Entanglement, partial set of measurements, and diagonality of the density matrix in the parton model

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Introduction: parton model

- Incoherent collection of partons; partons are "frozen" in infinite momentum frame
- Model (+ QCD factorization) describes many aspects of high energy experiments through universal parton distributions
- Only diagonal components of density matrix

  in number basis representation

Introduction: paradox of parton model

- Proton is a quantum object in a pure state; it has zero entropy

- Parton model: in high energy experiments, proton behaves
  like an incoherent ensemble of partons;
  as such it carries non-vanishing entropy

D. Kharzeev and E. Levin, 1702.03489

Introduction: possible resolution

- Entanglement of observed partons with unobserved proton degrees of freedom leads to lack of coherence and large entropy of partonic system

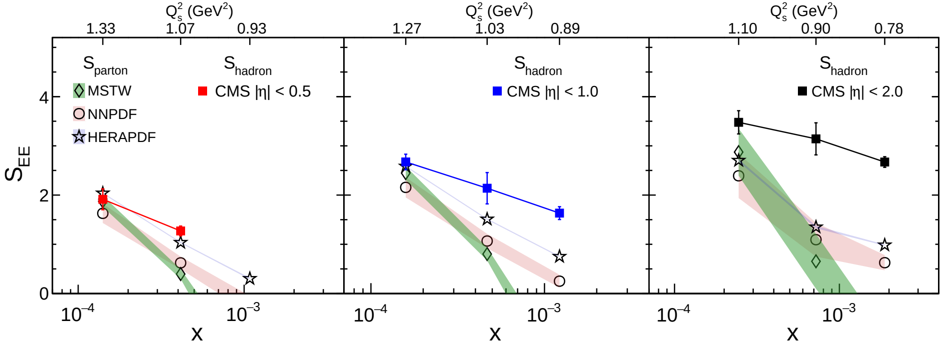
- That is $\rho_{\text{part. mod.}} \stackrel{?}{=} \text{Tr}_{\text{unobs}}\left[|P\rangle\langle P|\right]$; with $\rho_{\text{part. mod.}}$ being *diagonal* in the number of partons representation

- This natural proposal eliminates tension between pure nature of proton and incoherence nature of parton model

D. Kharzeev and E. Levin, 1702.03489

Introduction: application

- In DIS photon probes only a part of proton wave function
- Associated entanglement entropy $S_E = \ln xG$
- Entropy of final hadrons $S_h \geq S_E$
- Similar in p-p collisions



Z. Tu, D. Kharzeev, and T. Ullrich, Phys. Rev. Lett., 1904.11974 & later publications

Introduction: alternative

- A set of measurements described by parton model is not complete
- Most measurements in DIS are related to $\langle N \rangle = \mathrm{Tr}\Big[\int \frac{d^2 k}{(2\pi)^2} a^\dagger(\underline{k}) a(\underline{k})\, \hat{\rho}_{\mathrm{PM}}\Big]$
- Extending this to TMD's: $\langle a^\dagger(\underline{k}_1) a(\underline{k}_1)\, a^\dagger(\underline{k}_2) a(\underline{k}_2)\, \ldots \rangle$
- All of these are diagonal in number operator basis; no information about off diagonal elements
- $\rightsquigarrow$ infinite number of density matrices that are equivalent for the limited purpose of describing results of measurements

arXiv:2001.01726

Introduction: alternative

- Lack of knowledge can be characterized by an entropy: “the entropy of ignorance”
- Consider an incomplete defining set of observables $\{O_i\}$
- A density matrix reproducing this measurements $\hat{\rho}(\alpha_j)$ with α_j parametrizing measurements not covered by $\{O_i\}$
- Associated entropy $S(\alpha) = -\mathrm{Tr}\left[\hat{\rho}(\alpha) \ln \hat{\rho}(\alpha)\right]$
- Entropy of ignorance is the maximum of $S(\alpha)$ with respect to variation of α: $S_I = \max_\alpha S(\alpha)$
- In PM: the set of defining operators are products of particle density operators and thus only diagonal elements of the density matrix are determined by $\{O_i\}$
- In PM: α_j parametrize off-diagonal elements; it can be rigorously shown that parameters defining the entropy of ignorance corresponds to diagonal $\hat{\rho}$

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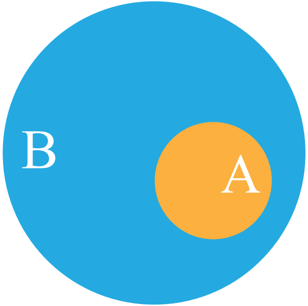
Entropy of entanglement and entropy of ignorance



- $\rho_A = \mathrm{Tr}_B \rho$
- ρ_{PM} – two alternatives: either ρ_A or obtained from ρ_A by dropping off-diagonal elements; which?
- We cannot answer this question in QCD
- Computable in CGC

Example: two fermion model I

- Two fermions, A and B in **pure** state

$$|\phi_{AB}\rangle = \frac{\sqrt{2}}{2}|0_A\rangle \otimes |0_B\rangle + \frac{1}{2}|1_A\rangle \otimes (|0_B\rangle + |1_B\rangle)$$

- Reduced density matrix for subsystem A and B are

$$\rho_A = \frac{1}{2}\begin{pmatrix} 1 & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 1 \end{pmatrix} \quad \rho_B = \frac{1}{4}\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}$$

Entanglement entropies for A and its complement are identical

$$S_E(\rho_A) = S_E(\rho_B) = \frac{3}{2}\ln 2 + \frac{1}{\sqrt{2}}\,\mathrm{acoth}\,\sqrt{2} \approx 0.416496$$

Example: two fermion model II

- Ignorance entropy depends on set of defining operators $\{O_i\}$
- First: $\{O_i\}$ as all operators diagonal in particle number basis. To calculate S_I: discard off-diagonal matrix elements in number basis $\rho_{AB} = \text{diag}\{1/2,\, 1/4,\, 0,\, 1/4\}$

$$S_I(\rho_{AB}) = -\sum_i p_i \ln p_i = \frac{3}{2}\ln 2 \approx 1.03972$$

- Entropy of ignorance for reduced density matrix ρ_A: measurable quantities are operators diagonal in Fock space of fermion A. Drop off-diagonal matrix elements of ρ_A: $\rho_A^I = \text{diag}\{1/2,\, 1/2\}$

$$S_I(\rho_A) = \ln 2 \approx 0.693147$$

- Similarly, $\rho_B^I = \text{diag}\{3/4,\, 1/4\}$, and corresponding entropy of ignorance is

$$S_I(\rho_B) = 2\ln 2 - \frac{3}{4}\ln 3 \approx 0.56233$$

Example: two fermion model III

- Entanglement: $S_E(\rho_A) = S_E(\rho_B)$
- Ignorance: $S_I(\rho_A) \neq S_I(\rho_B)$.
- Entanglement: $S_E(\rho_{AB}) = 0$
- Ignorance: $S_I(\rho_{AB}) \neq 0$

CGC wave function

- Wave function of slowly evolving valence charges and faster soft gluon degrees of freedom has the form

$$|\psi\rangle = |s\rangle \otimes |v\rangle$$

$|v\rangle =$ the state vector characterizing the valence dof;
$|s\rangle =$ the vacuum of the soft fields

- $|v\rangle$ is approximated by the McLerran-Venugopalan model

$$\langle\rho|v\rangle\langle v|\rho\rangle = \mathcal{N}\, e^{-\int_{\underline{k}} \frac{1}{2\mu^2}\rho_a(\underline{k})\rho_a^*(\underline{k})}$$

Soft fields

- Leading order: the CGC soft vacuum

$$|s\rangle = \mathcal{C}|0\rangle; \quad \mathcal{C} = \exp\left\{2i\mathrm{tr}\int_{\underline{k}} b^i(\underline{k})\phi_i^a(\underline{k})\right\}; \quad \phi_i(\underline{k}) \equiv a_i^+(\underline{k}) + a_i(-\underline{k})$$

- Background field b_a^i is determined by valence color charge density ρ via:

$$b_a^i(\underline{k}) = g\rho_a(\underline{k})\frac{i\underline{k}_i}{\underline{k}^2} + \mathcal{O}(\rho^2)$$

Reduced density matrix

- Consider hadron density matrix:

$$\hat{\rho} = |v\rangle \otimes |s\rangle\langle s| \otimes \langle v|$$

- Integrate out the valence dof – reduced density matrix

$$\hat{\rho}_r = \mathrm{Tr}_\rho \hat{\rho} \equiv \int D\rho \, \langle \rho|\hat{\rho}|\rho\rangle = \int D\rho \, \langle \rho|v\rangle \, |s\rangle\langle s| \, \langle v|\rho\rangle$$

- A meaningful proxy? Common element with the real life PM is the natural bi-partitioning of the dof in the underlying wave function and integrating over the "environment"
- Number basis representation ($R = \left(1 + \frac{q^2}{2g^2\mu^2}\right)^{-1}$)

$$\langle n_c(\underline{q}), m_c(-\underline{q})|\hat{\rho}_r(\underline{q})|\alpha_c(\underline{q}), \beta_c(-\underline{q})\rangle = (1-R)\frac{(n+\beta)!}{\sqrt{n!m!\alpha!\beta!}}\left(\frac{R}{2}\right)^{n+\beta} \delta_{(n+\beta),(m+\alpha)}$$

- Includes off-diagonal elements; c.f. Kharzeev-Levin conjecture

$$\langle n_c(\underline{q}), n_c(-\underline{q})|\hat{\rho}_r(\underline{q})|0, 0\rangle = (1-R)\left(\frac{R}{2}\right)^n$$

Entanglement entropy

- Entanglement entropy = Von Neumann entropy of reduced matrix $S_E = -\mathrm{Tr}\,\hat{\rho}_r \ln \hat{\rho}_r$

$$S_E = \frac{1}{2}(N_c^2 - 1)S_\perp \int \frac{d^2q}{(2\pi)^2}\left[\ln\left(\frac{g^2\mu^2}{q^2}\right) + \sqrt{1 + 4\frac{g^2\mu^2}{q^2}}\ln\left(1 + \frac{q^2}{2g^2\mu^2} + \frac{q^2}{2g^2\mu^2}\sqrt{1 + 4\frac{g^2\mu^2}{q^2}}\right)\right]$$

- Amusingly this is equivalent to

$$S_E = (N_c^2 - 1)S_\perp \int \frac{d^2q}{(2\pi)^2}\Big[(1+f)\ln(1+f) - f\ln f\Big], f = \frac{1}{\exp(\beta\omega) - 1}$$

with

$$\omega\beta = 2\ln\left(\frac{q}{2g\mu} + \sqrt{1 + \left(\frac{q}{2g\mu}\right)^2}\right)$$

- The reduced density matrix is thermal in quasi-particle basis

Ignorance entropy

- Take reduced density matrix; drop off-diag. elements; compute Von Neumann entropy

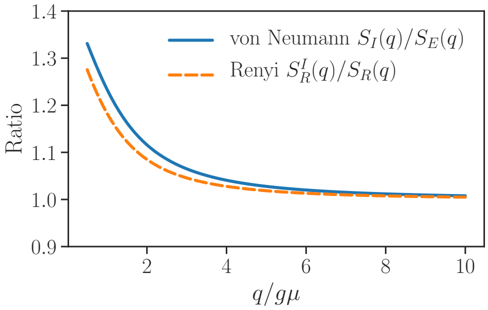
$$S_I = -\frac{1}{2}(N_c^2-1)S_\perp \int \frac{d^2q}{(2\pi)^2} \sum_{m,n} \left[(1-R)\frac{(m+n)!}{m!n!}\left(\frac{R}{2}\right)^{m+n}\right] \ln\left[(1-R)\frac{(m+n)!}{m!n!}\left(\frac{R}{2}\right)^{m+n}\right]$$

$$R=\left(1+\frac{q^2}{2g^2\mu^2}\right)^{-1}$$

- No analytical result;
  large q behaviour is known;
  small q cannot be computed analytically

- Numerical comparison of S_I to S_E

- Why large q behaviour is the same?!



UV behaviour

For large q

$$S_I(q) \simeq \frac{(N_c^2-1)g^2\mu^2 S_\perp}{q^2}\left[\ln\left(e\frac{q^2}{g^2\mu^2}\right) + \frac{g^2\mu^2}{q^2}\ln\frac{e}{2}\right]$$

vs

$$S_E(q) \simeq \frac{(N_c^2-1)g^2\mu^2 S_\perp}{q^2}\left[\ln\left(e\frac{q^2}{g^2\mu^2}\right) - \frac{g^2\mu^2}{q^2}\ln\left(e\frac{q^4}{g^4\mu^4}\right)\right]$$

Leading contribution originates from property of "vacuum state" $n = m = \alpha = \beta = 0$. Rather trivial effect: it does not probe the distribution of partons, but rather the probability that no partons are present.

Subleading contribution probes parton distribution and at this level is different for both entropies.

Fixed color charge configuration

- Study case: density matrix of soft modes at fixed configuration of valence charges.

- Entanglement entropy is trivial: the soft modes are in a pure state at fixed $\rho_a(\underline{q})$

$$\hat{\rho} = \mathcal{C}|0\rangle\langle 0|\mathcal{C}^\dagger$$

where $\mathcal{C}$ is unitary.

- Renyi entropy of ignorance

$$S_I = -\ln \mathrm{Tr}\hat{\rho}^2 = \frac{1}{2} S_\perp \int \frac{d^2q}{(2\pi)^2} \sum_a \left[4\frac{g^2}{q^2} \frac{\Delta^2}{(2\pi)^2} |\rho_a(q)|^2 - \ln I_0^2 \left(\frac{2g^2}{q^2} \frac{\Delta^2}{(2\pi)^2} |\rho_a(\underline{q})|^2 \right) \right]$$

- Typical configuration $\frac{\Delta^2}{(2\pi)^2}|\rho_a(q)|^2 \sim \mu^2$

$$S_I^{\mathrm{typ}} = -\ln \mathrm{Tr}\hat{\rho}^2 = \frac{1}{2}(N_c^2-1) S_\perp \int \frac{d^2q}{(2\pi)^2} \left[4\frac{g^2\mu^2}{q^2} - \ln I_0^2 \left(\frac{2g^2\mu^2}{q^2} \right) \right]$$

Conclusions

- Practical application of parton model assumes incoherent collection of partons with the diagonal density matrix in the number representation. Parton model is successful in describing a large number of observables.
- Does it mean that the actual physical partonic system is described by a diagonal density matrix?
- Is there a compelling argument? Instead: introduce a new form of entropy – ignorance entropy – which besides the reduction of the density matrix reflects our inability to perform a complete set of measurements. $S_I \geq S_E$
- In this talk, I considered a computable model based on CGC.
  It manifests the difference between entanglement and ignorance entropies.


BACKUP

Fixed color charge configuration

- At hight momentum the integrand behaves as

$$4\frac{g^2\mu^2}{q^2} - 2\left(\frac{g^2\mu^2}{q^2}\right)^2$$

- Compare this to ignorance entropy

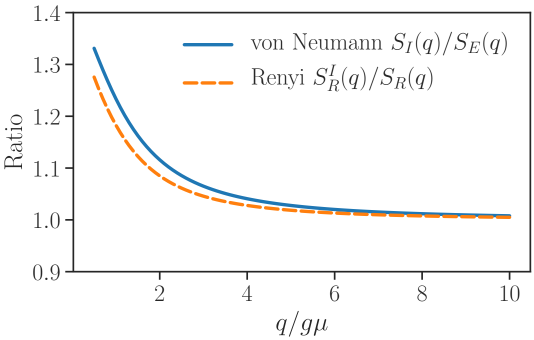
$$4\frac{g^2\mu^2}{q^2} - 6\left(\frac{g^2\mu^2}{q^2}\right)^2$$

of the reduced density matrix

- For a typical configuration of $\rho_a(\underline{q})$, the ignorance entropy is close to the ignorance entropy of reduced density matrix.

- On the other hand S_E crucially depends on reducing the density matrix – it vanishes for fixed configuration of $\rho_a(\underline{q})$, but is nonzero for $\hat{\rho}_r$.

UV behaviour



- Small difference in UV is solely due to small occupation numbers; at intermediate and low momenta, where the occupation numbers are of order unity, difference becomes significant.
- Expectation: similar feature in real parton model of QCD?!

Reduced density matrix

Finally, reduced density:

$$\hat{\rho}_r = \mathcal{N} \int D\rho \; e^{-\int_{\underline{k}} \frac{1}{2\mu^2} \rho_a(\underline{k}) \rho_a^*(\underline{k})} \mathcal{C}(\rho_b, \phi_b^i) |0\rangle \langle 0| \mathcal{C}^\dagger(\rho_c, \phi_c^j)$$

Coherent operator on soft gluon vacuum:

$$\mathcal{C}|0\rangle = e^{i\int_{\underline{k}} b_c^i(\underline{k})[a_c^{i\,+}(\underline{k})+a_c^i(-\underline{k})]}|0\rangle = e^{i\int_{\underline{k}} b_c^i(\underline{k})a_c^{i\,+}(\underline{k})} e^{-\frac{1}{2}\int_{\underline{k}}\frac{g^2}{k^2}|\rho_c(\underline{k})|^2}|0\rangle$$

$$\hat{\rho}_r = \mathcal{N}\int\prod_{\underline{k}}\prod_a d\rho_a(\underline{k})\ e^{-\frac{\Delta^2}{(2\pi)^2}\left(\frac{1}{2\mu^2}+\frac{g^2}{k^2}\right)\rho_a(\underline{k})\rho_a^*(\underline{k})} e^{ib_a^i(\underline{k})a_{ia}^\dagger(\underline{k})\frac{\Delta^2}{(2\pi)^2}}|0\rangle\langle 0|e^{-ib_a^{*i}(\underline{k})a_{ia}(\underline{k})\frac{\Delta^2}{(2\pi)^2}}$$

Thus density matrix element is

$$\langle n_c(\underline{q}), m_c(-\underline{q})|\hat{\rho}_r(\underline{q})|\alpha_c(\underline{q}), \beta_c(-\underline{q})\rangle = (1-R)\frac{(n+\beta)!}{\sqrt{n!m!\alpha!\beta!}}\left(\frac{R}{2}\right)^{n+\beta}\delta_{(n+\beta),(m+\alpha)}$$

where

$$R = \left(1+\frac{q^2}{2g^2\mu^2}\right)^{-1}$$

Reduced density matrix

$$\langle n_c(\underline{q}), m_c(-\underline{q})|\hat{\rho}_r(\underline{q})|\alpha_c(\underline{q}), \beta_c(-\underline{q})\rangle = (1-R)\frac{(n+\beta)!}{\sqrt{n!m!\alpha!\beta!}}\left(\frac{R}{2}\right)^{n+\beta}\delta_{(n+\beta),(m+\alpha)}$$

- Clearly includes off-diagonal elements
- As an example,

$$\langle n_c(\underline{q}), n_c(-\underline{q})|\hat{\rho}_r(\underline{q})|0,0\rangle = (1-R)\left(\frac{R}{2}\right)^{n}$$

Entanglement entropy

$$S_E = \frac{1}{2}(N_c^2-1)S_\perp \int \frac{d^2q}{(2\pi)^2}\left[\ln\left(\frac{g^2\mu^2}{q^2}\right) + \sqrt{1+4\frac{g^2\mu^2}{q^2}}\ln\left(1+\frac{q^2}{2g^2\mu^2}+\frac{q^2}{2g^2\mu^2}\sqrt{1+4\frac{g^2\mu^2}{q^2}}\right)\right]$$

- Proportional to $S_\perp g^2\mu^2 \sim \frac{1}{\alpha_s}S_\perp Q_s^2$
- In general, for arbitrary q, it is not obvious that it has the form
$$(n+1)\ln(n+1) - n\ln n = \ln(n+1) + n\ln(1+1/n)$$
- Consider small momentum

$$\approx 2\ln\left(\frac{g\mu}{q}+1\right) + 2\frac{g\mu}{q}\ln\left(1+\frac{q}{g\mu}\right)$$

or, identifying $n = \frac{g\mu}{q}$, one obtains $(N_c^2-1)S_\perp \int_q [\ln(n+1) + n\ln(1+1/n)]$
It makes sense, as $n = 1/[\exp(q/g\mu)-1] \approx \frac{g\mu}{q}$

Entanglement entropy

$$S_E = \frac{1}{2}(N_c^2 - 1)S_\perp \int \frac{d^2q}{(2\pi)^2}\left[\ln\left(\frac{g^2\mu^2}{q^2}\right) + \sqrt{1 + 4\frac{g^2\mu^2}{q^2}}\ln\left(1 + \frac{q^2}{2g^2\mu^2} + \frac{q^2}{2g^2\mu^2}\sqrt{1 + 4\frac{g^2\mu^2}{q^2}}\right)\right]$$

- In general, for arbitrary q, it is not obvious that it has the form

$$(n+1)\ln(n+1) - n\ln n$$

- Consider large momentum

$$S_E(q) \simeq -(N_c^2 - 1)S_\perp \frac{g^2\mu^2}{q^2}\ln\left(\frac{g^2\mu^2}{q^2}\right)$$

or, identifying $n = \frac{g^2\mu^2}{q^2}$, one obtains $-(N_c^2 - 1)S_\perp \int_q n \ln n$