Exact equilibrium distributions for massive and massless fermions with rotation and acceleration

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in collaboration with
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Main result

Wigner function for free fermions at general global equilibrium:

$$W(x,k)=\frac{1}{(2\pi)^3}\int\frac{\mathrm{d}^3p}{2\varepsilon}\sum_{n=1}^{\infty}(-1)^{n+1}\mathrm{e}^{-n\widetilde{\beta}(in\phi)\cdot p}\times$$
$$\left[\mathrm{e}^{-in\frac{\phi:\Sigma}{2}}(m+\not{p})\delta^4\left(k-\frac{\Lambda^n p+p}{2}\right)+(m-\not{p})\mathrm{e}^{in\frac{\phi:\Sigma}{2}}\delta^4\left(k+\frac{\Lambda^n p+p}{2}\right)\right]$$

Compute all quantum corrections in relativistic fluids at global equilibrium.

Exact **distribution functions** and **spin vector** at all orders in thermal vorticity.

Global equilibrium

Density operator at global equilibrium:

$$\widehat{\rho} = \frac{1}{Z} \exp\left[-b_\mu \widehat{P}^\mu + \frac{1}{2}\varpi_{\mu\nu}\widehat{J}^{\mu\nu}\right] \qquad \langle \widehat{O} \rangle = \mathrm{Tr}\left(\widehat{\rho}\widehat{O}\right)$$

The vector b is constant and the thermal vorticity ϖ is a constant antisymmetric tensor. The four-temperature β vector is a Killing vector:

$$\beta^\mu(x) = b^\mu + \varpi^{\mu\nu}x_\nu \equiv \frac{u^\mu}{T}$$

At global equilibrium:

$$\frac{A^\mu}{T} = \varpi^{\mu\nu}u_\nu$$

Acceleration

$$\frac{\omega^\mu}{T} = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\varpi_{\nu\rho}u_\sigma$$

Angular velocity

The generators of the Poincaré group appear in the density operator.

Analytic continuation of the thermal vorticity: $\varpi \mapsto -i\phi$

$$\widehat{\rho} = \frac{1}{Z} \exp\left[-b_\mu \widehat{P}^\mu - \frac{i}{2}\phi_{\mu\nu}\widehat{J}^{\mu\nu}\right]$$

P $\mapsto$ translations
J$\mapsto$ Lorentz transformations

Factorization of the density operator:

$$\widehat{\rho} = \frac{1}{Z} \exp\left[-\widetilde{b}_\mu(\phi)\widehat{P}^\mu\right] \exp\left[-i\frac{\phi_{\mu\nu}}{2}\widehat{J}^{\mu\nu}\right] \equiv \frac{1}{Z} \exp\left[-\widetilde{b}_\mu(\phi)\widehat{P}^\mu\right]\widehat{\Lambda}$$

$$\widetilde{b}^\mu(\varpi) = \sum_{k=0}^{\infty} \frac{1}{(k+1)!} \underbrace{(\phi^\mu{}_{\alpha_1}\phi^{\alpha_1}{}_{\alpha_2}\dots\phi^{\alpha_{k-1}}{}_{\alpha_k})}_{k \text{ times}} b^{\alpha_k} \qquad \widehat{\Lambda} \equiv \mathrm{e}^{-i\frac{\phi_{\mu\nu}}{2}\widehat{J}^{\mu\nu}}$$

We can use **group theory** to calculate **thermal expectation values.**

Any thermal expectation value in a free quantum field theory is obtained from:

$$\langle \widehat{a}_s^\dagger(p)\widehat{a}_t(p')\rangle = \frac{1}{Z}\mathrm{Tr}\left(\exp[-\tilde{b}_\mu(\phi)\widehat{P}^\mu]\widehat{\Lambda}\;\widehat{a}_s^\dagger(p)\widehat{a}_t(p')\right)$$

$$[\widehat{a}_s^\dagger(p),\widehat{a}_t(p')]_\pm = 2\varepsilon\delta^3(\boldsymbol{p}-\boldsymbol{p'})\delta_{st}$$

Using Poincaré transformation rules and (anti)commutation relations (particle with spin S):

$$\begin{aligned}\langle \widehat{a}_s^\dagger(p)\widehat{a}_t(p')\rangle =&(-1)^{2S}\sum_r D^S(W(\Lambda,p))_{rs}\mathrm{e}^{-\widetilde{b}\cdot\Lambda p}\langle \widehat{a}_r^\dagger(\Lambda p)\widehat{a}_t(p')\rangle+\\ &+2\varepsilon\,\mathrm{e}^{-\widetilde{b}\cdot\Lambda p}D^S(W(\Lambda,p))_{ts}\delta^3(\Lambda\boldsymbol{p}-\boldsymbol{p}')\end{aligned}$$

$\mathrm{D(W)}$ is the “Wigner rotation” in the S-spin representation.

We find a solution by iteration:

I
$$\langle \widehat{a}_s^\dagger(p)\widehat{a}_t(p')\rangle \sim 2\varepsilon\, \mathrm{e}^{-\widetilde{b}\cdot\Lambda p} D^S(W(\Lambda,p))_{ts}\delta^3(\Lambda\boldsymbol{p}-\boldsymbol{p}')$$

II
$$\langle \widehat{a}_s^\dagger(p)\widehat{a}_t(p')\rangle \sim 2\varepsilon\, (-1)^{2S} D^S(W(\Lambda^2,p))_{ts}\mathrm{e}^{-\widetilde{b}\cdot(\Lambda p+\Lambda^2 p)}\delta^3(\Lambda^2\boldsymbol{p}-\boldsymbol{p}')+ + 2\varepsilon\, \mathrm{e}^{-\widetilde{b}\cdot\Lambda p} D^S(W(\Lambda,p))_{ts}\delta^3(\Lambda\boldsymbol{p}-\boldsymbol{p}')$$

∞
$$\langle \widehat{a}_s^\dagger(p)\widehat{a}_t(p')\rangle = 2\varepsilon' \sum_{n=1}^{\infty}(-1)^{2S(n+1)}\delta^3(\Lambda^n\boldsymbol{p}-\boldsymbol{p}')D^S(W(\Lambda^n,p))_{ts}\mathrm{e}^{-\widetilde{b}\cdot\sum_{k=1}^{n}\Lambda^k p}$$

For vanishing vorticity (i.e. $\Lambda = \mathrm{I}$):

$$\langle \widehat{a}_s^\dagger(p)\widehat{a}_t(p')\rangle = 2\varepsilon' \sum_{n=1}^{\infty}(-1)^{2S(n+1)}\delta^3(\boldsymbol{p}-\boldsymbol{p}')\delta_{ts}\,\mathrm{e}^{-nb\cdot p} = \frac{2\varepsilon\,\delta^3(\boldsymbol{p}-\boldsymbol{p}')\delta_{ts}}{\mathrm{e}^{b\cdot p}+(-1)^{2S+1}}$$

Wigner function

The Wigner for free fermions:

$$W(x,k) = -\frac{1}{(2\pi)^4}\int \mathrm{d}^4y\, \mathrm{e}^{-ik\cdot y}\langle : \Psi(x-y/2)\overline{\Psi}(x+y/2) :\rangle$$

Expectation values:

$$j^\mu(x) = \int \mathrm{d}^4k\, \mathrm{tr}\left(\gamma^\mu W(x,k)\right)$$

Wigner equation, a constraint:

$$\left(m - \not{k} - \frac{i\hbar\not{\partial}}{2}\right) W(x,k) = 0$$

Holds regardless of the density operator

Exact Wigner function for free fermions at global equilibrium:

$$W(x,k) = \frac{1}{(2\pi)^3}\int \frac{\mathrm{d}^3p}{2\varepsilon}\sum_{n=1}^{\infty}(-1)^{n+1}\mathrm{e}^{-n\tilde{\beta}(in\phi)\cdot p}\times$$
$$\left[\mathrm{e}^{-in\frac{\phi:\Sigma}{2}}(m+\not{p})\delta^4\left(k-(\Lambda^n p+p)/2\right)+(m-\not{p})\mathrm{e}^{in\frac{\phi:\Sigma}{2}}\delta^4\left(k+(\Lambda^n p+p)/2\right)\right]$$

Where $\Lambda = \mathrm{e}^{-i\frac{\phi}{2}:J}$ is in the four-vector representation.
Solves the Wigner equation! Full summation of the "$\hbar$ expansion".

Differs from previous **ansatz:**
[F.Becattini, V.Chandra, L.Del Zanna, E.Grossi, Annals Phys. 338 (2013) 32-49]

$$W_A(x,k) = \int \frac{\mathrm{d}^3p}{2\varepsilon}\delta^4(k-p)(\not{p}+m)\left(\mathrm{e}^{\beta\cdot p}e^{-\frac{\varpi:\Sigma}{2}}+\mathbb{I}\right)^{-1}(\not{p}+m)$$
$$+\,\delta^4(k+p)(m-\not{p})\left(e^{\beta\cdot p}e^{\frac{\varpi:\Sigma}{2}}+\mathbb{I}\right)(m-\not{p})$$

We can use the exact solution to compute **exact expectation values.**

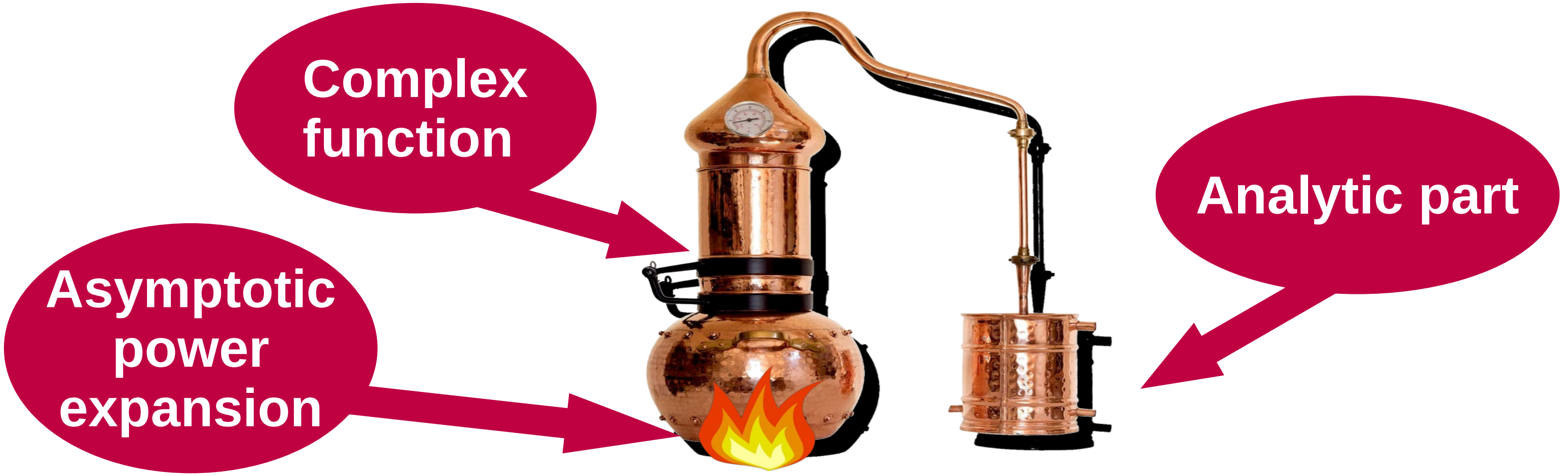
Energy density for massless fermions, equilibrium with acceleration ($\phi = ia/T$)

$$\rho = \frac{3T^4}{8\pi^2} \sum_{n=1}^{\infty} (-1)^{n+1} \phi^4 \frac{\sinh n\phi}{\sinh^5(n\phi/2)}$$

The series is finite as long as ϕ is real. For real thermal vorticity it diverges!

The series includes terms which are non analytic at $\phi=0$.

Analytic distillation:



The series boils down to polynomials: $\alpha^\mu = \frac{A^\mu}{T}$ $\quad w^\mu = \frac{\omega^\mu}{T}$

$$\rho = \frac{7\pi^2}{60\beta^4} - \frac{\alpha^2}{24\beta^4} - \frac{17\alpha^4}{960\pi^2\beta^4}$$

Expectation values vanish at the Unruh temperature $T_U = \sqrt{-A\cdot A}/2\pi$
[G.Prokhorov, O. Teryaev, V. Zakharov, JHEP03(2020)137]

Axial current under rotation: [V. Ambrus, E. Winstanley, 1908.10244]

$$j_A^\mu = T^2\left(\frac{1}{6} - \frac{w^2}{24\pi^2} - \frac{\alpha^2}{8\pi^2}\right)\frac{w^\mu}{\sqrt{\beta^2}}$$

First exact results at equilibrium with **both rotation and acceleration.**
[V. Ambrus, E. Winstanley 2107.06928]

$$\rho = T^4\left(\frac{7\pi^2}{60} - \frac{\alpha^2}{24} - \frac{w^2}{8} - \frac{17\alpha^4}{960\pi^2} + \frac{w^4}{64\pi^2} + \frac{23\alpha^2 w^2}{1440\pi^2} + \frac{11(\alpha\cdot w)^2}{720\pi^2}\right)$$

New contribution!

Current and distribution function

The current of particles is (massive or massless):

$$j_+^\mu(x) = \frac{1}{(2\pi)^3}\sum_{st}\int \frac{\mathrm{d}^3p}{2\varepsilon}\frac{\mathrm{d}^3p'}{2\varepsilon'}\,\mathrm{e}^{i(p'-p)\cdot x}\langle\widehat{a}_s^\dagger(p')\widehat{a}_t(p)\rangle\,\bar{u}_s(p')\gamma^\mu u_t(p)$$

Compare with the previous ansatz
[F.Becattini, V.Chandra, L.Del Zanna, E.Grossi, Annals Phys. 338 (2013) 32-49]

$$j_+^\mu(x) = \int\frac{\mathrm{d}^3p}{\varepsilon}p^\mu \mathrm{tr}\,(X) \qquad X = \left(\mathrm{e}^{\beta\cdot p}\mathrm{e}^{-\frac{\varpi}{2}:\Sigma}+\mathbb{I}\right)^{-1}$$

The exact current is parallel to p^μ only if $\langle\widehat{a}_s^\dagger(p')\widehat{a}_t(p)\rangle \propto \delta_{st}\delta^3(\boldsymbol{p}-\boldsymbol{p}')$

The ansatz does not take into account the neither the non-diagonal terms nor the $\tilde{\beta}$ transform.

The current of particles is (massive or massless):

$$j_+^\mu(x) = \frac{1}{(2\pi)^3}\int \frac{\mathrm{d}^3 p}{2\varepsilon}\sum_{n=1}^{\infty}(-1)^{n+1}\mathrm{e}^{-n\widetilde{\beta}_n\cdot p}\left[p^\mu \mathrm{tr}\left(\mathrm{e}^{-in\frac{\phi}{2}:\Sigma}\right) + 2ip_\nu \mathrm{tr}\left(\Sigma^{\mu\nu}\mathrm{e}^{-in\frac{\phi}{2}:\Sigma}\right)\right]$$

We define a distribution function from:

$$j_+^\mu(x)\int\frac{\mathrm{d}^3 p}{\varepsilon}p^\mu f(x,p) + N^\mu(x,p)$$

The distribution function reads:

$$f(x,p) = \frac{1}{2(2\pi)^3}\sum_{n=1}^{\infty}(-1)^{n+1}\mathrm{e}^{-n\widetilde{\beta}(-n\varpi)\cdot p}\mathrm{tr}\left(\mathrm{e}^{n\frac{\varpi}{2}:\Sigma}\right)$$

The ansatz misses all **quantum corrections** due to the tilde transform:

$$\tilde{\beta}^\mu(-n\varpi) = \beta^\mu(x) + \sum_{k=1}^{\infty}\frac{(-in)^k}{(k+1)!}(\varpi^{\mu\alpha_1}\varpi_{\alpha_1\alpha_2}\dots\varpi^{\alpha_{k-1}\alpha_k})\beta_{\alpha_k}(x)$$

In the massless case $\bar{u}_\lambda(p)\gamma^\mu u_\lambda(p)$ is a light-like complex vector. Given **an arbitrary lightlike vector** q, non orthogonal to p.

$$\bar{u}_\lambda(p')\gamma^\mu u_\lambda(p) = \frac{\bar{u}_\lambda(p')\not{q} u_\lambda(p)}{q\cdot p} p^\mu + \frac{\overbrace{\bar{u}_\lambda(p')\not{p} u_\lambda(p)}^{=0}}{q\cdot p} q^\mu + N^\mu(p,q)$$

We have:

$$f_\lambda(x,p)_{(q)} \equiv \frac{1}{(2\pi)^3}\int \frac{\mathrm{d}^3 p'}{2\varepsilon'}\, \mathrm{e}^{i(p'-p)\cdot x}\langle \widehat{a}^\dagger_\lambda(p')\widehat{a}_\lambda(p)\rangle \frac{\bar{u}_\lambda(p')\not{q} u_\lambda(p)}{2q\cdot p}$$

Exact distribution function at global equilibrium:

$$f_\lambda(x,p)_{(q)} = \frac{1}{(2\pi)^3}\frac{1}{2p\cdot q}\sum_{n=1}(-1)^{n+1}\mathrm{e}^{-n\widetilde{\beta}(in\phi)\cdot p}\mathrm{tr}\left(\frac{I+2\lambda\gamma_5}{2}\mathrm{e}^{-in\frac{\phi:\Sigma}{2}}\not{p}\not{q}\right)$$

It is not a function of collisional invatiants only: $f(x,p) \neq f(p\cdot u \pm \Sigma_n : \varpi)$
[J.Y.Chen, D.Son, M.Stephanov, H.Y.Yee, Y.Yin, P.R.L. 113 (2014) 182302]

Spin vector

Spin vector of massive particles:

$$S^\mu(p) = \frac{1}{2}\frac{\int \mathrm{d}\Sigma \cdot p \; \mathrm{tr}\left(\gamma^\mu \gamma_5 W_+(x,p)\right)}{\int \mathrm{d}\Sigma \cdot p \; \mathrm{tr}\left(W_+(x,p)\right)}$$

Exact spin vector at global equilibrium:

$$S^\mu(p) = \frac{1}{2m}\frac{\sum_{n=1}^{\infty}(-1)^{n+1}\mathrm{e}^{-n\widetilde{b}(in\phi)\cdot p}\mathrm{tr}\left(\gamma^\mu\gamma_5\mathrm{e}^{-in\frac{\phi:\Sigma}{2}}\not{p}\right)}{\sum_{n=1}^{\infty}(-1)^{n+1}\mathrm{e}^{-n\widetilde{b}(in\phi)\cdot p}\mathrm{tr}\left(\mathrm{e}^{-in\frac{\phi:\Sigma}{2}}\right)}$$

Reproduces the known formula in the literature.

Conclusions & Outlook

Exact Wigner function at general global equilibrium with thermal vorticity.

New operation on complex functions: the **analytic distillation.**

Expressions for the **spin polarization vector** and the **distribution functions** to all orders in $\hbar$ and in thermal vorticity.

Outlook:
- Comparison between exact solution and commonly used ansatz

Thank you for the attention!