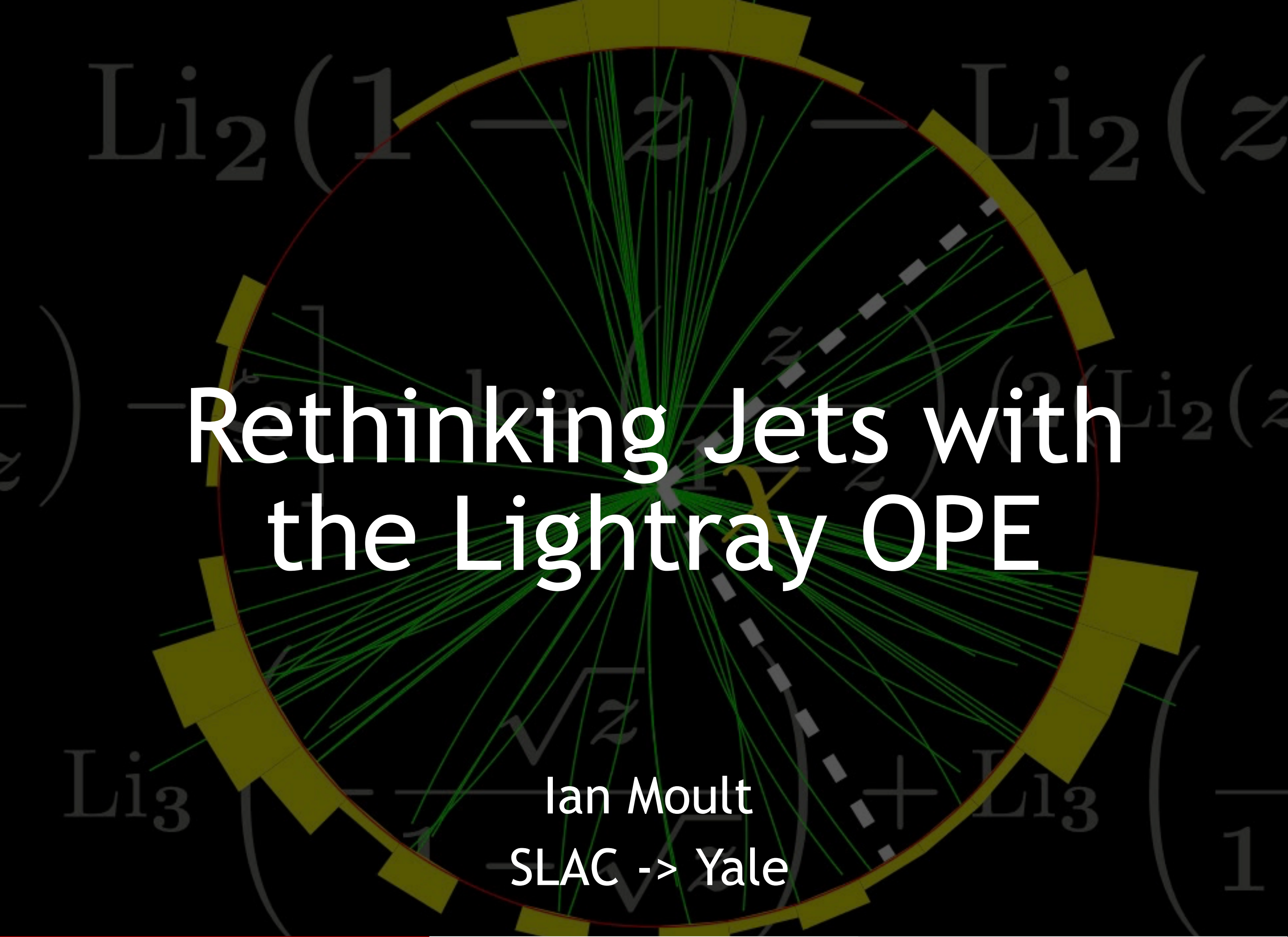


Rethinking Jets with the Lightray OPE

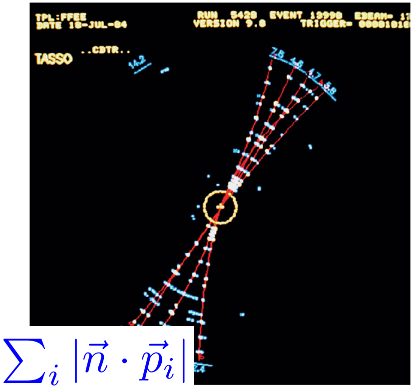
Ian Moult

SLAC -> Yale

Jet Shapes

- The classic approach to studying jets is to use "jet shapes", which measure the spread of radiation.
- Dates back to the introduction of "thrust" by Farhi in 1977.

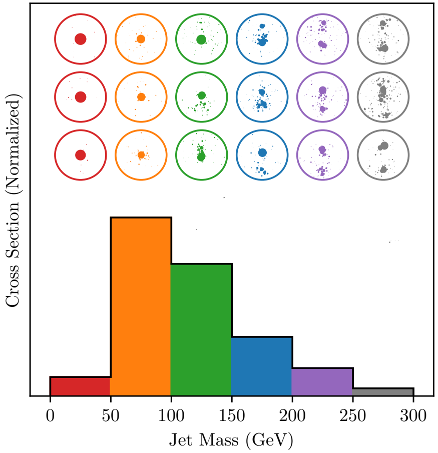
LEP

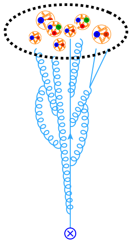


$$T = \max_{\vec{n}} \frac{\sum_i |\vec{n} \cdot \vec{p}_i|}{Q}$$

LHC

(Metodiev, Komiske, Thaler)



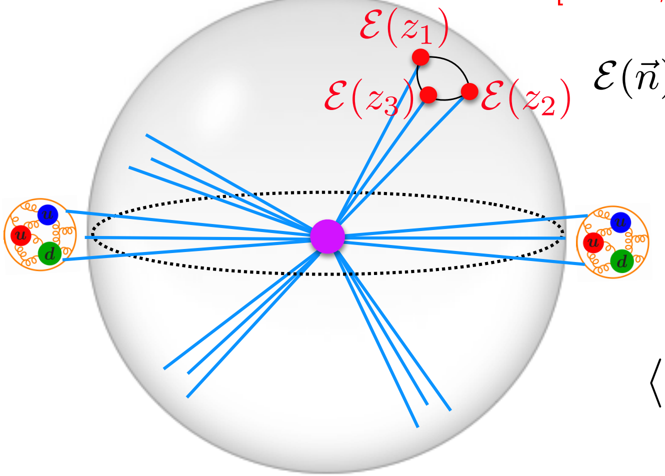


- Despite the intuitive nature of "jet shapes", their lack of direct connection to the underlying field theory has inhibited connections to other areas in field theory (amplitudes, CFT correlators, ...)

Energy Flow Operators

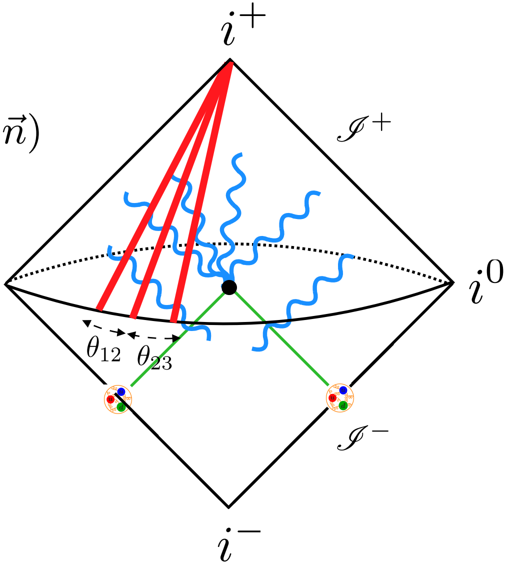
- Goal of this talk is to rethink jet substructure at the LHC in terms of correlation functions of operators living on the celestial sphere.

[Sveshnikov, Tkachov; Korchemsky, Sterman]
[Hofman, Maldacena]



$$\mathcal{E}(\vec{n}) = \int_0^\infty dt \lim_{r\to\infty} r^2 n^i T_{0i}(t, r\vec{n})$$

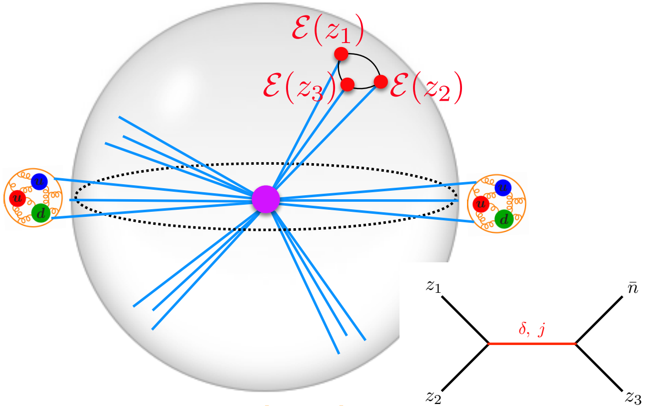
$$\langle\Psi|\mathcal{E}(\hat{n}_1)\cdots\mathcal{E}(\hat{n}_k)|\Psi\rangle$$



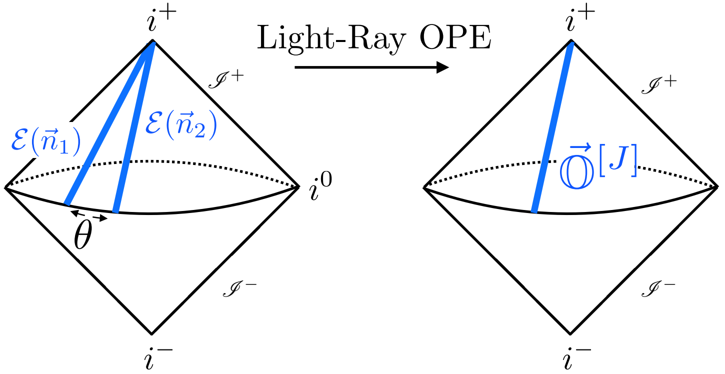
- Rephrasing jet substructure in this manner
  1. Gives a new perspective on how to study QCD at colliders in a way that is more closely connected to field theory.
  2. Allows the use of new theoretical tools developed in other areas of field theory for QCD calculations.

The Lightray OPE

- Situations of interest at the LHC involve highly non-generic configurations of lightray ops: The small angle or OPE limit.



$$g(z,\bar{z}) = \sum_{\delta,j} \overset{\text{dynamics}}{\boxed{c(\delta,j)}}\ \underset{\text{symmetries}}{\boxed{g_{\delta,j}(z,\bar{z})}}$$



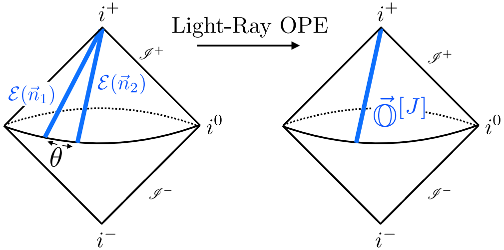
$$\mathcal{E}(\hat{n}_1)\mathcal{E}(\hat{n}_2) \sim \sum \theta^{\tau_i - 4} \mathbb{O}_i(\hat{n}_1)$$

[Hofman, Maldacena]
[Chang, Kologlu, Kravchuk, Simmons Duffin, Zhiboedov]

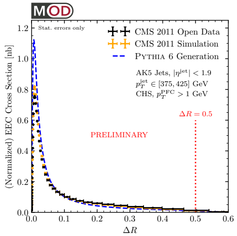
- Reformulates jet substructure as the study of the symmetry and OPE structure of these operators.

Outline

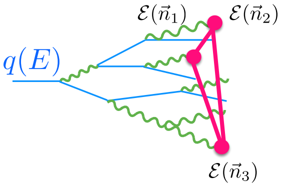
- The Leading Twist QCD Lightray OPE



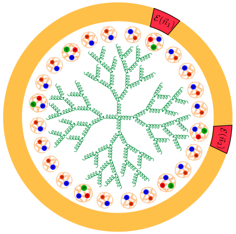
- Energy Flow Operators with LHC Data



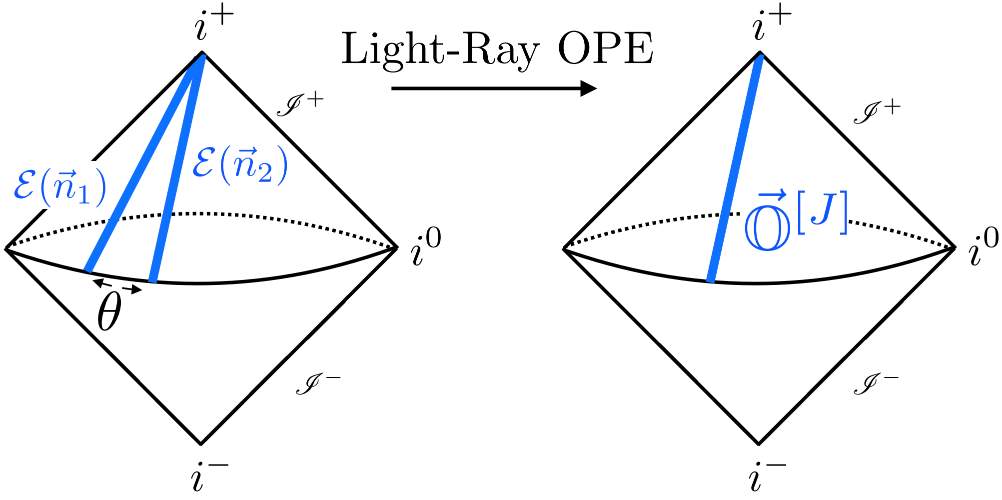
- The Three Point Correlator and Celestial Blocks



- Practical Offshoots
    - Generalized Fragmentation Functions and Calculations for Tracks
    - Spin Correlations in Jets



The Leading Twist QCD Lightray OPE

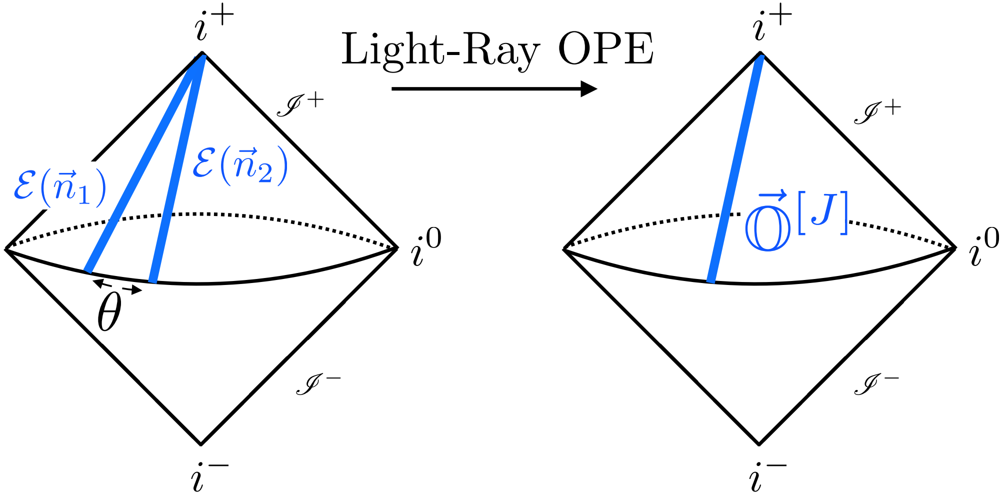


Chen, IM, Zhu

The Lightray OPE

- In CFTs, the lightray OPE is a convergent, and rigorous expansion.
  [Hofman, Maldacena; Chang, Kologlu, Kravchuk, Simmons Duffin, Zhiboedov]

$$\mathcal{E}(\hat{n}_1)\mathcal{E}(\hat{n}_2) = \sum c_i \theta^{\tau_i - 4} \mathbb{O}_i(\hat{n}_1)$$



- To describe the leading scaling at the LHC, we can restrict to the leading term in the OPE $\implies$ twist-2 light ray operators.

The Leading Twist Lightray OPE

[Hofman, Maldacena]

- The twist-2 operators in QCD are characterized by a spin-J and a transverse spin $j = 0, 2$.
- These can be light-transformed to obtain a vector of twist-2 lightray operators parametrized by spin-J:

[Kravchuk, Simmons Duffin]

Local Operators

transverse spin-0

$$\mathcal{O}_q^{[J]} = \frac{1}{2^J}\bar{\psi}\gamma^+(iD^+)^{J-1}\psi$$

$$\mathcal{O}_g^{[J]} = -\frac{1}{2^J}F_a^{\mu+}(iD^+)^{J-2}F_a^{\mu+}$$

transverse spin-2

$$\mathcal{O}_{\tilde{g}\lambda}^{[J]} = -\frac{1}{2^J}F_a^{\mu+}(iD^+)^{J-2}F_a^{\nu+}\epsilon_{\lambda,\mu}\epsilon_{\lambda,\nu}$$

helicity ±

$$\xrightarrow{\lim_{r\to\infty} r^2\int_0^\infty dt} \quad \vec{\mathbb{O}}^{[J]}(\vec{n}) = \begin{bmatrix} \mathbb{O}_q^{[J]}(\vec{n}) \\ \mathbb{O}_g^{[J]}(\vec{n}) \\ \mathbb{O}_{\tilde{g},+}^{[J]}(\vec{n}) \\ \mathbb{O}_{\tilde{g},-}^{[J]}(\vec{n}) \end{bmatrix}$$

unpolarized

polarized

- The anomalous dimensions of these operators,

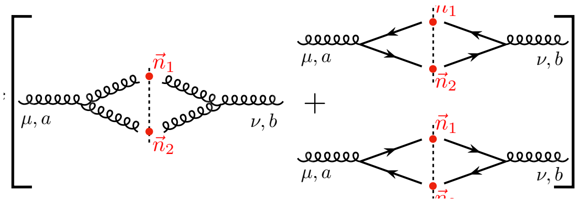
$$\frac{d}{d\ln\mu^2}\vec{\mathbb{O}}^{[J]}(\hat{n}_1) = \hat{\gamma}(J)\vec{\mathbb{O}}^{[J]}(\hat{n}_1)$$

determines the leading behavior of jet substructure at the LHC.

Explicit Structure

[Chen, IM, Zhu]

- The explicit structure of the perturbative OPE can be derived by matching calculations



- The OPE coefficients can be expressed in terms of a matrix, $\hat{C}_\phi(J)$, whose entries are analytic functions of J:

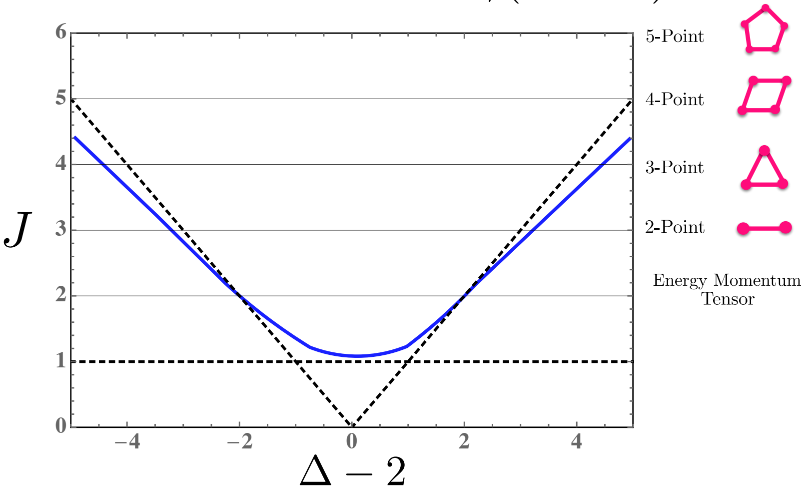
$$\mathcal{E}(\hat{n}_1)\mathcal{E}(\hat{n}_2) = -\frac{1}{2\pi}\frac{2}{\theta^2}\vec{\mathcal{J}}\left[\widehat{C}_\phi(2) - \widehat{C}_\phi(3)\right]\vec{\mathbb{O}}^{[3]}(\hat{n}_1) + \cdots ,$$

$$\vec{\mathbb{O}}^{[J]}(\hat{n}_1)\mathcal{E}(\hat{n}_2) = -\frac{1}{2\pi}\frac{2}{\theta^2}\left[\widehat{C}_\phi(J) - \widehat{C}_\phi(J+1)\right]\vec{\mathbb{O}}^{[J+1]}(\hat{n}_1) + \cdots$$

- Iterated application of these OPEs allows derivation of leading twist scaling behavior for arbitrary correlators.

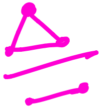
Unpolarized Scaling: Schematic

- Since the LHC has unpolarized beams, overall scaling of correlators probes transverse spin $j = 0$ operators.
- An N-point correlator's leading scaling is determined by the twist-2 spin-$N+1$ anomalous dimension $\gamma(N+1)$

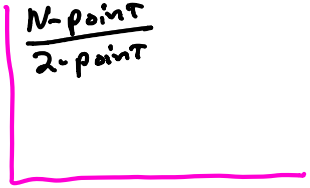


$\gamma(N+1) > \gamma(N)$

- To isolate anomalous scaling it is convenient to take the ratio to the two-point correlator:

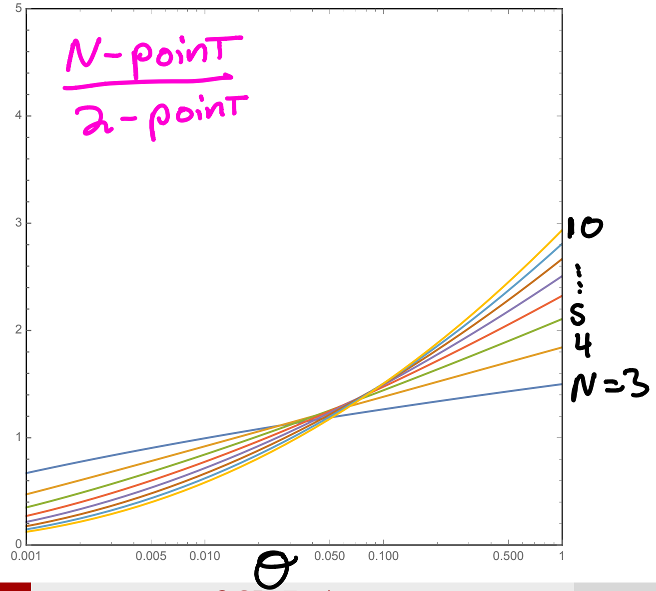


$\sim \theta^{\gamma(4)-\gamma(3)}$

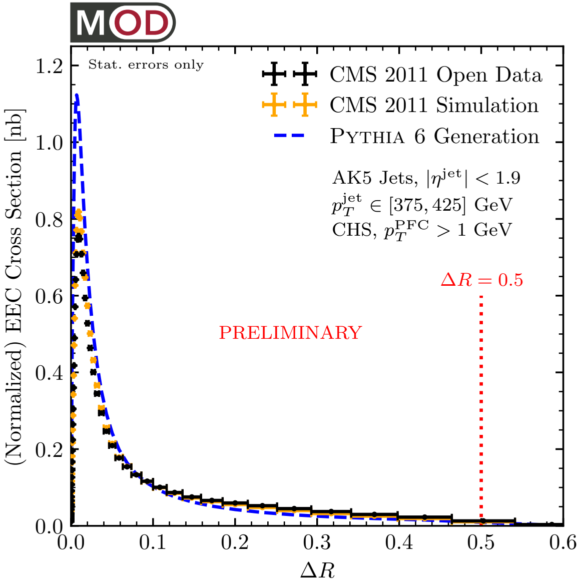


Unpolarized Scaling: Reality

- The previous picture is a little schematic. In real world QCD:
  - $\widehat{\gamma}(J) = \begin{pmatrix} \gamma_{qq}(J) & \gamma_{qg}(J) \\ \gamma_{gq}(J) & \gamma_{gg}(J) \end{pmatrix}$ is a matrix.
  - Non-vanishing β function: $\theta^{\gamma} \to \exp\left(\frac{\widehat{\gamma}(J)}{2\beta_0} \ln \frac{\alpha_s(\theta Q)}{\alpha_s(Q)}\right) + \cdots$
- These are quantitatively important, particularly at small J where $\gamma(J) \sim \beta$, but don't qualitatively change the picture for these ratios:



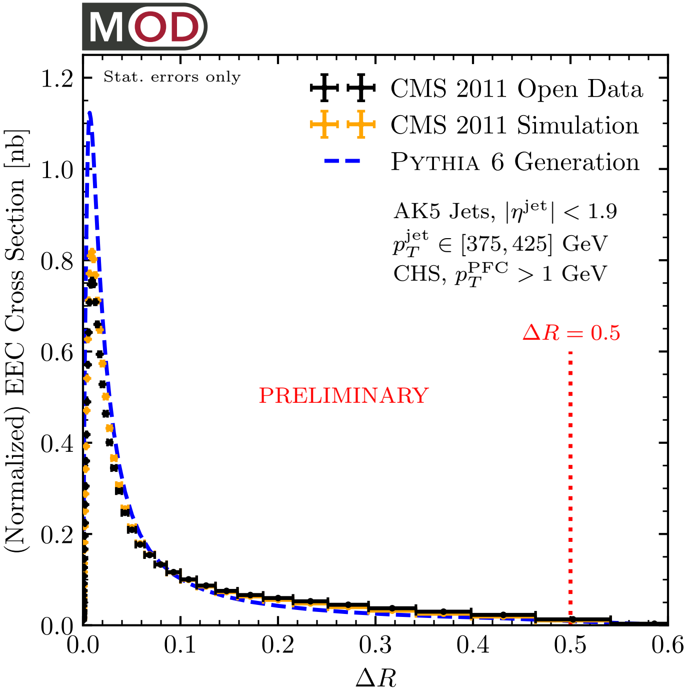
Energy Flow Operators with LHC Data

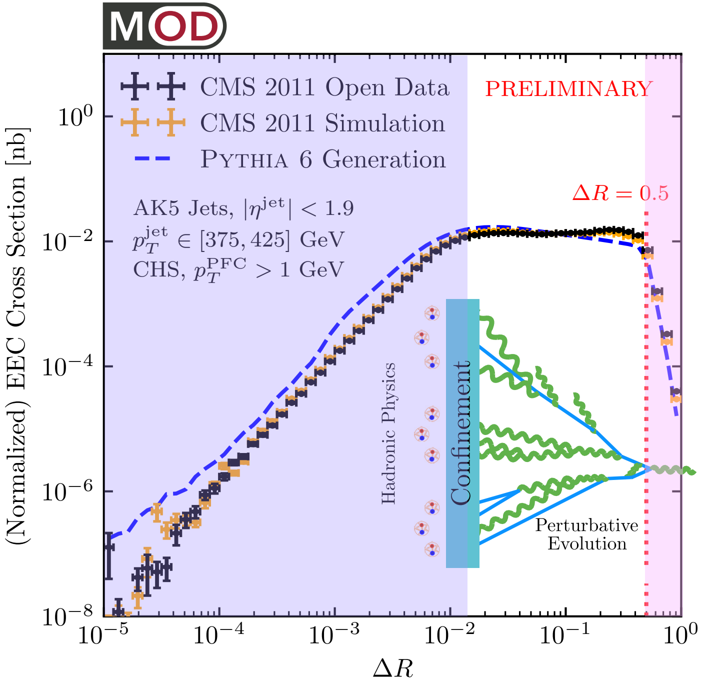


Komiske, Moult, Thaler, Zhu (forthcoming)

Two-point Correlator with Data

- At LHC energies, there is a wide perturbative regime where we can study anomalous scalings in detail.

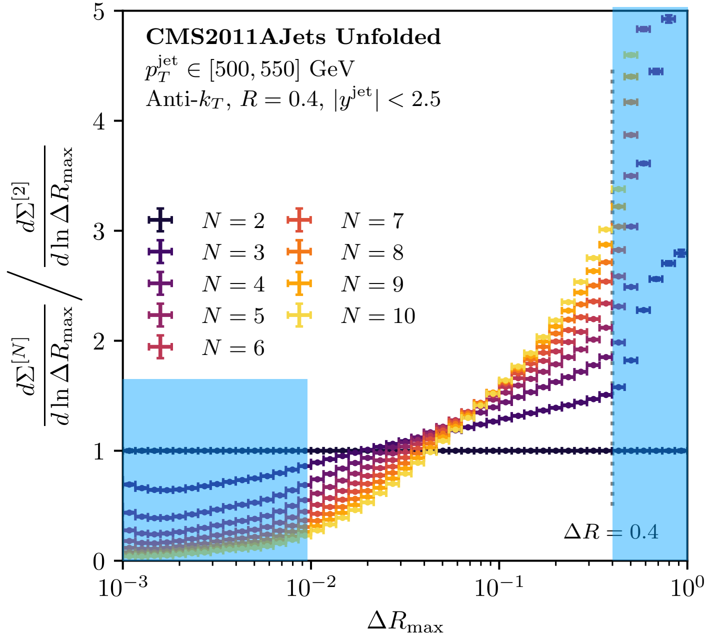


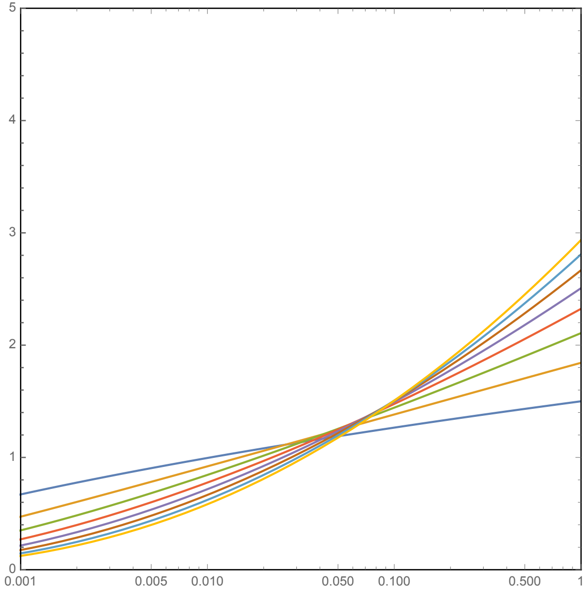


- Scaling in non-perturbative regime corresponds to that of uniformly distributed particles.

Anomalous Scaling in Data

- Anomalous scaling of energy correlators up to 10 points:

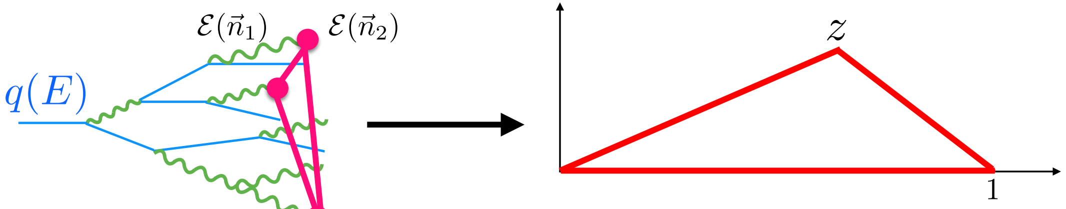




- Lorentzian scaling of lightray operators at the quantum level!!
- Basic features, e.g. convexity of anomalous dimensions clearly seen.
- Proper treatment of exp/theory uncertainties, etc. ongoing.

The Three-Point Energy Correlator

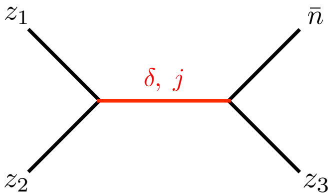
- The first correlator with non-trivial shape dependence in the collinear limit is the three-point correlator.
- Depends on a scaling variable x, and cross ratio variables $(z, \bar{z})$ or (u, v).



$$G_i(z, \bar{z})$$

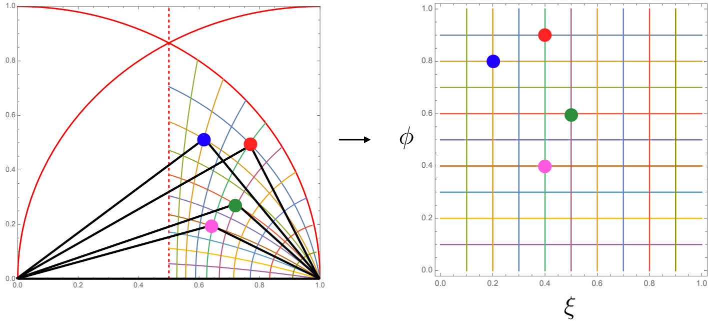
- As I will explain later, $G(z, \bar{z})$ is a true correlation function living on the celestial sphere. e.g. it admits a decomposition into blocks

$$g(z, \bar{z}) = \sum_{\delta, j} \overset{\text{dynamics}}{\boxed{c(\delta, j)}} \underset{\text{symmetries}}{\boxed{g_{\delta, j}(z, \bar{z})}}$$

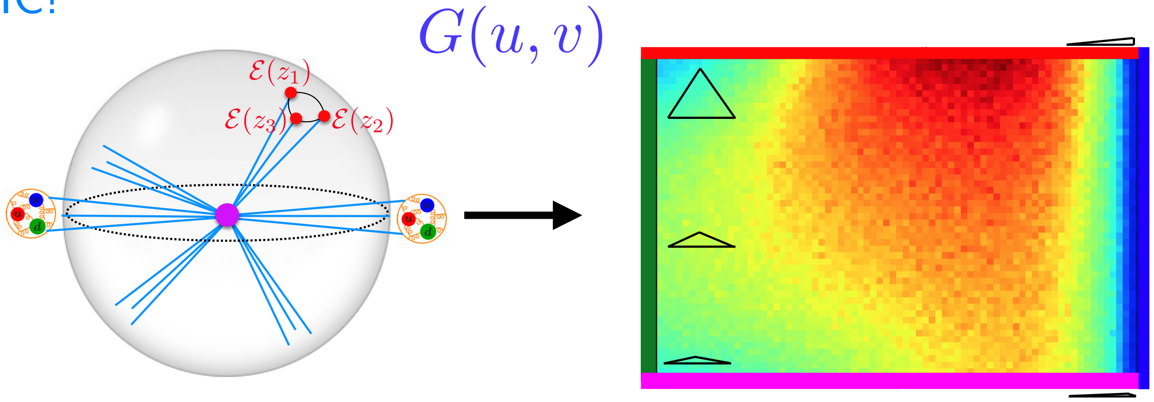


Shape Dependence in Data

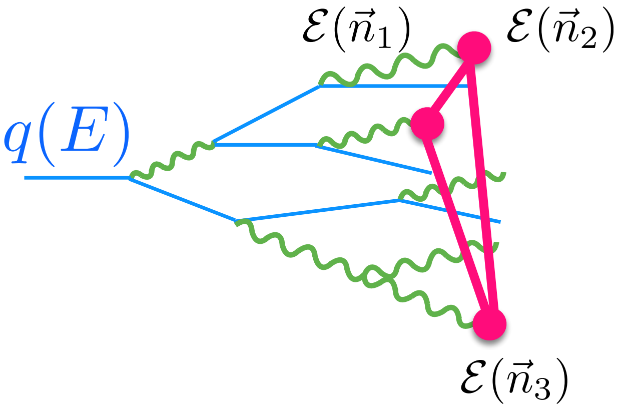
- Standard parametrization is inconvenient for experimental binning: We can map to a square grid by “blowing up” the OPE region.



- Can directly measure correlation functions of lightray operators at the LHC!



The Three Point Correlator and Celestial Blocks



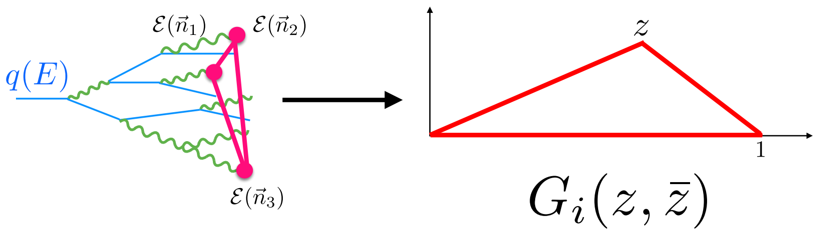
Chen, IM, Zhu
Chen, IM, Sandor, Zhu

Multipoint Correlators

- Only explicit results for correlators with $N > 2$ are the strong coupling expansion of Hofman and Maldacena:

$$\langle \mathcal{E}(\vec{n}_1)\cdots\mathcal{E}(\vec{n}_n)\rangle = \left(\frac{q}{4\pi}\right)^n \Bigg[\boxed{1} + \sum_{i<j}\frac{6\pi^2}{\lambda}[(\vec{n}_i.\vec{n}_j)^2 - \frac{1}{3}] + \\ + \frac{\beta}{\lambda^{3/2}}[\sum_{i<j<k}(\vec{n}_i.\vec{n}_j)(\vec{n}_j.\vec{n}_k)(\vec{n}_i.\vec{n}_k) + \cdots] + o(\lambda^{-2})\Bigg]$$

- Qualitatively very different from what is expected at weak coupling.
- Due to the phenomenological motivation, will focus on collinear limit. We can start with the three-point correlator:



Result in $\mathcal{N} = 4$ Super Yang-Mills

- The three-point correlator turns out to be remarkably simple.
- e.g. for $\mathcal{N} = 4$ SYM, it is given by

$$G^{\mathcal{N}=4}_{\mathcal{EEE}}(z) = \frac{1+u+v}{2uv} - \frac{1+v}{2uv}\log(u) - \frac{1+u}{2uv}\log(v)$$
$$-(1+u+v)(\partial_u + \partial_v)\Phi(z) + \frac{(1+u^2+v^2)}{2uv}\Phi(z) + \frac{(z-\bar{z})^2(u+v+u^2+v^2+u^2v+uv^2)}{4u^2v^2}\Phi(z)$$
$$+\frac{(u-1)(u+1)}{2uv^2}D_2^+(z) + \frac{(v-1)(v+1)}{2u^2v}D_2^+(1-z) + \frac{(u-v)(u+v)}{2uv}D_2^+\left(\frac{z}{z-1}\right) + \frac{1+u+v}{2uv}\zeta_2$$

- Here Φ and D_2^+ are polylogarithmic functions

$$\Phi(z) = \frac{1}{z-\bar{z}}\left(2\text{Li}_2(z) - 2\text{Li}_2(\bar{z}) + (\log(1-z) - \log(1-\bar{z}))\log(z\bar{z})\right),$$
$$D_2^+(z) = \text{Li}_2\left(1-|z|^2\right) + \frac{1}{2}\log\left(|1-z|^2\right)\log\left(|z|^2\right)$$

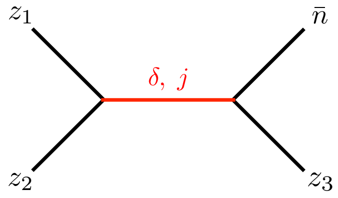
- Real world QCD involves up to $(\partial_u + \partial_v)^5\Phi$, but is otherwise similar.

Celestial Blocks

[Chen, IM, Sandor, Zhu]

- This collinear limit of interest for jet substructure turns out to be a simple limit for studying the structure of the lightray OPE.
- Lightray operators behave like primaries on celestial sphere with $(\delta = \Delta - 1, j)$.
- For the two point correlator Kologlu, Kravchuk, Simmons Duffin and Zhiboedov showed how to organize the lightray OPE into "celestial blocks" associated with the action of the Lorentz symmetry on the celestial sphere.
- We can derive similar blocks for the three-point function in the collinear limit.

$$g(z,\bar{z}) = \sum_{\delta,j} \overset{\text{dynamics}}{\boxed{c(\delta,j)}}\,\underset{\text{symmetries}}{\boxed{g_{\delta,j}(z,\bar{z})}}$$



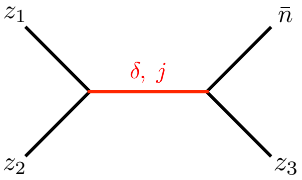
- Due to the simplified limit, the celestial blocks are precisely 2D conformal blocks!

Casimir Differential Equations

- Celestial blocks derived by solving Casimir differential equations

$$G(z_1, z_2, z_3, \bar{n}) = \frac{1}{(z_1 \cdot z_2)^3} \frac{1}{(z_3 \cdot \bar{n})^4} \left(\frac{z_1 \cdot z_3}{z_1 \cdot \bar{n}} \right) g(z, \bar{z})$$

Symmetry: **Lorentz Group**



$$\boxed{g(z,\bar{z}) = \sum_{\delta,j} c_{\delta,j}\, g_{\delta,j}(z,\bar{z})}$$

Representation labels:

δ **celestial dimension** ------ $\vec{n} \cdot \vec{\mathbf{K}}$

j **transverse spin** ------ $\vec{n} \cdot \vec{\mathbf{J}}$

Quadratic Casimir: $\boxed{\frac{1}{2} M_{\mu\nu} M^{\mu\nu}} \xrightarrow{\text{eigenvalue}} -(\delta(\delta-2)+j^2)$

Casimir Equation: acting Casimir operator on $z_1,\ z_2$

$$\mathcal{L}^{\mu\nu}(z_1, z_2)\mathcal{L}_{\mu\nu}(z_1, z_2)\boxed{G_{\delta,j}} = -(\delta(\delta-2)+j^2)\boxed{G_{\delta,j}}$$

[Dolan, Osborn, 2003]

$$\mathcal{L}^{\mu\nu}(z_1, z_2) \equiv \sum_{i=1,2} \left(z_i^\mu \frac{\partial}{\partial z_{i\nu}} - z_i^\nu \frac{\partial}{\partial z_{i\mu}} \right)$$

Solutions (2D Conformal Blocks):

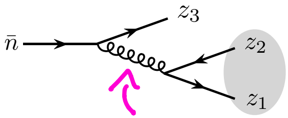
$$g_{\delta,j}(z,\bar{z}) = \frac{1}{1+\delta_{j,0}} \left(k_{\delta-j}(z) k_{\delta+j}(\bar{z}) + k_{\delta+j}(z) k_{\delta-j}(\bar{z}) \right)$$

[Notations] In our case, $a=0,\, b=-1$

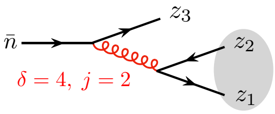
$$k_\beta(x) \equiv x^{\beta/2}\ {}_2F_1\left(\frac{\beta}{2}+a, \frac{\beta}{2}+b, \beta, x\right)$$

Celestial Blocks

- Interesting interplay with structure of Feynman diagrams.
- e.g. In QCD can choose color structures to isolate internal parton states.



- Only $j = 0, 2$ contribute. Leading twist $j = 2$ block uncontaminated by higher twist contributions.



Contributing Operator

$F_a^{\mu+}(iD^+)F_a^{\nu+}\epsilon_{\lambda,\mu}\epsilon_{\lambda,\nu}$

twist-2, transverse spin-2 gluonic operator

highest transverse spin series

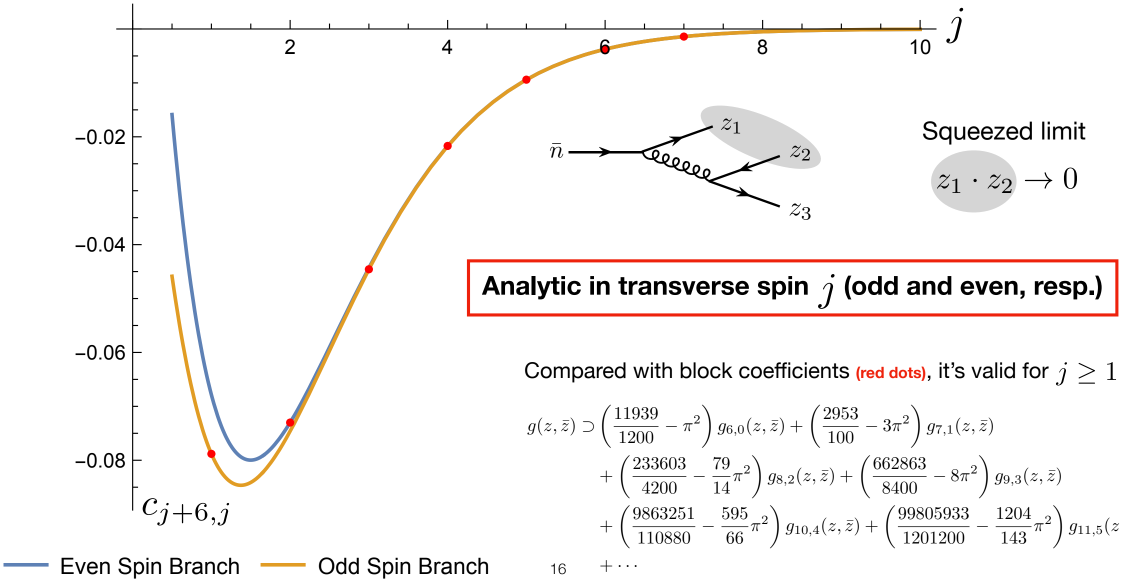
$-z^3\bar{z}\ {}_2F_1\left(3,2,6,z\right)$

$$\begin{array}{llllll}
\underset{\cos 2\phi}{-z^3\bar{z}} & +\frac{39}{10}z^2\bar{z}^2 & -z\bar{z}^3 & & & \\
\underset{\cos 3\phi}{-z^4\bar{z}} & +\frac{39}{20}z^3\bar{z}^2 & +\frac{39}{20}z^2\bar{z}^3 & -z\bar{z}^4 & & \\
\underset{\cos 4\phi}{-\frac{6}{7}z^5\bar{z}} & +\frac{229}{140}z^4\bar{z}^2 & -\frac{211}{140}z^3\bar{z}^3 & +\frac{229}{140}z^2\bar{z}^4 & -\frac{6}{7}z\bar{z}^5 & \\
\underset{\cos 5\phi}{-\frac{5}{7}z^6\bar{z}} & +\frac{207}{140}z^5\bar{z}^2 & -\frac{233}{140}z^4\bar{z}^3 & -\frac{233}{140}z^3\bar{z}^4 & +\frac{207}{140}z^2\bar{z}^5 & -\frac{5}{7}z\bar{z}^6 \\
\ldots & & \ldots & & & \ldots
\end{array}$$

- Full tower of higher twist corrections uniquely fixed by symmetry!
- Highly non-trivial interplay with structure of transcendental functions.

Analyticity in Transverse Spin

- In crossed channels, the complete spectrum of contributing operators can be extracted using Lorentzian Inversion. [Caron-Huot] [Simmons Duffin, Stanford, Witten]
- OPE data are analytic functions of the transverse spin j.



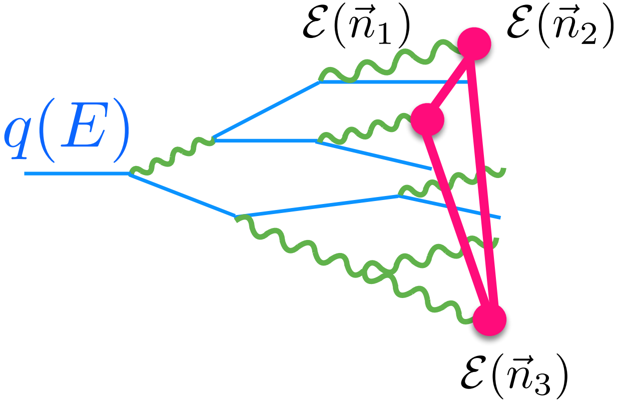
Analytic in transverse spin j (odd and even, resp.)

Compared with block coefficients **(red dots)**, it's valid for $j \geq 1$

$$
\begin{aligned}
g(z,\bar{z}) \supset & \left(\frac{11939}{1200} - \pi^2\right) g_{6,0}(z,\bar{z}) + \left(\frac{2953}{100} - 3\pi^2\right) g_{7,1}(z,\bar{z}) \\
& + \left(\frac{233603}{4200} - \frac{79}{14}\pi^2\right) g_{8,2}(z,\bar{z}) + \left(\frac{662863}{8400} - 8\pi^2\right) g_{9,3}(z,\bar{z}) \\
& + \left(\frac{9863251}{110880} - \frac{595}{66}\pi^2\right) g_{10,4}(z,\bar{z}) + \left(\frac{99805933}{1201200} - \frac{1204}{143}\pi^2\right) g_{11,5}(z,\bar{z}) \\
& + \cdots
\end{aligned}
$$

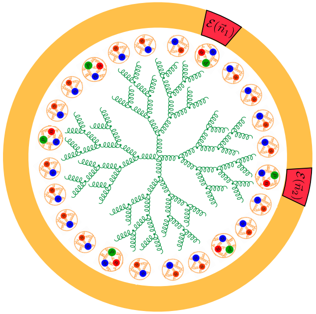
- New tools for understanding the higher twist collinear limit.

Practical Offshoots



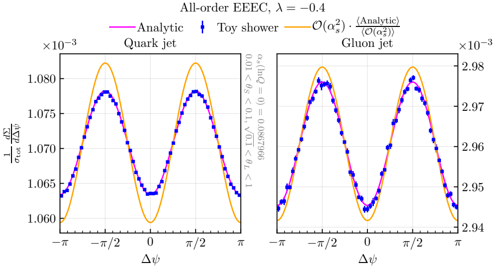
Practical Offshoots

- Although the motivation for studying these observables was to make connections to developments in CFT, they have provided insight into some (perhaps surprising) “practical” issues for LHC phenomenology

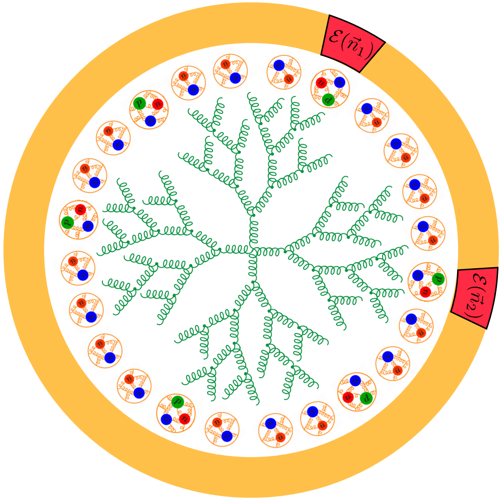


  - Generalized Fragmentation Functions

  - Perturbative Spin Correlations in Jets



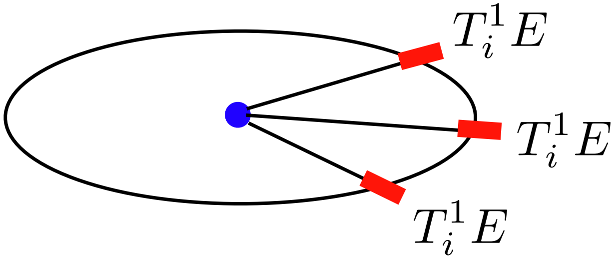
Extending Precision Perturbative QCD with Generalized Fragmentation Functions



Li, IM, Schrijnder van Velzen, Waalewijn, Zhu (forthcoming)

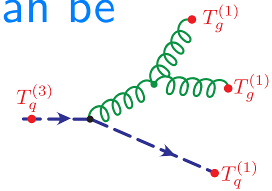
Tracks and Energy Correlators

- Energy correlators are weighted by energy flow through detector cells as a function of angle.
- How to go from full calorimeter to tracks? simply multiply by first moment of a track function!



$$E_i \to \int dx_i \, x_i T_i(x_i) E_i = T_i^{(1)} E_i$$

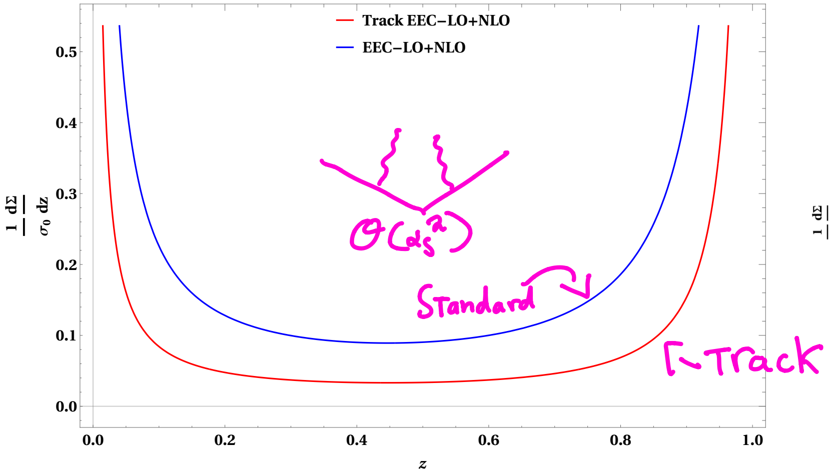
- Upshot: Any analytic calculation of energy correlators can be upgraded to tracks!
- Exhibit interesting evolution. See Wouter's talk!



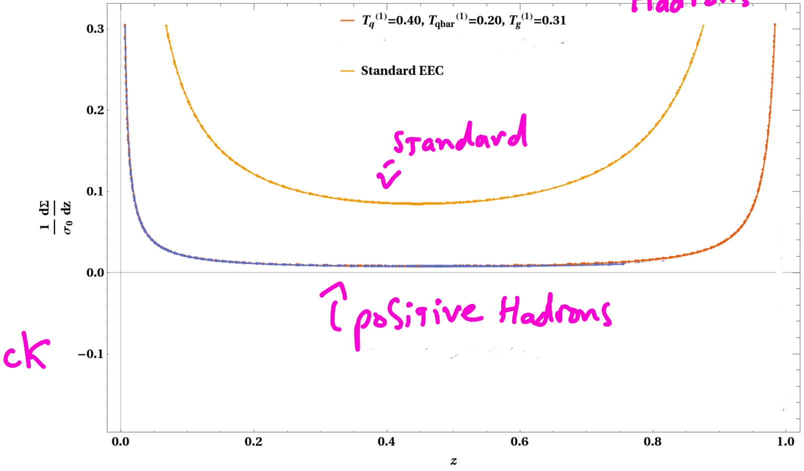
Energy Correlators on Tracks at $\mathcal{O}(\alpha_s^2)$

- Full angle two-point correlator at $\mathcal{O}(\alpha_s^2)$ on generic hadronic states, R. (Note that this matches the state of the art for analytic event shape calculations period, but now can do on tracks, neutrals, whatevers!).

Track EEC

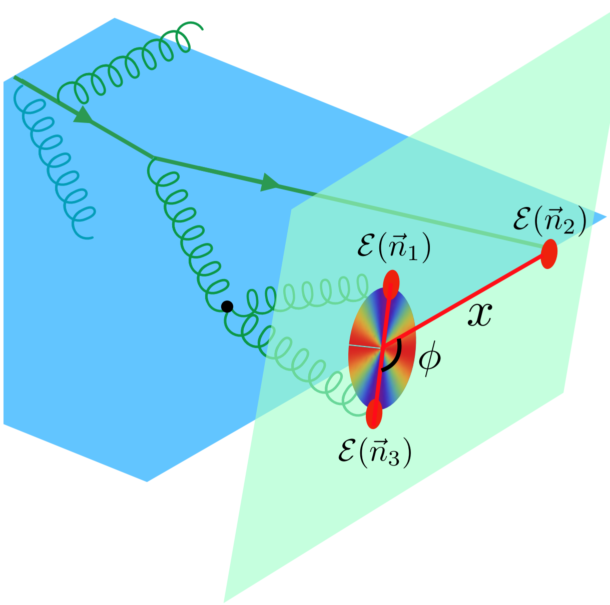


EEC on Positively Charged Hadrons



- Many phenomenological advantages! See Wouter's talk.

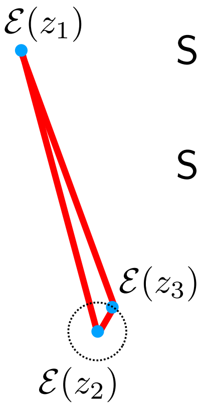
Perturbative Spin Correlations in Jets



Chen, IM, Zhu

Transverse Spin $j = 2$

- With unpolarized beams, the easiest way to see transverse spin $j = 2$ operators is in squeezed limits of higher point correlators.
- As an explicit numerical example, consider the OPE limit of the three-point correlator:



$$\mathsf{Sq}_q^{(0)} = C_F n_f \left(\frac{39/2 - 10\cos(2\phi)}{225}\right) + C_F C_A \left(\frac{273 + 10\cos(2\phi)}{225}\right) + C_F^2 \frac{16}{5}\,,$$

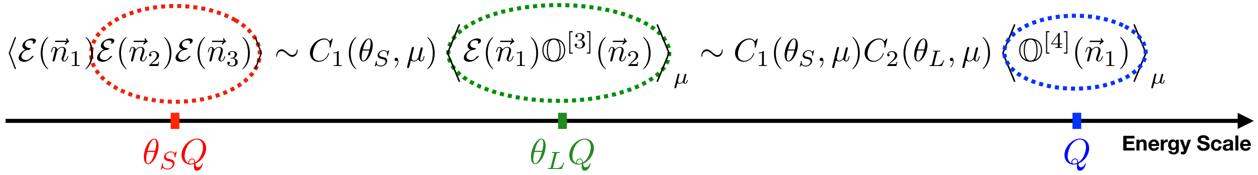
$$\mathsf{Sq}_g^{(0)} = C_A n_f \left(\frac{63 - 10\cos(2\phi)}{225}\right) + C_A^2 \left(\frac{882 + 10\cos(2\phi)}{225}\right) + C_F n_f \frac{3}{10}$$

$$\left(\frac{\alpha_s}{4\pi}\right)^2 \frac{\mathsf{Sq}^{(0)}}{|\omega|^2} + \cdots\,,$$

- Leading twist $j = 2$ operators manifest as $\cos(2\phi)$ interference patterns in the detector.

Resummation

- Resummation including transverse spin can be performed using lightray OPE: Perform iterative OPE and then RG evolve the



$$\langle \mathcal{E}(\vec{n}_1)\mathcal{E}(\vec{n}_2)\mathcal{E}(\vec{n}_3)\rangle \sim C_1(\theta_S,\mu)\left\langle \mathcal{E}(\vec{n}_1)\mathbb{O}^{[3]}(\vec{n}_2)\right\rangle_\mu \sim C_1(\theta_S,\mu)C_2(\theta_L,\mu)\left\langle \mathbb{O}^{[4]}(\vec{n}_1)\right\rangle_\mu$$

transverse spin-0

$$\mathcal{O}_q^{[J]} = \frac{1}{2^J}\bar{\psi}\gamma^+(iD^+)^{J-1}\psi$$

$$\mathcal{O}_g^{[J]} = -\frac{1}{2^J}F_a^{\mu+}(iD^+)^{J-2}F_a^{\mu+}$$

transverse spin-2

$$\mathcal{O}_{\tilde{g},ij}^{[J]} = -\frac{1}{2^J}F_a^{(i+}(iD^+)^{J-2}F_a^{j)+}$$

$$\left[\mathcal{O}_{\tilde{g},\lambda}^{[J]} = -\frac{1}{2^J}F_a^{\mu+}(iD^+)^{J-2}F_a^{\nu+}\epsilon_{\lambda,\mu}\epsilon_{\lambda,\nu}\right]$$

CANNOT MIX

RG equation:

$$\frac{d}{d\ln\mu^2}\vec{\mathcal{O}}^{[J]} = -\widehat{\gamma}(J)\cdot\vec{\mathcal{O}}^{[J]}$$

$$\widehat{\gamma}(J) = \begin{pmatrix} \gamma_{qq}(J) & 2n_f\gamma_{qg}(J) & 0 \\ \gamma_{gq}(J) & \gamma_{gg}(J) & 0 \\ 0 & 0 & \gamma_{\tilde{g}\tilde{g}}(J)\mathbf{1} \end{pmatrix}$$

$$\mathcal{E}(\hat{n}_1)\mathcal{E}(\hat{n}_2)\mathcal{E}(\hat{n}_3) = \frac{1}{(2\pi)^2}\frac{2}{\theta_S^2}\frac{2}{\theta_L^2}\vec{\mathcal{J}}\left[\widehat{C}_{\phi_S}(2)-\widehat{C}_{\phi_S}(3)\right]\left[\frac{\alpha_s(\theta_L Q)}{\alpha_s(\theta_S Q)}\right]^{\frac{\widehat{\gamma}^{(0)}(3)}{\beta_0}}$$

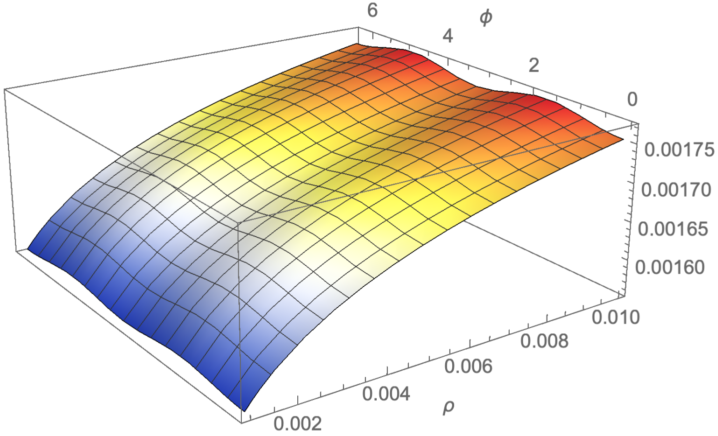
$$\cdot\left[\widehat{C}_{\phi_L}(3)-\widehat{C}_{\phi_L}(4)\right]\left[\frac{\alpha_s(Q)}{\alpha_s(\theta_L Q)}\right]^{\frac{\widehat{\gamma}^{(0)}(4)}{\beta_0}}\vec{\mathbb{O}}^{[4]}(\hat{n}_1)+\cdots$$

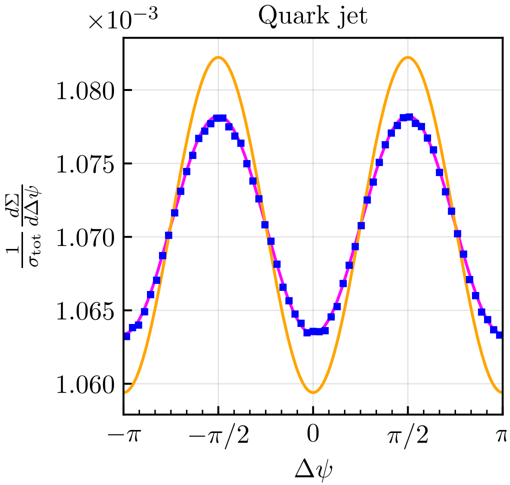
Resummation

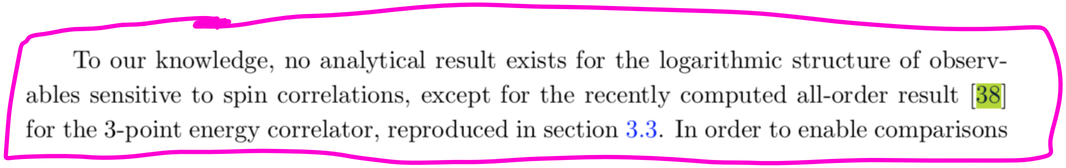
- Allows for validation of recent implementations of spin correlations in parton showers.
- Nice interplay between analytic calculations and parton shower development.

[Karlberg, Salam, Scyboz, Verheyen]

[Chen, IM, Zhu]





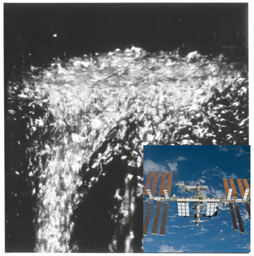


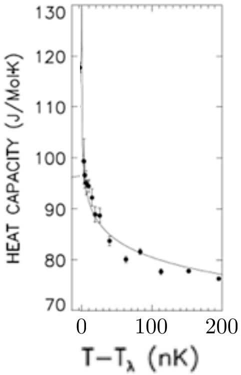
To our knowledge, no analytical result exists for the logarithmic structure of observables sensitive to spin correlations, except for the recently computed all-order result [38] for the 3-point energy correlator, reproduced in section 3.3. In order to enable comparisons

Conclusions

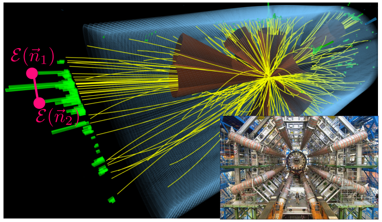
- Jet Substructure is the study of the lightray OPE.

Local OPE

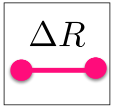


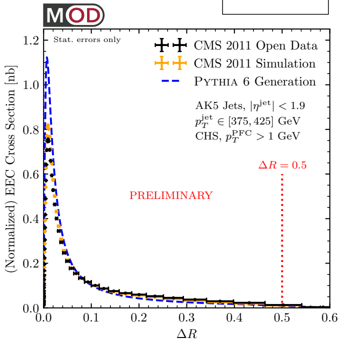


Lightray OPE

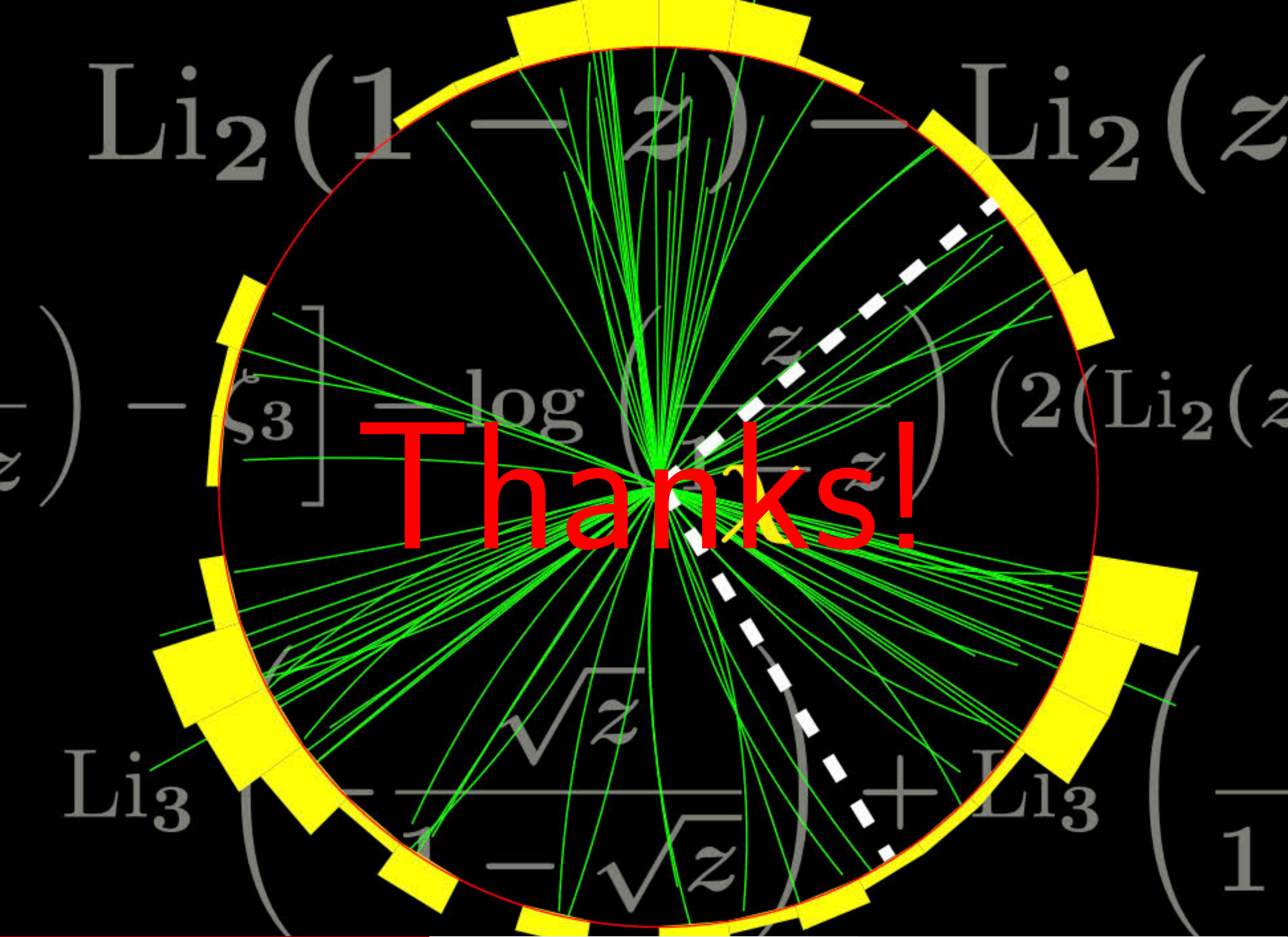


$$\frac{d\sigma}{d\Delta R}$$





- Formulating Jet Substructure as the study of correlation functions of lightray operators allows the use of powerful field theoretic techniques for real world phenomenology.



Thanks!