Track functions and their QCD evolution

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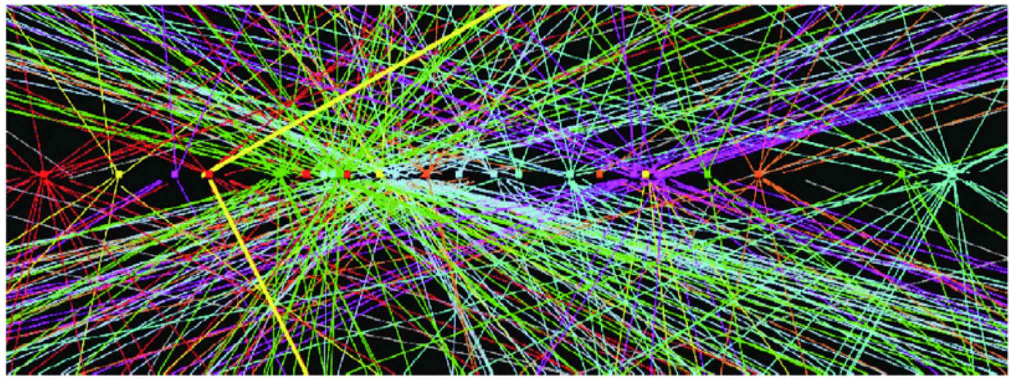
Why this talk?

- Advantages of track-based measurements:
  - Superior angular resolution.
  - Pile-up removal.



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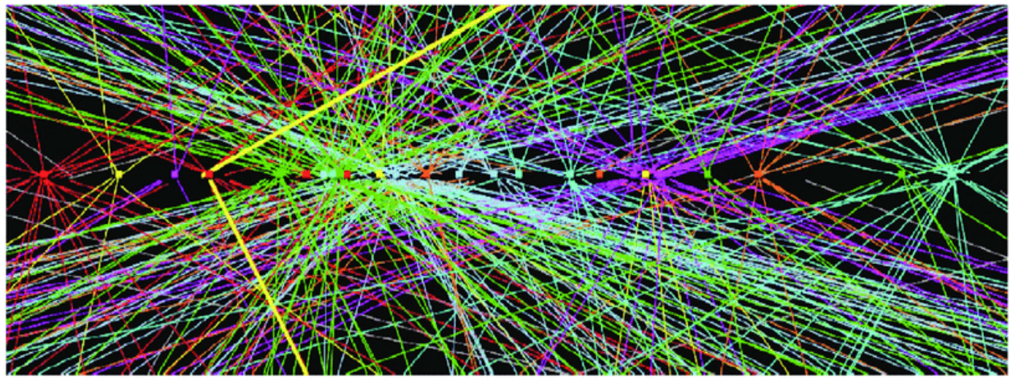

- Disadvantage: not IR safe → describe by track functions.
- Track functions: QFT approach to track-based calculations.
  - Certain observables minimally sensitive to track functions.
    [see Ian Moult's and Rudi Rahn's talks]
- This talk: extend track functions to order α_s^2
  - Higher precision.
  - Strong check on formalism.

Outline

1. Track functions
   [arXiv:1303.6637 - Chang, Procura, Thaler, WW]

2. Evolution at order α_s^2
   [arXiv:2106.xxxxx - Li, Moult, Schrijnder van Velzen, WW, Zhu]

3. Conclusions and outlook

1. Track functions

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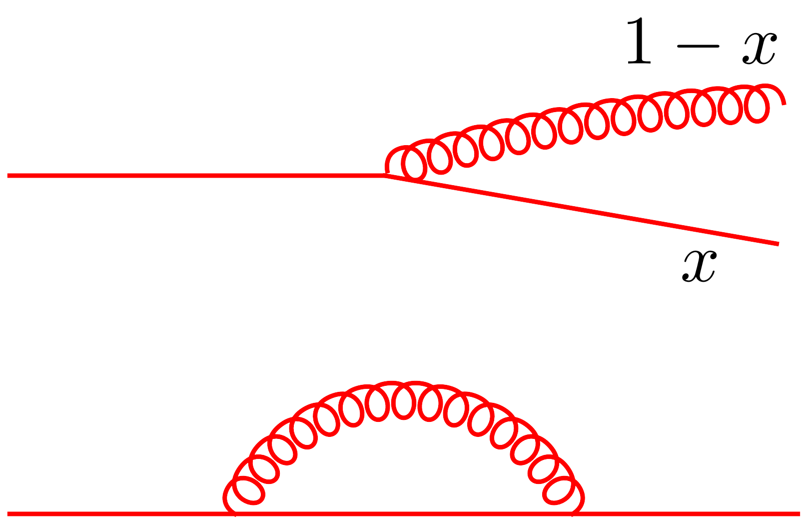
1. Track-based calculations are not IR safe



track fraction:

x

1

- E.g. partonic calculation of energy fraction of tracks in a jet.
- IR divergences of real and virtual located at different x values → don't cancel.

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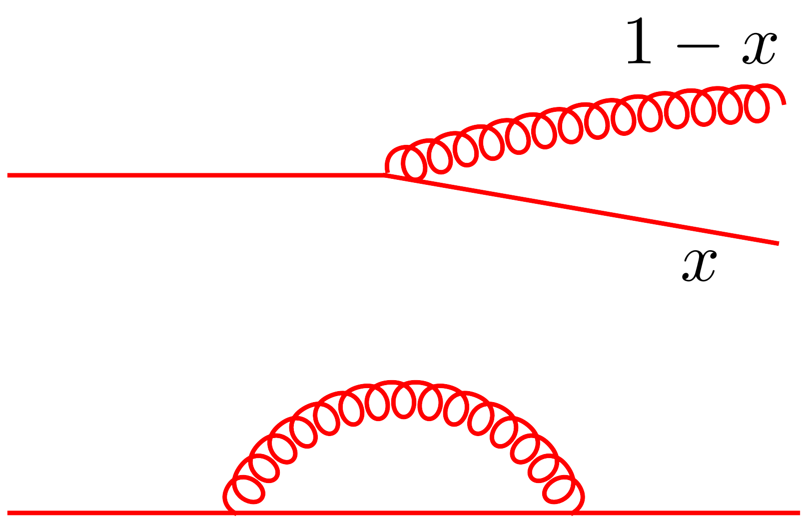

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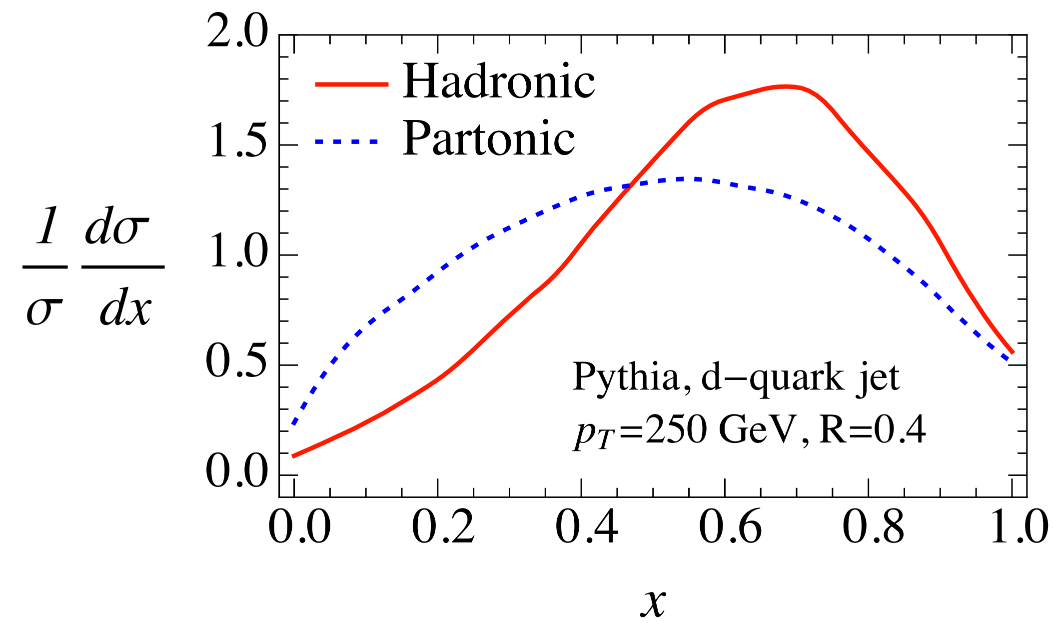
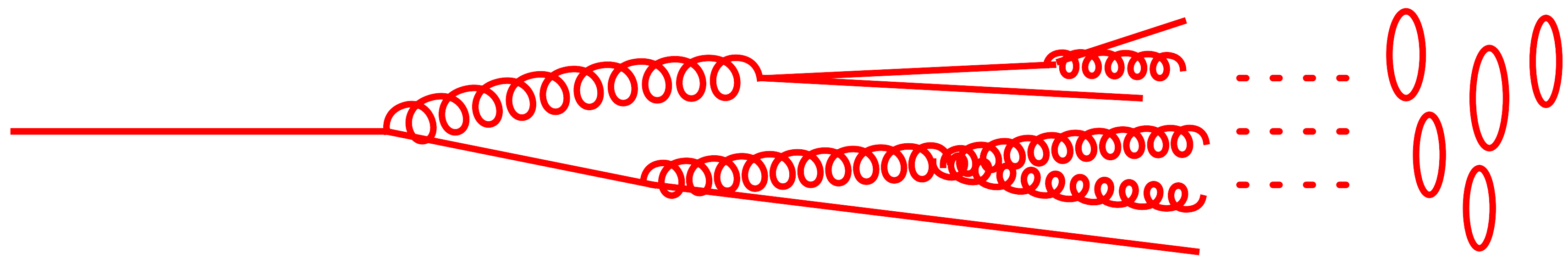


- E.g. partonic calculation of energy fraction of tracks in a jet.
- IR divergences of real and virtual located at different x values → don't cancel.
- Large hadronization corrections in Pythia.

1. Track function



- $T_i(x, \mu)$ describes momentum fraction x of initial parton i converted to tracks, i.e. $\bar{p}^\mu = x p^\mu + \mathcal{O}(\Lambda_{\rm QCD})$
- This is a generalization of a fragmentation function:
  - Also independent of hard process.
  - Satisfies **different** sum rule: $\int_0^1 \mathrm{d}x \, T_i(x, \mu) = 1$

1. Track-based calculations

- Consider a generic cross section, differential in observable e:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}e} = \sum_N \int \mathrm{d}\Pi_N \, \frac{\mathrm{d}\sigma_N}{\mathrm{d}\Pi_N} \, \delta[e - \hat{e}(\{p_i^\mu\})]$$

- At leading order, the track-based measurement $\bar{e}$ is:

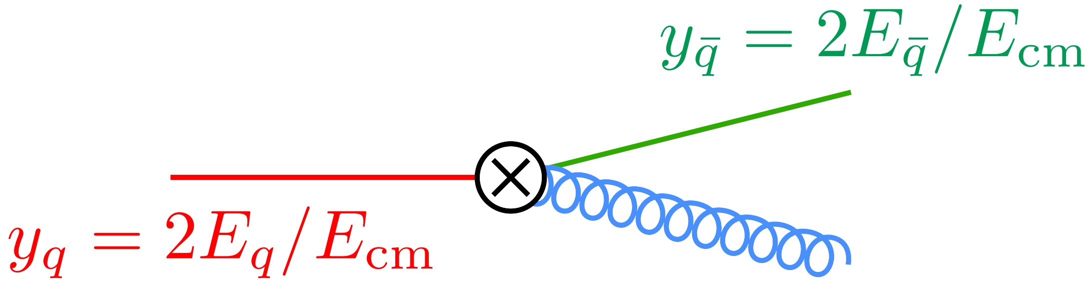
$$\frac{\mathrm{d}\sigma}{\mathrm{d}\bar{e}} = \sum_N \int \mathrm{d}\Pi_N \, \frac{\mathrm{d}\sigma_N}{\mathrm{d}\Pi_N} \, \underbrace{\int \prod_{i=1}^{N} \mathrm{d}x_i \, T_i(x_i)}_{\text{hadronization}} \, \delta[e - \hat{e}(\{x_i p_i^\mu\})]$$

- Beyond leading order, there is a cancellation of IR divergences, $\mathrm{d}\sigma_N \to \mathrm{d}\bar{\sigma}_N$, similar to cross sections for fragmentation.

1. Example: track fraction in e+e-

- Cross section differential in track fraction w up to NLO:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}w} = \int \mathrm{d}y_q \mathrm{d}y_{\bar{q}} \, \frac{\mathrm{d}\bar{\sigma}}{\mathrm{d}y_q \mathrm{d}y_{\bar{q}}} \int \mathrm{d}x_q \, T_q(x_q) \int \mathrm{d}x_{\bar{q}} \, T_q(x_{\bar{q}}) \int \mathrm{d}x_g \, T_g(x_g)$$
$$\times \, \delta\{w - [x_q y_q + x_{\bar{q}} y_{\bar{q}} + x_g(2 - y_q - y_{\bar{q}})]/2\}$$



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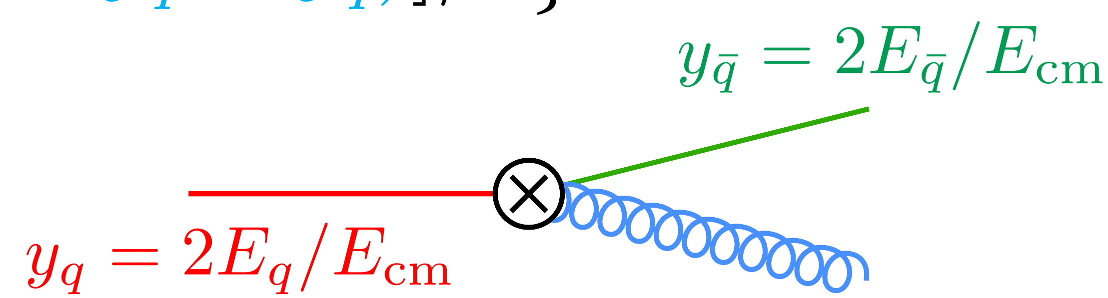


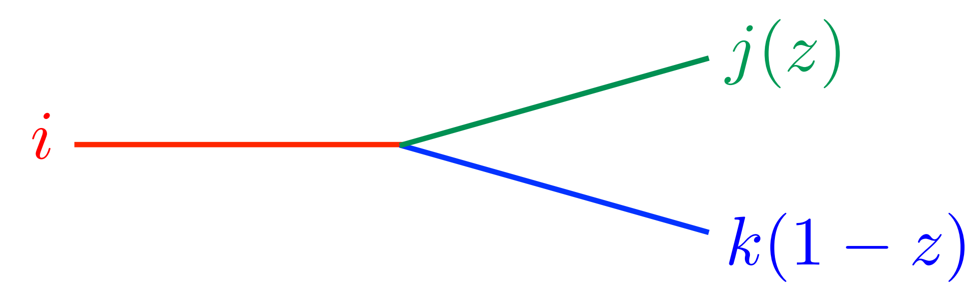
- $1/\epsilon_{\mathrm{IR}}$ pole in partonic cross section $\mathrm{d}\sigma$ cancels against pole in **partonic** track function (next slide), resulting in finite $\mathrm{d}\bar{\sigma}$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}y_q \mathrm{d}y_{\bar{q}}} = \sigma^{(0)} \left\{ \delta(1 - y_q)\delta(1 - y_{\bar{q}}) + \frac{\alpha_s C_F}{2\pi} \left[-\frac{1}{\epsilon_{\mathrm{IR}}} P_{qq}(y_q)\delta(1 - y_{\bar{q}}) + \dots \right] \right\}$$

1. Track function at order α_s

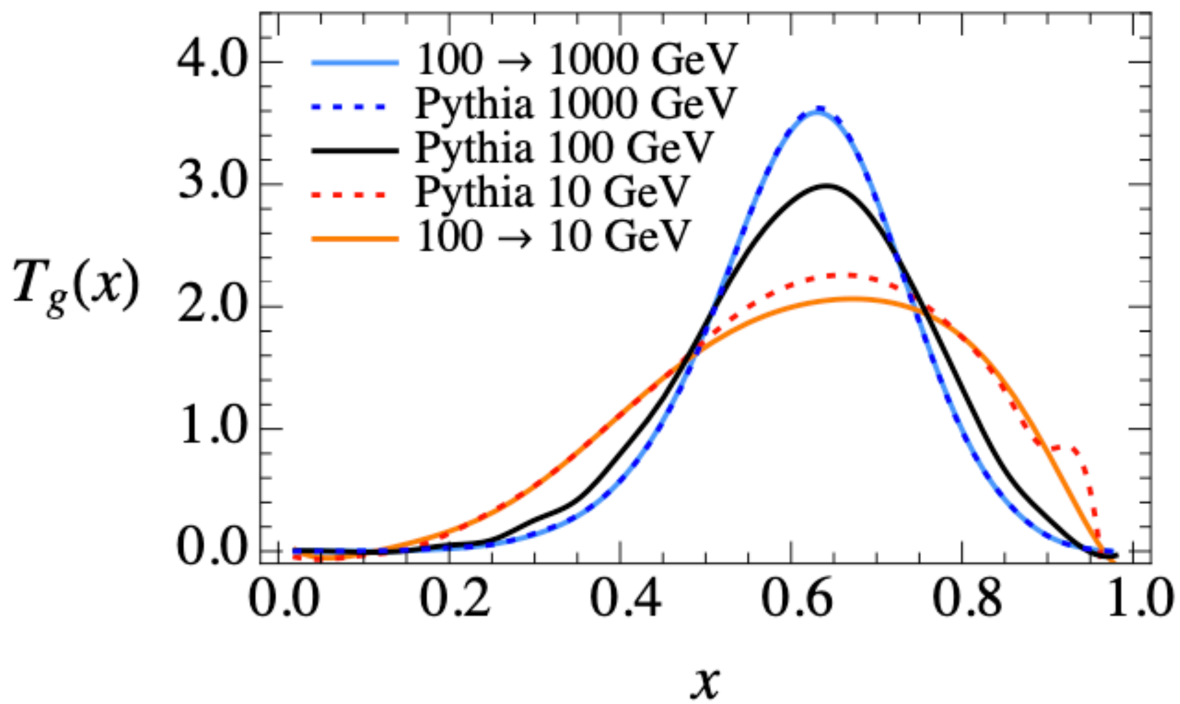
$$T^{(1)}_{i,\mathrm{bare}}(x) = \sum_j \int \mathrm{d}z \left[\frac{\alpha_s}{4\pi}\Big(\frac{1}{\epsilon_{\mathrm{UV}}} - \frac{1}{\epsilon_{\mathrm{IR}}}\Big) P_{ji}(z)\right] \int \mathrm{d}x_1\, T^{(0)}_j(x_1,\mu)$$

$$\times \int \mathrm{d}x_2\, T^{(0)}_k(x_2,\mu)\,\delta\big[x - z x_1 - (1-z)x_2\big]$$



- Similar to fragmentation function:
  - $1/\epsilon_{\mathrm{UV}}$ is renormalized, leads to evolution of track function.
  - $1/\epsilon_{\mathrm{IR}}$ cancels against IR pole in partonic cross section.
- Nonlinear structure because observable is simultaneously sensitive to all final-state particles.

1. Track function evolution



- Nonlinear evolution (leading order):

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} T_i(x,\mu) = \sum_{j,k} \int \mathrm{d}z\, \frac{\alpha_s}{2\pi} P_{ji}(z) \int \mathrm{d}x_1\, T_j(x_1,\mu) \int \mathrm{d}x_2\, T_k(x_2,\mu)$$
$$\times\, \delta[x - z x_1 - (1-z) x_2]$$

- Consistent with extraction from Pythia at different energies.

1. Track function evolution: moments

- Taking **integer** moments of evolution equation:

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} T_i[N,\mu] = \sum_{j,k} \int \mathrm{d}z\, \frac{\alpha_s}{2\pi} P_{ji}(z) \int \mathrm{d}x_1\, T_j(x_1,\mu) \int \mathrm{d}x_2\, T_k(x_2,\mu) [z x_1 - (1-z) x_2]^N$$

$$= \sum_{j,k} \int \mathrm{d}z\, \frac{\alpha_s}{2\pi} P_{ji}(z) \sum_n T_j[n,\mu]\, T_k[N-n,\mu] \binom{N}{n} z^n (1-z)^{N-n}$$

  - $N = 1$ related to frag. functions: $T_i[1,\mu] = \sum_{\mathrm{charged}\ h} d_{i\to h}[1,\mu]$
  - Moments needed in (projected) energy correlators

    [Chen, Moult, Zhang, Zhu]

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    [Chen, Moult, Zhang, Zhu]

- Beyond leading order (ignoring flavor)

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} T[3,\mu] = c_3 T[3,\mu] + c_{21} T[2,\mu] T[1,\mu] + c_{111} T[1,\mu] T[1,\mu] T[1,\mu]$$

  - Goal to determine unknown coefficients

2. Evolution at order α_s^2

[arXiv:2106.xxxxx - Li, Moult, Schrijnder van Velzen, WW, Zhu]

2. First method: track jet function

- Direct track function calculation in dimensional regularization results in scaleless integrals $1/\epsilon_{\mathrm{UV}} - 1/\epsilon_{\mathrm{IR}} = 0$

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- Direct track function calculation in dimensional regularization results in scaleless integrals $1/\epsilon_{\rm UV} - 1/\epsilon_{\rm IR} = 0$
- Track jet function $\mathcal{G}_i(s,x)$ differential in invariant mass s of all particles and momentum fraction x of charged particles
  - Same renormalization as invariant mass jet function $J_i(s)$ from consistency of factorization in SCET
  - Remaining $1/\epsilon$ poles are infrared and cancel when matching $\mathcal{G}$ onto track functions.

$$\mathcal{G}_i^{(2)} = T_i^{(2)} + \sum_j J_{i\to jk}^{(1)} \otimes [T_j^{(1)} T_k^{(0)}] + \sum_{j,k} J_{i\to jk\ell}^{(2)} \otimes [T_j^{(0)} T_k^{(0)} T_\ell^{(0)}]$$

- Analogous to fragmenting jet function

[Procura, Stewart; Jain, Procura, WW; …]

2. Track jet function calculation

- Calculate $\mathcal{G}$ from integrating collinear splitting amplitudes [Ritzmann, WW]

$$\mathcal{G}(s,z) = \sum_N \int \mathrm{d}\Pi_N^c \, \sigma_N^c \, \delta(s - s_{12\dots N}) \Big[\prod_{i=1}^N \int \mathrm{d}x_i \, T^{(0)}(x_i)\Big] \delta(z - \sum_i z_i x_i)$$

  - Perform integrals over s_{ij} [Kosower, Uwer]
  - For z_i integrals switch to moments (avoids plus distributions).

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- Illustration at order α_s

$$\mathcal{G}_q^{(1)}[s,N] \propto \frac{1}{s^{1+\epsilon}} \int \mathrm{d}z \left[\frac{1+z^2}{1-z} - \epsilon(1-z)\right]$$

$$\sum_n \binom{N}{n} z^{N-n-\epsilon} (1-z)^{n-\epsilon} \, T_q^{(0)}[N-n] \, T_g^{(0)}[n]$$

  - Agrees with earlier expression for $T^{(1)}$, after including renormalization of $\mathcal{G}$ (only affects $n=0$).

2. Track jet function simplifications

- Can directly calculate track jet function $\mathcal{G}$ at order α_s^2
- Large amount of symmetry relating moments.
  E.g. for $g \to ggg$

$$\begin{aligned} 2z_1 z_2 &= (z_1 + z_2)^2 - z_1^2 - z_2^2 \\ &= (1 - z_3)^2 - z_1^2 - z_2^2 \\ &\to 1 - 2z_i - z_i^2 \end{aligned}$$

  - Less trivial example: $6z_1 z_2 z_3 \to 1 - 9z_i^2 + 6z_i^3$

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  - Less trivial example: $6z_1 z_2 z_3 \to 1 - 9z_i^2 + 6z_i^3$
  - Moments involving only one variable are related to moments of splitting functions.
  - More complicated when including flavor, but can choose a convenient basis of integrals that avoids soft singularities.

2. Second method: energy correlation functions

- Energy-energy correlation function

$$\frac{\mathrm{d}\sigma}{\mathrm{d}z} = \sum_N \int \mathrm{d}\Pi_N \, \sigma_N \sum_{i,j} \frac{E_i E_j}{Q^2} \delta\Big(z - \frac{1-\cos\chi_{ij}}{2}\Big)$$

- Track-based measurement

$$E_i \to \bar{E}_i = T_i[1,\mu] E_i \,, \quad E_i^2 \to \bar{E}_i^2 = T_i[2,\mu] E_i^2$$

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- Applying this:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}z} = \sum_N \int \mathrm{d}\Pi_N \, \bar{\sigma}_N \Big[\sum_{i\neq j} \frac{E_i E_j}{Q^2} \delta\Big(z - \frac{1-\cos\chi_{ij}}{2}\Big) T_i[1,\mu] T_j[1,\mu] + \sum_i \frac{E_i^2}{Q^2} \delta(z) T_i[2,\mu]\Big]$$

$$= \sum_N \int \mathrm{d}\Pi_N \, \sigma_N \Big[\sum_{i\neq j} \frac{E_i E_j}{Q^2} \delta\Big(z - \frac{1-\cos\chi_{ij}}{2}\Big) T_{i,\mathrm{bare}}[1,\mu] T_{j,\mathrm{bare}}[1,\mu] + \sum_i \frac{E_i^2}{Q^2} \delta(z) T_{i,\mathrm{bare}}[2,\mu]\Big]$$

using that $T_{i,\mathrm{bare}} = T_i^{(0)} +$ scaleless integrals.

2. Extracting track functions from EEC

- Requiring the poles to cancel in the resulting expression for the EEC in the collinear $z \to 0$ limit fixes evolution of $T_i[2, \mu]$
- Illustration at order α_s :

$$\frac{1}{\sigma_0}\frac{\mathrm{d}\sigma}{\mathrm{d}z} = T_{q,\mathrm{bare}}[2]\,\delta(z)\left(\frac{1}{2} + \frac{\alpha_s C_F}{4\pi}\frac{25}{12}\frac{1}{\epsilon}\right) + \dots$$

$$\to \quad T_{q,\mathrm{bare}}^{(1)}[2] = \frac{\alpha_s C_F}{4\pi}\left(-\frac{25}{12}\right)\frac{1}{\epsilon_{\mathrm{UV}}} T_q^{(0)}[2,\mu] + \dots$$

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- Extended this to α_s^2 for e^+e^- and Higgs.
- Obtained partial results for third moment from projected three-point energy correlation function.
- Importantly, both methods for extracting track functions **agree**!

2. Selected results at α_s^2

$$\frac{d}{d\ln\mu^2}T_g[1,\mu] = \gamma_{gg}[1,\mu]T_g[1,\mu] + \sum_i \gamma_{qg}[1,\mu](T_{q_i}[1,\mu] + T_{\bar{q}_i}[1,\mu])\,,$$

$$\frac{d}{d\ln\mu^2}T_g[2,\mu] = \gamma_{gg}[2,\mu]T_g[2,\mu] + \left(\frac{\alpha_s}{4\pi}\right)^2\left[C_A^2\left(-8\zeta_3 + \frac{26}{45}\pi^2 + \frac{2158}{675}\right) - \frac{4}{9}C_A n_f T_F\right]T_g[1,\mu]T_g[1,\mu] + \ldots$$

$$\frac{d}{d\ln\mu^2}T_g[3,\mu] = \gamma_{gg}[3,\mu]T_g[3,\mu] + \left(\frac{\alpha_s}{4\pi}\right)^2\left[C_A^2\left(24\zeta_3 - \frac{278}{15}\pi^2 + \frac{767263}{4500}\right) - \frac{2}{3}C_A n_f T_F\right]T_g[2,\mu]T_g[1,\mu]$$

$$+ \left(\frac{\alpha_s}{4\pi}\right)^2\left[C_A T_f\left(\frac{23051}{1125} - \frac{28}{15}\pi^2\right) - C_F T_f\frac{28}{15}\right]T_g[1,\mu]\sum_i T_{q_i}[1,\mu]T_{\bar{q}_i}[1,\mu] + \ldots\,.$$

- γ is anomalous dimension for fragmentation functions. (other papers use different convention for moment and sign)
- First moment same as fragmentation function evolution.
- Higher moments can involve two or three track functions.

3. Conclusions and outlook

- Tracks are appealing because of their superior angular resolution and reduced effect of pile-up.
- Track functions offer a QFT approach to calculating track-based observables.
- Formalism fully studied at order α_s. **New:** extending to α_s^2
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Track you!