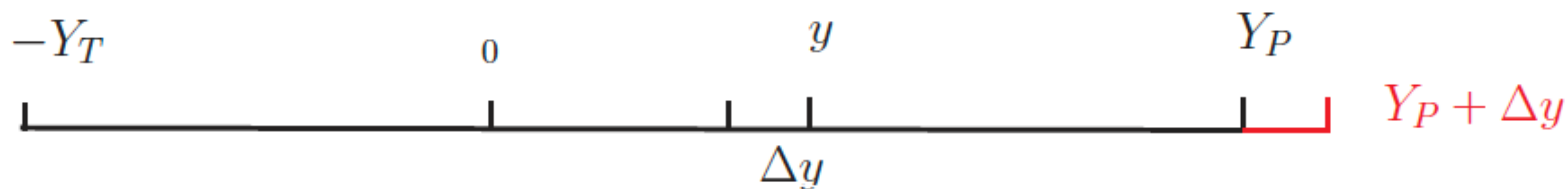
- Consider the bipartite system of soft gluons and hard gluons



<u>Assuming the initial density matrix is separable</u>

$$\hat{\rho} = \hat{\rho}_h \otimes |0\rangle\langle 0|$$

- Boost the system by a finite rapidity



For observables that only depend on color charge density of hard gluons

$$\left\langle \mathcal{O}(\hat{j}^a) \right\rangle = \mathrm{Tr}\left[\mathcal{O}(\hat{j}^a)\hat{\rho}\right] \longrightarrow \left\langle \mathcal{O}(\hat{j}^a) \right\rangle(\Delta y) = \mathrm{Tr}\left[\mathcal{O}(\hat{j}^a + \hat{j}^a_{\mathrm{soft}})\hat{\rho}(\Delta y)\right]$$

$$\hat{\rho}(\Delta y) = \hat{\Omega}|0\rangle\hat{\rho}_h\langle 0|\hat{\Omega}^\dagger \qquad \hat{\Omega} \equiv \Omega[\hat{j}^a, \hat{a}_i^{a\dagger}, \hat{a}_i^a; \Delta y] = \hat{\mathcal{C}}\hat{\mathcal{B}}$$

- Trace over soft gluons, integrating out all degrees of freedom except the color charge density

$$\hat{\rho}_h(\Delta y) = \mathrm{Tr}_s[\hat{R}\hat{\rho}(\Delta y)\hat{R}^\dagger] = \sum_n \langle n|\hat{R}\hat{\Omega}|0\rangle\hat{\rho}_h\langle 0|\hat{\Omega}^\dagger\hat{R}^\dagger|n\rangle = \sum_n \hat{M}_n\hat{\rho}_h\hat{M}_n^\dagger$$

<u>Color Charge Shift Operator</u> $\quad \hat{R}^\dagger\mathcal{O}(\hat{j}^a)\hat{R} = \mathcal{O}(\hat{j}^a + \hat{j}^a_{\mathrm{soft}}) \quad$ <u>**$\hat{R}$ must be a Unitary operator**</u>

Extended $SU(N_c)$ Algebra

$$\hat{R}^\dagger \hat{j}^a(\mathbf{x}_\perp)\hat{R} = \hat{j}^a(\mathbf{x}_\perp) + \hat{j}^a_{\rm soft}(\mathbf{x}_\perp) \qquad \hat{R} = {\rm Exp}\left\{-i\int d^2\mathbf{x}_\perp \hat{j}^a_{\rm soft}(\mathbf{x}_\perp)\hat{\Phi}^a(\mathbf{x}_\perp)\right\}$$

Angular Momentum Renormalization (Coarse Graining) Algebra

$$\boxed{\begin{aligned}&\left[\hat{j}^a, \hat{j}^b\right] = igf^{abc}\hat{j}^c\,. \\ &\left[\hat{\Phi}^a, \hat{\Phi}^b\right] = 0\,, \\ &\left[\hat{\Phi}^a, \hat{j}^b\right] = -i\left[\frac{\chi}{2}\coth\frac{\chi}{2} - \frac{\chi}{2}\right]^{ab}, \quad \text{with} \quad \chi = igT^e\hat{\Phi}^e\,.\end{aligned}}$$

<u>Taylor Expansion:</u> $M^{ab}(\chi) = \left[\frac{\chi}{2}\coth\frac{\chi}{2} - \frac{\chi}{2}\right]^{ab} = \sum_N [\chi^N]^{ab}$

$$c_0 = 1,\ c_1 = -\frac{1}{2},\ c_2 = \frac{1}{12},\ c_3 = 0,\ c_4 = -\frac{1}{6!}\,, c_5 = 0,\ c_6 = \frac{1}{6}\frac{1}{7!},\ c_7 = 0,\ c_8 = -\frac{1}{30}\frac{1}{8!},\ \ldots$$

<u>Consistence Check:</u> **Jacobi Identity** $\left[\left[\hat{\Phi}^a, \hat{j}^b\right], \hat{j}^c\right] + \left[\left[\hat{j}^b, \hat{j}^c\right], \hat{\Phi}^a\right] + \left[\left[\hat{j}^c, \hat{\Phi}^a\right], \hat{j}^b\right] = 0$ **Checked up to c_4**

Constructing a representation of this new algebra?

$$\hat{j}^a(\mathbf{x}) = g\int_{\Lambda^+}\frac{dk^+}{2\pi}\hat{a}_i^{\dagger b}(k^+, \mathbf{x})T^a_{bc}\hat{a}^c_i(k^+, \mathbf{x})$$

$$\hat{\Phi}^a(\mathbf{x}) = \mathcal{F}[\hat{a}_i^{\dagger a}, \hat{a}_i^b]$$

Analogy with Heisenberg Lie algebra

$$[\hat{x}, \hat{p}] = i, \quad e^{ia\hat{p}}\hat{x}e^{-ia\hat{p}} = \hat{x} + a.$$

The Lindblad Form

Differential Evolution Equation in the Leading Logarithmic Approimation (LLA)

$$\boxed{\alpha_s \Delta y \sim 1}$$

$$\hat{\rho}_h(\Delta y) = \sum_n \hat{M}_n \hat{\rho}_h \hat{M}_n^\dagger \quad \longrightarrow \quad \frac{d\hat{\rho}_h}{dy} = \alpha_s(?)$$

In the Dilute Regime

$$\hat{M}_n = \langle n|\hat{R}\hat{\Omega}|0\rangle = \langle n|\hat{R}\mathrm{Exp}\left\{-\frac{\Delta y}{2\pi} b_\alpha b_\alpha\right\} \mathrm{Exp}\left\{i\sqrt{2} b_\alpha \int \frac{d\eta}{2\pi} \hat{a}^\dagger_\alpha(\eta)\right\} |0\rangle$$

The Lindblad High Energy Evolution Equation

$$\frac{d\hat{\rho}_h}{dy} = -2\alpha_s(\bar{b}_\alpha \bar{b}_\alpha \hat{\rho}_h + \hat{\rho}_h \bar{b}_\alpha \bar{b}_\alpha - 2\bar{b}_\alpha \hat{\rho}_h \bar{b}_\alpha)$$

$$\bar{b}_\alpha = \mathcal{R}^\dagger_{\alpha\beta} b_\beta$$

$$\hat{R}\hat{a}_i^b(k^+, \mathbf{y}_\perp)\hat{R}^\dagger = [\hat{\mathcal{R}}(\mathbf{y}_\perp)]_{bd}\, \hat{a}_i^d(k^+, \mathbf{y}_\perp)$$

$$\hat{\mathcal{R}}(\mathbf{y}_\perp) = e^{igT^a \hat{\Phi}^a(\mathbf{y}_\perp)}$$

The LLA automatically picks up terms that are linear in Δy

1. In LLA, the "time" scale for the change of density matrix is $\Delta y \sim \mathcal{O}(1/\alpha_s)$

2. Soft gluons do not interact with each other and they propagate freely.
The correlation "time" is related to the free propagator $y_{corr} \sim 1/k^+ \sim e^{-y} \sim \mathcal{O}(1)$

$$\boxed{y_{corr} \ll \Delta y}$$

Markovian condition is satisfied.

Lindblad Form in the Dilute Regime

In the Dilute Regime

The Lindblad Equation for High Energy Evolution

$$\frac{d\hat{\rho}_h}{dy} = -2\alpha_s(\bar{b}_\alpha \bar{b}_\alpha \hat{\rho}_h + \hat{\rho}_h \bar{b}_\alpha \bar{b}_\alpha - 2\bar{b}_\alpha \hat{\rho}_h \bar{b}_\alpha)$$

$$\hat{\mathcal{R}}(\mathbf{y}_\perp) = e^{igT^a \hat{\Phi}^a(\mathbf{y}_\perp)}$$

$$b_i^\alpha(x) = \frac{\partial_i}{\partial^2} j^a(y)$$

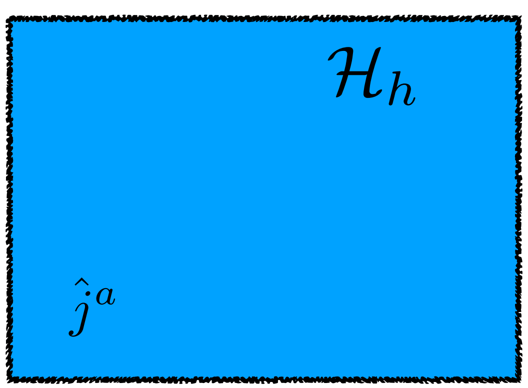
$$\bar{b}_\alpha = \mathcal{R}^\dagger_{\alpha\beta} b_\beta$$

Virtual terms,
color rotation within
each representation

Real term,
Changing weights of representations.

What is the meaning of the equation? Evolution of the Hilbert space

$$\mathcal{H}_h = \bigoplus_j \mathcal{H}_j$$



Hilbert space of hard gluons is completely determined by color charge operator.

By analogy with *Angular Momentum Addition,* adding one gluon at each step of the evolution is equivalent to understanding it as adding one adjoint representation of the color SU(3) algebra to the Hilbert space

$$\mathbf{8} \otimes \mathbf{8} = \mathbf{1} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{10} \oplus \overline{\mathbf{10}} \oplus \mathbf{27}$$

Increasing the weights of higher color charge representations in the density matrix.

Lindblad Form in the Dense Regime

Differential Evolution Equation in the Leading Logarithmic Approximation (LLA)

$$\hat{\rho}_h(\Delta y) = \sum_n \hat{M}_n \hat{\rho}_h \hat{M}_n^\dagger \longrightarrow \frac{d\hat{\rho}_h}{dy} = \alpha_s(?)$$

$$\boxed{\alpha_s \Delta y \sim 1}$$

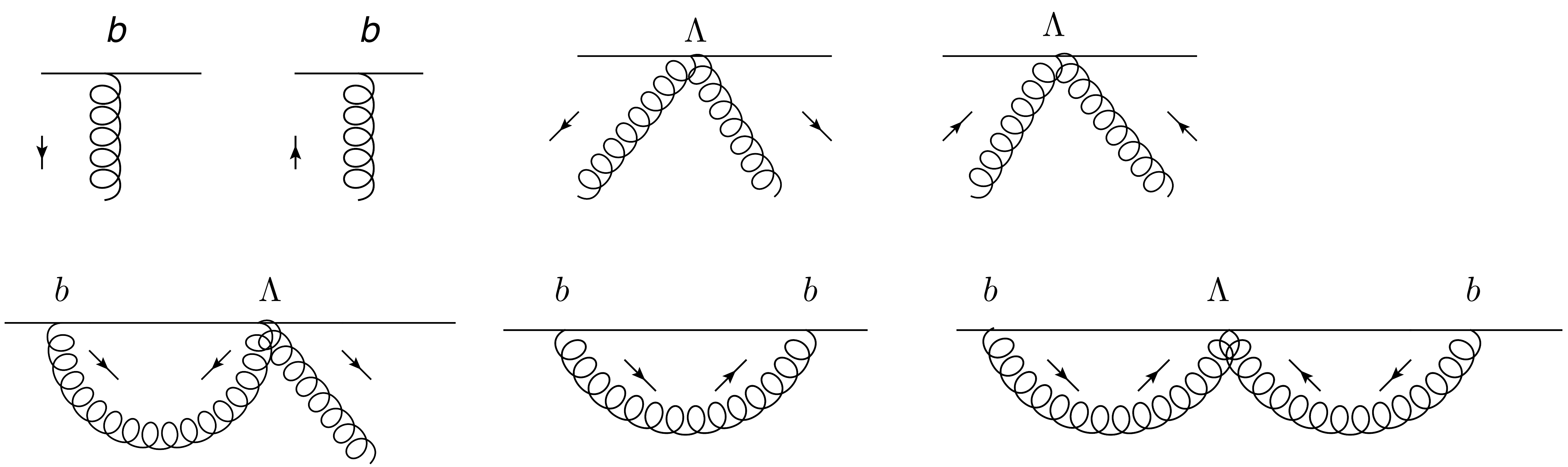
In the Dense Regime

$$\hat{M}_n = \langle n|\hat{R}\hat{\Omega}|0\rangle = \mathrm{Exp}\left\{-\frac{\Delta y}{2\pi} b_\alpha b_\beta (1-\Lambda_0)_{\alpha\beta}\right\} \mathcal{N}(\Lambda)$$

$$\langle n|\hat{R}\,\mathrm{Exp}\left\{i\sqrt{2} b_\alpha (1-\Lambda_0)_{\alpha\beta} \int \frac{d\eta}{2\pi} \hat{a}^\dagger_\beta(\eta)\right\} \mathrm{Exp}\left\{-\frac{1}{2}\int \frac{d\eta}{2\pi}\frac{d\xi}{2\pi} \Lambda_{\alpha\beta}(\eta,\xi) \hat{a}^\dagger_\alpha(\eta) \hat{a}^\dagger_\beta(\xi)\right\} |0\rangle$$

$$\boxed{\Lambda_{0,\alpha\beta} = \int_{-\Delta y}^{\Delta y} \frac{d\zeta}{2\pi} \Lambda_{\alpha\beta}(\zeta)}$$

Single-gluon, double-gluon emission and absorption vertices.



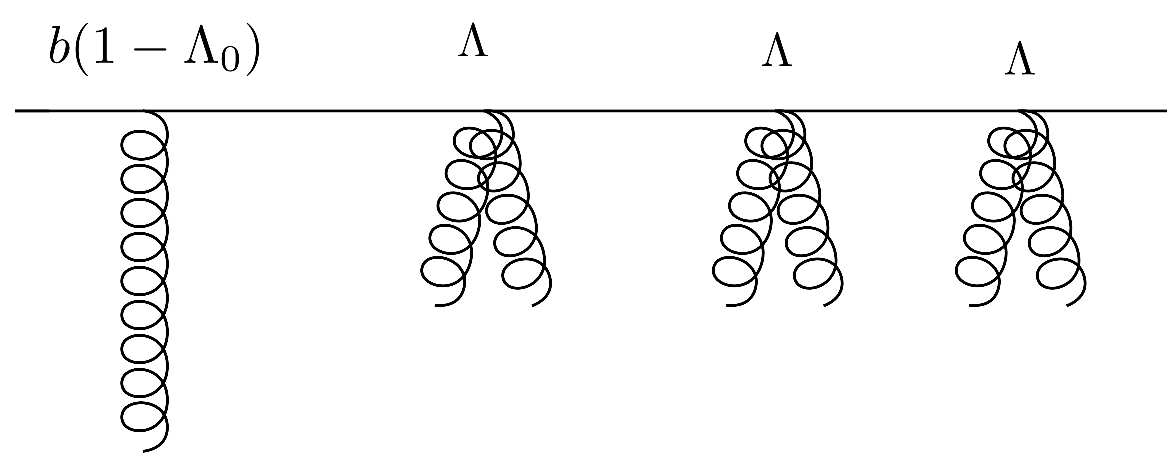
Jump Operators in the Dense Regime

In the Dense Regime

$$\hat{M}_n = \langle n|\hat{R}\hat{\Omega}|0\rangle = \mathrm{Exp}\left\{-\frac{\Delta y}{2\pi} b_\alpha b_\beta (1-\Lambda_0)_{\alpha\beta}\right\}\mathcal{N}(\Lambda)$$

$$\langle n|\hat{R}\,\mathrm{Exp}\left\{i\sqrt{2}b_\alpha(1-\Lambda_0)_{\alpha\beta}\int\frac{d\eta}{2\pi}\hat{a}^\dagger_\beta(\eta)\right\}\mathrm{Exp}\left\{-\frac{1}{2}\int\frac{d\eta}{2\pi}\frac{d\xi}{2\pi}\Lambda_{\alpha\beta}(\eta,\xi)\hat{a}^\dagger_\alpha(\eta)\hat{a}^\dagger_\beta(\xi)\right\}|0\rangle$$

Odd numbers of gluons



$$\hat{M}^{(O)}_{2m+1_{\{\alpha_i,\mathbf{w}_i,\eta_i;i=1,\ldots,2m+1\}}} = \sum_{\{i_1,\ldots,i_{2m+1}\}}^{P}[ib(1-\Lambda_0)]_{\beta_{i_1}}(\mathbf{w}_{i_1})\prod_{k\neq l\neq 1}^{m}\Lambda_{\beta_{i_k}\beta_{i_l}}(\mathbf{w}_{i_k},\eta_{i_k};\mathbf{w}_{i_l},\eta_{i_l})\prod_{q=1}^{2m+1}\mathcal{R}_{\beta_{i_q}\alpha_{i_q}}(\mathbf{w}_{i_q})$$

Overall factor in Δy

$$\hat{M}^{(O)}_{2m+1}\hat{\rho}_h\hat{M}^{(O)\dagger}_{2m+1} = \boxed{\frac{\Delta y}{2\pi}}[\bar{b}_L(1-\bar{\Lambda}_{0,L})](\bar{\Lambda}^\dagger_{0,R}\bar{\Lambda}_{0,L})^m[(1-\bar{\Lambda}^\dagger_{0,R})\bar{b}_R]\hat{\rho}_h$$

$$\sum_{m=0}\hat{M}^{(O)}_{2m+1}\hat{\rho}_h\hat{M}^{(O)\dagger}_{2m+1} = \frac{\Delta y}{2\pi}[\bar{b}_L(1-\bar{\Lambda}_{0,L})](1-\bar{\Lambda}^\dagger_{0,R}\bar{\Lambda}_{0,L})^{-1}[(1-\bar{\Lambda}^\dagger_{0,R})\bar{b}_R]\hat{\rho}_h\,.$$

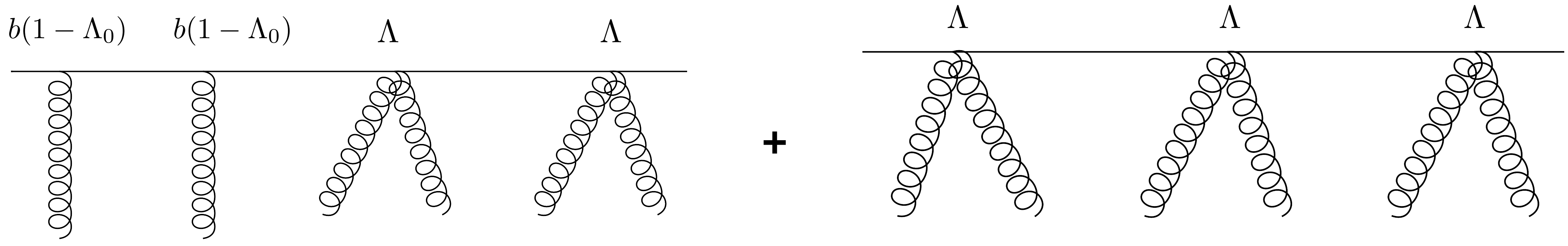
Geometric Series

Jump Operators in the Dense Regime

In the Dense Regime

$$\hat{M}_n = \langle n|\hat{R}\hat{\Omega}|0\rangle = \mathrm{Exp}\left\{-\frac{\Delta y}{2\pi} b_\alpha b_\beta (1-\Lambda_0)_{\alpha\beta}\right\}\mathcal{N}(\Lambda)$$
$$\langle n|\hat{R}\,\mathrm{Exp}\left\{i\sqrt{2} b_\alpha (1-\Lambda_0)_{\alpha\beta}\int \frac{d\eta}{2\pi}\hat{a}^\dagger_\beta(\eta)\right\}\mathrm{Exp}\left\{-\frac{1}{2}\int\frac{d\eta}{2\pi}\frac{d\xi}{2\pi}\Lambda_{\alpha\beta}(\eta,\xi)\hat{a}^\dagger_\alpha(\eta)\hat{a}^\dagger_\beta(\xi)\right\}|0\rangle$$

Even numbers of gluons



$$\begin{aligned}
&\hat{M}_0\hat{\rho}_h\hat{M}_0^\dagger + \sum_{n>0}\hat{M}_n^{(E)}\hat{\rho}_h\hat{M}_n^{(E)\dagger}\\
=&\hat{\rho}_h - \left(\frac{\Delta y}{2\pi}\right)\left[\bar{b}_L(1-\bar{\Lambda}_{0,L})(1-\bar{\Lambda}^\dagger_{0,R}\bar{\Lambda}_{0,L})^{-1}(1-\bar{\Lambda}^\dagger_{0,R})\bar{b}_L\right]\hat{\rho}_h\\
&-\left(\frac{\Delta y}{2\pi}\right)\left[\bar{b}_R(1-\bar{\Lambda}_{0,L})(1-\bar{\Lambda}^\dagger_{0,R}\bar{\Lambda}_{0,L})^{-1}(1-\bar{\Lambda}^\dagger_{0,R})\bar{b}_R\right]\hat{\rho}_h\,.
\end{aligned}$$

Lindblad Form in the Dense Regime

Infinitesimal changes (linear in Δy) of density matrix

$$\hat{\rho}_h(\Delta y) = \hat{\rho}_h - \left(\frac{\Delta y}{2\pi}\right)\left[(\bar{b}_L - \bar{b}_R)(1-\bar{\Lambda}^\dagger_{0,R})(1-\bar{\Lambda}_{0,L}\bar{\Lambda}^\dagger_{0,R})^{-1}(1-\bar{\Lambda}_{0,L})(\bar{b}_L-\bar{b}_R)\right]\hat{\rho}_h$$

In the leading logarithmic approximation, only terms that are of order α_s need to be retained

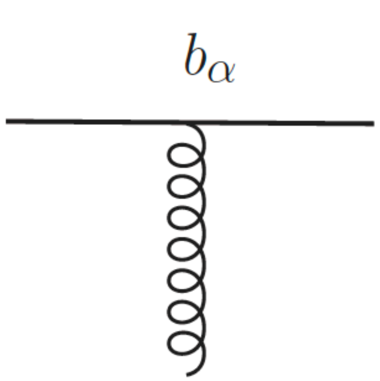
$$(1-\bar{\Lambda}^\dagger_0)(1-\bar{\Lambda}_0\bar{\Lambda}^\dagger_0)^{-1}(1-\bar{\Lambda}_0) = \bar{N}^\dagger_\perp \bar{N}_\perp \qquad N_\perp = \left[\delta_{ij} - \partial_i \frac{1}{\partial^2}\partial_j - D_i\frac{1}{D^2}D_j\right]^{ab}(\mathbf{x}_\perp, \mathbf{y}_\perp)$$

$$\longrightarrow \quad \hat{\rho}_h(\Delta y) = \hat{\rho}_h - \frac{\Delta y}{2\pi}(\bar{b}_L - \bar{b}_R)\bar{N}^\dagger_\perp \bar{N}_\perp(\bar{b}_L - \bar{b}_R)\hat{\rho}_h$$

Lose information on operator orderings

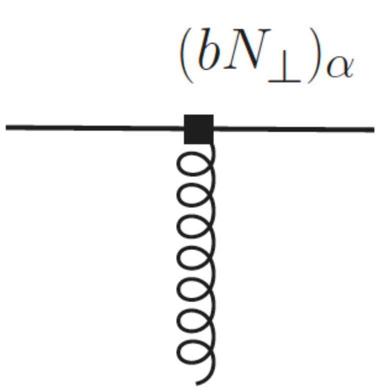
Expressing it into a Lindblad Form

$$\boxed{\begin{aligned} \frac{d\hat{\rho}_h}{dy} &= -\frac{1}{2\pi}\left(\bar{B}^\dagger_\alpha \bar{B}_\alpha \hat{\rho}_h + \hat{\rho}_h \bar{B}^\dagger_\alpha \bar{B}_\alpha - 2\bar{B}_\alpha \hat{\rho}_h \bar{B}^\dagger_\alpha\right), \\ \bar{B}_\alpha &= \hat{\mathcal{R}}^\dagger_{\alpha\beta} B_\beta\,, \quad B_\beta = (bN_\perp)_\beta\,. \end{aligned}}$$



Dilute limit

$\Longrightarrow$



Dense limit

effective vertex of single gluon emission

Kovner, Lublinsky and Skokov (2017)

From Lindblad Equation to JIMWLK Equation

$$\frac{d\hat{\rho}_h}{dy} = -\frac{1}{2\pi}\left(\bar{B}^\dagger_\alpha \bar{B}_\alpha \hat{\rho}_h + \hat{\rho}_h \bar{B}^\dagger_\alpha \bar{B}_\alpha - 2\bar{B}_\alpha \hat{\rho}_h \bar{B}^\dagger_\alpha\right),$$

$$\bar{B}_\alpha = \hat{\mathcal{R}}^\dagger_{\alpha\beta} B_\beta\,, \quad B_\beta = (bN_\perp)_\beta\,.$$

Quantum Operator Equation for Density Matrix

Mapping from Hilbert space to classical Phase space

$$\frac{\partial \mathcal{W}_y[j^a]}{\partial y} = \frac{1}{2}\frac{\delta}{\delta j^a(\mathbf{x}_\perp)}\eta^{ab}[\mathbf{x}_\perp, \mathbf{y}_\perp; j]\frac{\delta}{\delta j^b(\mathbf{y})}\mathcal{W}_y[j^a]$$

Classical Functional Fokker-Planck Equation for Probability Functional

$$\frac{\partial \mathcal{W}_y[V]}{\partial y} = -\mathcal{H}_{\text{JIMWLK}}\mathcal{W}_y[V]$$

$$\mathcal{H}_{\text{JIMWLK}} = \frac{1}{2\pi}\int d^2\mathbf{z}_\perp\, Q^{a\dagger}_i(\mathbf{z}_\perp) Q^a_i(\mathbf{z}_\perp)$$

$$Q^a_i(\mathbf{z}_\perp) = \frac{g}{2\pi}\int d^2\mathbf{x}_\perp \frac{(\mathbf{z}_\perp - \mathbf{x}_\perp)_i}{(\mathbf{z}_\perp - \mathbf{x}_\perp)^2}\left[V(\mathbf{z}_\perp) - V(\mathbf{x}_\perp)\right]^{ab} \mathcal{J}^b_R(\mathbf{x}_\perp)$$

$$\mathcal{J}^a_R(\mathbf{x}_\perp) = -\mathrm{Tr}\left(V(\mathbf{x}_\perp) T^a \frac{\delta}{\delta V^\dagger(\mathbf{x}_\perp)}\right)$$

Quantum Mechanics in Phase Space

Hillery, O'Connell, Scully and Wigner (1984)
Agarwal and Wolf (1970)

Hilbert Space		Phase Space
$\hat{q}, \hat{p}$	$\Delta_w(q,p;\hat{q},\hat{p})$ $\rightleftarrows$	q, p
$[\hat{q}, \hat{p}] = i$		$\{q, p\} = 1$
$A(\hat{q}, \hat{p})$		$A_w(q,p)$
$\hat{\rho}$		$W(q,p)$

Wigner Transformations

$$W(q,p) = \int dz e^{ipz} \langle q - \frac{z}{2} | \hat{\rho} | q + \frac{z}{2} \rangle$$

$$A_w(q,p) = \int dz e^{ipz} \langle q - \frac{z}{2} | A(\hat{q},\hat{p}) | q + \frac{z}{2} \rangle$$

$$\langle A(\hat{q},\hat{p}) \rangle = \mathrm{Tr}(A(\hat{q},\hat{p})\hat{\rho}) = \int dq dp A_w(q,p) W(q,p)$$

Weyl's correspondence rules : fully symmetrization, basis independent transformations

$$G(\hat{p},\hat{q}) = \int \frac{dp}{2\pi} \frac{dp}{2\pi} F(p,q) \Delta_w(q,p;\hat{q},\hat{p}) \,,$$

$$F(p,q) = \mathrm{Tr}\big(G(\hat{p},\hat{q}) \Delta_w(q,p;\hat{q},\hat{p})\big) \,,$$

Weyl's Mapping Kernel $$\Delta_w(q,p;\hat{q},\hat{p}) = \int du dv e^{-i(up+vq)} e^{i(u\hat{p}+v\hat{q})} \,.$$

Quantum Mechanics in Phase Space

Hillery, O'Connell,Scully and Wigner (1984)

For a product of two operators

Using ***Moyal Product***

$$\Gamma = \frac{\overleftarrow{\partial}}{\partial p}\frac{\overrightarrow{\partial}}{\partial q} - \frac{\overleftarrow{\partial}}{\partial q}\frac{\overrightarrow{\partial}}{\partial p}$$

$$F(\hat{q},\hat{p}) = A(\hat{q},\hat{p})B(\hat{q},\hat{p}) \longrightarrow F_w(q,p) = A_w(q,p)e^{\frac{\Gamma}{2i}}B_w(q,p) = B_w(q,p)e^{-\frac{\Gamma}{2i}}A_w(q,p)$$

An equivalent approach using *Bopp operators*

$$Q_L = q + \frac{i}{2}\frac{\partial}{\partial p}, \quad P_L = p - \frac{i}{2}\frac{\partial}{\partial q},$$
$$Q_R = q - \frac{i}{2}\frac{\partial}{\partial p}, \quad P_R = p + \frac{i}{2}\frac{\partial}{\partial q}.$$

$$F_w(q,p) = A(Q_L,P_L)B_w(q,p) = B(Q_R,P_R)A_w(q,p).$$

An example: Mapping the quantum Liouville equation onto the phase space

$$\frac{d\hat{\rho}}{dt} = -i[\hat{H},\hat{\rho}] \longrightarrow \frac{df(q,p,t)}{dt} = -i(H_w(q,p)e^{\frac{\Gamma}{2i}}f(q,p,t) - f(q,p,t)e^{\frac{\Gamma}{2i}}H_w(q,p))$$
$$= 2H_w(q,p)\sin\left(\frac{\Gamma}{2i}\right)f(q,p,t)$$

$$\longrightarrow \frac{df(q,p,t)}{dt} = -i\left(H(Q_L,P_L) - H(Q_R,P_R)\right)f(q,p,t)$$

Quantum-Classical Correspondence in CGC

Hilbert Space

$\hat{j}^a$

$[\hat{j}^a, \hat{j}^b] = igf^{abc}\hat{j}^c$

$\mathcal{O}(\hat{j}^a)$

$\hat{\rho}_h$

$\overset{?}{\rightleftarrows}$

Phase Space

j^a

$\mathcal{O}_w(j^a)$

$W[j^a]$

Fully symmetrized expression in $\hat{j}^a$.

From phase space to Hilbert space:

$$G_s[\hat{\mathbf{j}}] = \int d\mathbf{j}\, F[\mathbf{j}]\Delta_W(\mathbf{j}, \hat{\mathbf{j}})\,,$$

$$\Delta_W(\mathbf{j}, \hat{\mathbf{j}}) = \int d\alpha\, e^{-i\alpha\cdot\mathbf{j}} e^{i\alpha\cdot\hat{\mathbf{j}}}\,.$$

Density matrix and quasi-probability distribution

$$\mathrm{Tr}(\hat{\rho}G_s[\hat{\mathbf{j}}]) = \int d\mathbf{j}\, F[\mathbf{j}]\,\mathrm{Tr}(\hat{\rho}\Delta_W(\mathbf{j}, \hat{\mathbf{j}})) = \int d\mathbf{j}\, F[\mathbf{j}]\, W[\mathbf{j}]\,.$$

$$\boxed{\begin{aligned} &W[\mathbf{j}] = \mathrm{Tr}\left(\hat{\rho}\Delta_W(\mathbf{j}, \hat{\mathbf{j}})\right)\,, \\ &\int d\mathbf{j} W[\mathbf{j}] = 1\,. \end{aligned}}$$

For a Gaussian weight functional

$$W[j^a] \sim \exp\left\{-\frac{j^2}{\mu^2}\right\}$$

$$\hat{\rho}_h \nsim \exp\left\{-\frac{\hat{j}^a\hat{j}^a}{\mu^2}\right\}$$

From Hilbert space to phase space:

$$G_s[\mathbf{j}] = \mathrm{Tr}\left(G_s[\hat{\mathbf{j}}]\tilde{\Delta}_W(\mathbf{j}, \hat{\mathbf{j}})\right)\,, \quad \tilde{\Delta}_W[\mathbf{j}, \hat{\mathbf{j}}] = \int dg_\alpha e^{i\alpha\cdot\mathbf{j}} d_r(\hat{\mathbf{j}}) e^{i\alpha\cdot\hat{\mathbf{j}}}\,.$$

$$\mathrm{Tr}\left(\Delta_W[\mathbf{j}_1, \hat{\mathbf{j}}]\tilde{\Delta}_W[\mathbf{j}_2, \hat{\mathbf{j}}]\right) = \delta(\mathbf{j}_1 - \mathbf{j}_2)\,.$$

from the Peter-Weyl Theorem of Group Representation

From Lindblad Equation to JIMWLK Equation

Lindblad high energy evolution equation:

$$\frac{d\hat{\rho}_h}{dy} = -\frac{1}{2\pi}(\bar{B}^\dagger_\alpha \bar{B}_\alpha \hat{\rho}_h + \hat{\rho}_h \bar{B}^\dagger_\alpha \bar{B}_\alpha - 2\bar{B}_\alpha \hat{\rho}_h \bar{B}^\dagger_\alpha)\,,$$

$$\bar{B}_\alpha = \hat{\mathcal{R}}^\dagger_{\alpha\beta} B_\beta\,, \quad B_\beta = (bN_\perp)_\beta\,.$$

Bopp operators for $SU(N)$ Lie algebra (*Kovner and Lublinsky (2005, 2013)*):

$$\boxed{\begin{aligned} j^a_L &= j^b\left[\frac{\tau}{2}\coth\frac{\tau}{2} + \frac{\tau}{2}\right]^{ba}, \\ j^a_R &= j^b\left[\frac{\tau}{2}\coth\frac{\tau}{2} - \frac{\tau}{2}\right]^{ba}. \end{aligned}}$$

$$\left[j^a_R, j^b_R\right] = igf^{abc}j^c_R$$

$$\left[j^a_L, j^b_L\right] = -igf^{abc}j^c_L$$

$$\tau = gT^e\frac{\delta}{\delta j^e}$$

Mapping Rules

$$\boxed{\begin{aligned} &\hat{\rho}_h \longrightarrow W[j^a], \\ &B_\alpha[\hat{j}^a] \longrightarrow B^\alpha_L \equiv B_\alpha[j^a_L], \quad B^\alpha_R \equiv B_\alpha[j^a_R], \\ &\hat{\mathcal{R}}_{\alpha\gamma}\hat{O}[\hat{j}^a]\hat{\mathcal{R}}^\dagger_{\gamma\beta} = \hat{O}[\hat{j}^a + gT^a]_{\alpha\beta} \longrightarrow [\mathcal{R}_p O(j^a)]_{\alpha\beta}, \quad \mathcal{R}_p = e^\tau. \end{aligned}}$$

Lindblad high energy evolution equation in phase space

$$\frac{dW[j^a]}{dy} = -\frac{1}{2\pi}(\tilde{B}^\alpha_L - B^\alpha_R)^\dagger(\tilde{B}^\alpha_L - B^\alpha_R)W[j^a]$$

$$\tilde{B}^\alpha_L = \mathcal{R}^{\dagger\alpha\beta}_p B^\beta_L$$

From Lindblad Equation to JIMWLK Equation

Lindblad high energy evolution equation in phase space:

$$\frac{dW[j^a]}{dy} = -\frac{1}{2\pi}(\tilde{B}_L^\alpha - B_R^\alpha)^\dagger(\tilde{B}_L^\alpha - B_R^\alpha)W[j^a]\,,$$
$$\tilde{B}_L^\alpha = \mathcal{R}_p^{\dagger\alpha\beta}B_L^\beta\,.$$

Truncate $\mathcal{R}_p$ to first order in $\delta/\delta j^a$,
Expand b_L, b_R around j^a to first order.

$$\tilde{B}_L^\alpha - B_R^\alpha \simeq N_\perp(b_L\mathcal{R}_p - b_R)$$

$$\left(b_i^a[j] + \frac{\delta b_i^a}{\delta j^e}(j_L^e - j^e)\right)\left(\delta_{ab} + gT_{ab}^d\frac{\delta}{\delta j^d}\right) - \left(b_i^b[j] + \frac{\delta b_i^b}{\delta j^e}(j_R^e - j^e)\right)$$

Expressing the evolution equation in terms of spatial Wilson lines $U(\mathbf{x}_\perp) = \mathcal{P}\exp\left\{ig\int d\mathbf{y}_\perp \cdot b^c(\mathbf{y}_\perp)T^c\right\}$

After some algebra $\tilde{B}_L^\alpha(\mathbf{x}_\perp) - B_R^a(\mathbf{x}_\perp) = U^{\dagger ab}(\mathbf{x}_\perp)Q_i^b(\mathbf{x}_\perp)$

$$\mathcal{H}_{\text{JIMWLK}} = \frac{1}{2\pi}\int d^2\mathbf{z}_\perp\, Q_i^{a\dagger}(\mathbf{z}_\perp)Q_i^a(\mathbf{z}_\perp)$$

$$Q_i^a(\mathbf{z}_\perp) = \frac{g}{2\pi}\int d^2\mathbf{x}\frac{(\mathbf{z}-\mathbf{x})_i}{(\mathbf{z}-\mathbf{x})^2}\left[U(\mathbf{z}_\perp) - U(\mathbf{x}_\perp)\right]^{ab}\mathcal{J}_R^b(\mathbf{x}_\perp)$$

$$\mathcal{J}_R^b(\mathbf{z}_\perp) = -\mathrm{Tr}\left(U(\mathbf{z}_\perp)T^b\frac{\delta}{\delta U^\dagger(\mathbf{z}_\perp)}\right)$$

Summary

- The <u>*effective density matrix*</u> specified by the color charge density operators follows ***Lindblad type high energy evolution equation*** in the *leading logarithmic approximation* both in the dilute regime and in the dense regime. The Linblad operator turns out to be a product of the color charge shift operator and the (effective) Weizsacker-Williams field operator.

- This Lindblad type evolution equation can be mapped from the **Hilbert space** onto the **phase space** using **Weyl's correspondence rules** for color charge densities. In the dense regime, the JIMWLK equation is reproduced by truncating the phase space evolution equation to second order in gradient expansions.

- We have discovered a new algebra extending the general $SU(N)$ Lie algebra $[\hat{j}^a, \hat{j}^b] = igf^{abc}\hat{j}^c$, $[\hat{\Phi}^a, \hat{\Phi}^b] = 0$, $[\hat{\Phi}^a, \hat{j}^b] = -iM^{ab}[\hat{\Phi}]$. We are currently working on constructing representations of the new algebra and deriving the general mappings from Hilbert space determined by both $\hat{j}^a$ and $\hat{\Phi}^a$ operators onto the corresponding phase space.