Spin and Chirality in Hydrodynamics

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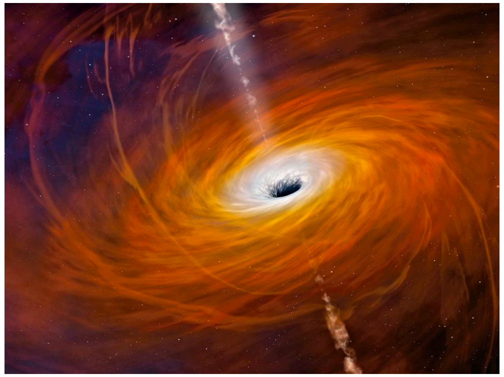
— The 6th Chirality, Vorticity and Magnetic Field in HIC —

Equilibrium with Spin / Angular Momentum

Conserved Charges

A black hole (maybe a QGP) has no hair:

Stable black holes are characterized by

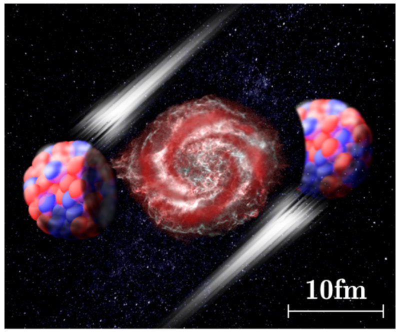


[From Forbes]

Mass ⟷ Temperature

Charge ⟷ Density

Angular Momentum



[From our review on Femto-Novae]

Equilibrium with Rotation

Distribution Functions [Vilenkin 1979]

$$\sum_p f^{(i)}(p) = n^{(i)} \qquad \sum_{i,p} f^{(i)}(p) j^{(i)} = J^z$$

$$\sum_{i,p} f^{(i)}(p) \varepsilon^{(i)} = e \qquad \sum_{i,p} f^{(i)}(p) p^{(i)} = P^z$$

Constraints from conservation laws

Lagrange multipliers

Entropy should be maximized under constraints

$$\frac{\delta}{\delta f^{(i)}(p)} \sum_j \Bigl[s^{(j)} - \sum_q f^{(j)}(q)(\alpha_j + \beta\varepsilon^{(j)} + \gamma j^{(j)} + \delta p^{(j)}) \Bigr] = 0$$

Once the entropy is given as a function of *f*, *f* can be fixed from the variation.

Equilibrium with Rotation

Distribution Functions [Vilenkin 1979]

Shannon entropy

$$s^{(i)} = \sum_p \Big[\pm(1 \pm f^{(i)}(p)) \ln(1 \pm f^{(i)}(p)) - f^{(i)}(p) \ln f^{(i)}(p) \Big]$$

$$\boxed{f^{(i)}(p) = \frac{1}{e^{\alpha_i + \beta\varepsilon^{(i)} + \gamma j^{(i)} + \delta p^{(i)}} \mp 1}}$$

Thermodynamic relations

$$\beta = \frac{1}{T}\,, \quad \alpha_i = -\frac{\mu_i}{T}\,, \quad \gamma = -\frac{\omega}{T}\,, \quad \delta = -\frac{v^z}{T}$$

Equilibrium with Rotation

Grand Canonical Descriptions

$$de = Tds + \mu dn + \omega_{\mu\nu} dJ^{\mu\nu}$$

$$dp = sdT + nd\mu + J^{\mu\nu} d\omega_{\mu\nu}$$

→

$$\langle n \rangle = \frac{\partial p}{\partial \mu}$$

$$\langle J^z \rangle = \frac{\partial p}{\partial \omega^z}$$

n : baryon density
μ : baryon chemical pot.

J : angular mom.
ω : spin chemical pot.

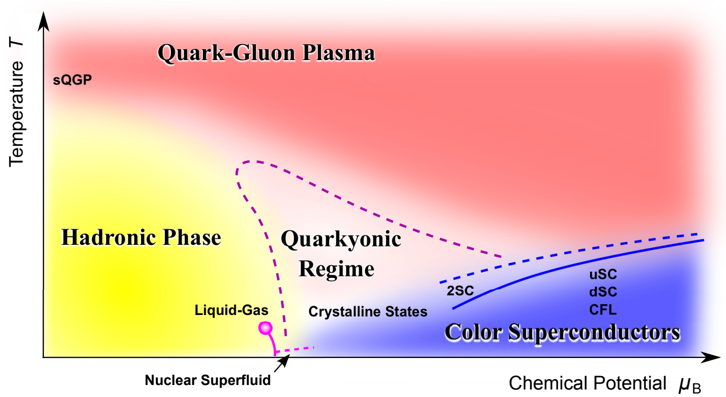
Phase Diagram
Hydro Trajectories

Counterparts?

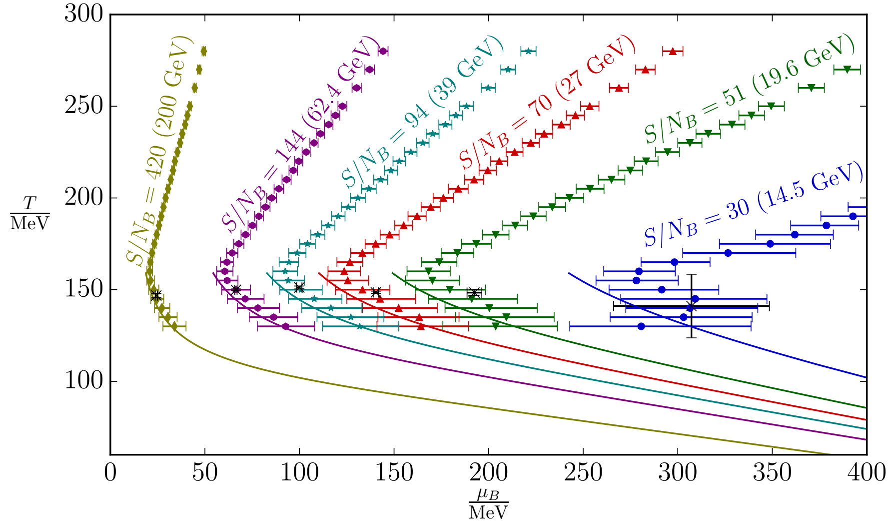
Equilibrium with Rotation

Baryon Density

Speculated Phase Diagram



Gunther et al. (2016)



Ideal hydrodynamics: $s^\mu = s u^\mu \quad j^\mu = n u^\mu$

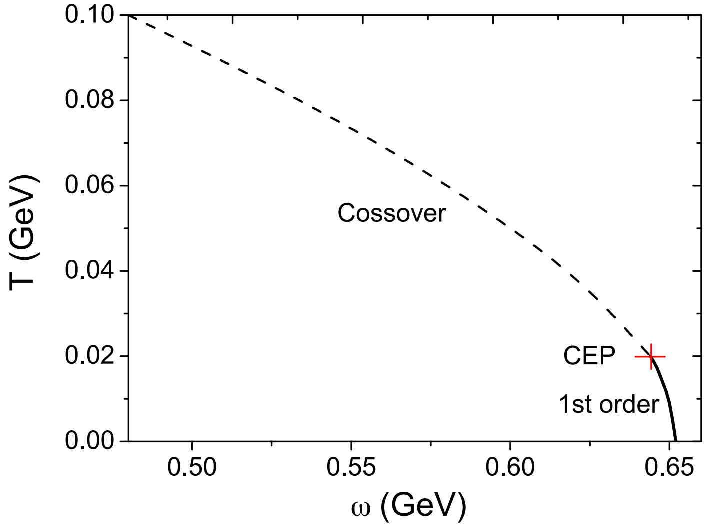
$$\partial_\mu T^{\mu\nu} = 0 \ \to \ \partial_\mu s^\mu = 0$$

Eliminating $(\partial \cdot u)$ ➔ $n\dot{s} - s\dot{n} = 0$ ➔ $s/n = (\text{const.})$

Equilibrium with Rotation

Angular Momentum

Speculated Phase Diagram
Jiang-Liao (2017)



Ideal hydrodynamics:

$$s^\mu = s u^\mu \quad j^\mu = n u^\mu$$

$$J^{\alpha\mu\nu} = J^{\mu\nu} u^\alpha$$

***s* and *n* and *J* all scale similarly with expansion**

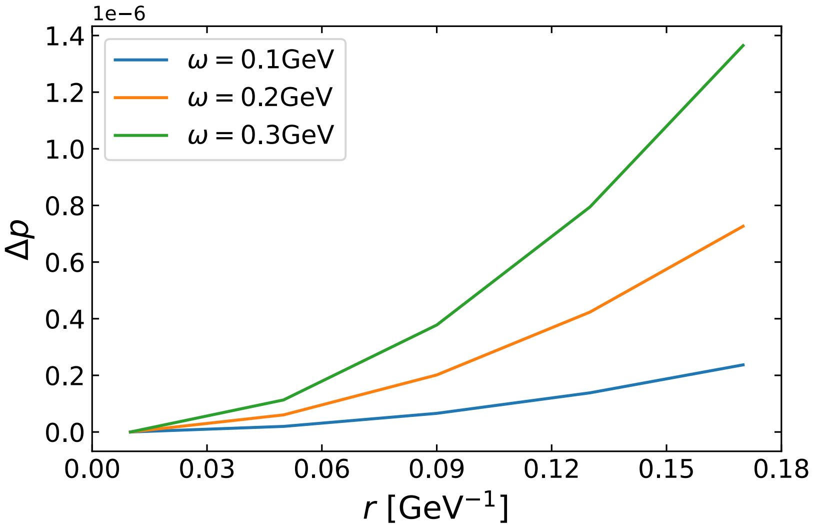
However, the angular momentum has some subtleties on /phase diagram/hydro counting/energy-mom. tensor

Rotating Thermodynamics

Rotating Hadron Resonance Gas

Pressure:

$$p_i^{\pm} = \pm \frac{T}{8\pi^2} \sum_{\ell=-\infty}^{\infty} \int_{(\Lambda_\ell^{\mathrm{IR}})^2} dk_r^2 \int dk_z \sum_{\nu=\ell}^{\ell+2S_i} J_\nu^2(k_r r)$$



$$\times \log\left\{1 \pm \exp[-(\varepsilon_{\ell,i} - \mu_i)/T]\right\}$$

$$\varepsilon_{\ell,i} = \sqrt{k_r^2 + k_z^2 + m_i^2} - (\ell + S_i)\omega$$

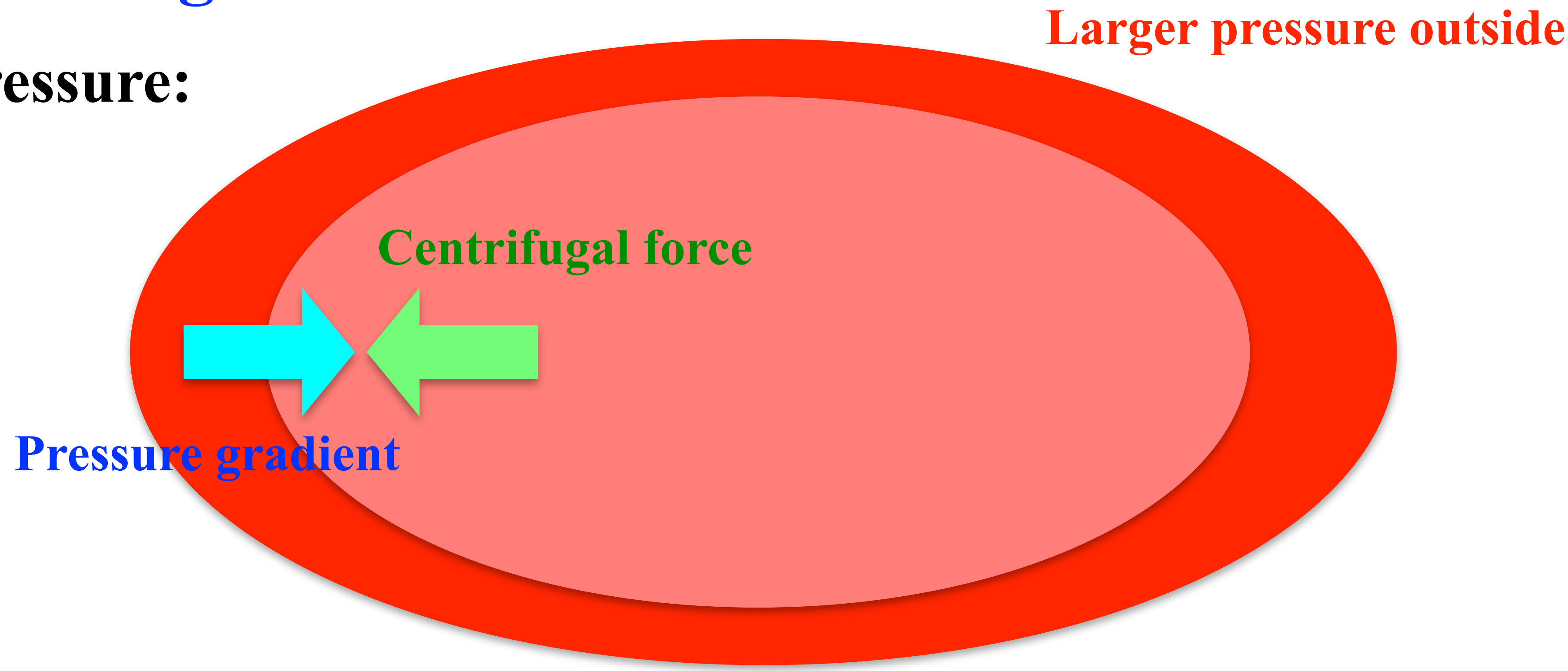
$$\Lambda_\ell^{\mathrm{IR}} = \xi_{\ell,1}\omega$$

Fujimoto-Fukushima-Hidaka (2021)

Rotating Thermodynamics

Rotating Hadron Resonance Gas

Pressure:



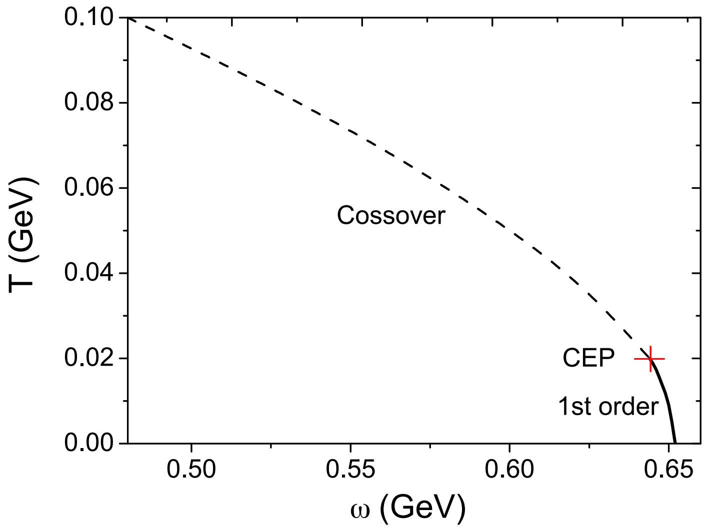
Pressure is always inhomogeneous

$$\langle j\rangle(r)\,dV \simeq dI(r)\omega \quad dI(r) = \rho r^2\,dV \longrightarrow \Delta p(r) \sim \# r^2\omega^2$$

Rotating Thermodynamics

To make thinking simplified

Thermodynamics near *r* = 0 simplifies the situation a lot.



Jiang-Liao (2017)

(Phase diagram with spin effects only)

The orbital angular momentum is dropped then, and only the spin remains finite.
(No need to worry about spatial inhomogeneity anymore.)

cf. Chiral Vortical Effect
(only the spin contributes)

Hydrodynamics with Spin / Angular Momentum

Entropy Argument

A classic work by Son-Surowka (2009) found:

(The notation is slightly changed here)

Ideal Hydrodynamics ~ $\mathcal{O}(\partial^0)$

Entropy Current

$$\mathcal{S}^\mu_{(0)} = s u^\mu = \frac{u_\nu}{T} T^{\mu\nu}_{(0)} + \frac{p}{T} u^\mu - \alpha\, j^\mu_{(0)} \qquad (\alpha = \mu/T)$$

$\partial_\mu \mathcal{S}^\mu_{(0)} = 0$ is easily concluded

Non-ideal Hydrodynamics ~ $\mathcal{O}(\partial)$

$$\mathcal{S}^\mu = \frac{u_\nu}{T} T^{\mu\nu} + \frac{p}{T} u^\mu - \alpha\, j^\mu \qquad \partial_\mu j^\mu = C_{\mathrm{anom}} E \cdot B$$

Entropy Argument

Non-ideal Hydrodynamics ~ $\mathcal{O}(\partial)$

$$\mathcal{S}^\mu = \frac{u_\nu}{T}T^{\mu\nu} + \frac{p}{T}u^\mu - \alpha\, j^\mu \qquad \partial_\mu j^\mu = C_{\rm anom} E\cdot B$$

$$\partial_\mu \mathcal{S}^\mu = T^{\mu\nu}_{(1)}\partial_\mu \frac{u_\nu}{T} + j^\mu_{(1)}\left(-\partial_\mu\alpha + \frac{E_\mu}{T}\right) \boxed{- C_{\rm anom}\alpha E\cdot B}$$

Viscosities **Ohm's law / Seebeck effect**

How to realize the second law of thermodynamics?

Missing terms $\delta j^\mu_{(1)} = \xi_V \omega^\mu + \xi_B B^\mu$ **(CVE+CME)**

Anomalously induced transport required thermodynamically!

Entropy Argument + S

Spin Hydrodynamics (Hattori-Hongo-Huang-Matsuo-Taya 2019)

$$\mathcal{S}^\mu = \frac{u_\nu}{T} T^{\mu\nu} + \frac{p}{T} u^\mu - \alpha\, j^\mu$$

$$\mathcal{S}^\mu_{\mathrm{can}} = \frac{u_\nu}{T} \Theta^{\mu\nu} + \frac{p}{T} u^\mu - \alpha j^\mu - \frac{\omega_{\rho\sigma}}{T} \Sigma^{\mu\rho\sigma}$$

"canonically" derived energy-momentum tensor

Induced by the spin potential

Straightforward generalization but the way to incorporate the spin should explained now.

Entropy Argument + S

Spin Hydrodynamics (Hattori-Hongo-Huang-Matsuo-Taya 2019)

Noether current from rotational symmetry

$$J_{\rm can}^{\alpha\mu\nu} = \underbrace{x^\mu \Theta^{\alpha\nu} - x^\nu \Theta^{\alpha\mu}}_{\text{Orbital}} + \underbrace{\Sigma^{\alpha\mu\nu}}_{\text{Spin}}$$

$$\partial_\alpha J_{\rm can}^{\alpha\mu\nu} = 0\ ;\ \partial_\mu T^{\mu\nu} = 0 \longrightarrow \boxed{\partial_\alpha \Sigma^{\alpha\mu\nu} = -2\Theta_{\rm (a)}^{\mu\nu}}$$

Anti-symmetric part of the energy-momentum tensor is the source of the spin (from the orbital part).

Entropy Argument + S

Spin Hydrodynamics (Hattori-Hongo-Huang-Matsuo-Taya 2019)

Previously…

$$\partial_\mu \mathcal{S}^\mu = T^{\mu\nu}_{(1)} \partial_\mu \frac{u_\nu}{T} + j^\mu_{(1)} \left(-\partial_\mu \alpha + \frac{E_\mu}{T} \right) - C_{\text{anom}} \alpha E \cdot B$$

In spin hydro…

$$\partial_\mu S^\mu_{\text{can}} = \Theta^{\mu\nu}_{(1)} \partial_\mu \frac{u_\nu}{T} - j^\mu_{(1)} \partial_\mu \alpha - \frac{\omega_{\rho\sigma}}{T} \partial_\mu \Sigma^{\mu\rho\sigma} \sim \Theta^{\rho\sigma}_{(a)}$$

$$\Theta^{\rho\sigma}_{(a)} \left(2 \frac{\omega_{\rho\sigma}}{T} + \partial_{[\mu} \frac{u_{\nu]}}{T} \right)$$

Becattini et al. (2013~)

Thermal Vorticity $\sim \Omega_{\mu\nu}$

Equilibrium (non-dissipative) limit: $\omega_{\mu\nu}\Big|_{\text{eq}} = -\frac{T}{2} \partial_{[\mu} \frac{u_{\nu]}}{T}$

Entropy Argument + S

Spin Hydrodynamics (Hattori-Hongo-Huang-Matsuo-Taya 2019)

$$\Theta^{\rho\sigma}_{(\mathrm{a})}\left(2\frac{\omega_{\rho\sigma}}{T}+\partial_{[\mu}\frac{u_{\nu]}}{T}\right)\geq 0$$

Tensor decomposition of dissipative (viscous) terms

$$\longrightarrow \quad \Theta^{\mu\nu}_{(1\mathrm{a})} = 2q^{[\mu}u^{\nu]}+\phi^{\mu\nu}$$

$$q^{\mu} = \lambda\left[T^{-1}\Delta^{\mu\alpha}\partial_{\alpha}T+(u\cdot\partial)u^{\mu}-4\omega^{\mu\nu}u_{\nu}\right]$$

$$\phi^{\mu\nu} = -\gamma(\Omega^{\mu\nu}-2T^{-1}\Delta^{\mu\alpha}\Delta^{\nu\beta}\omega_{\alpha\beta})$$

(Positive) Transport Coefficients introduced

Symmetric Form

Pseudo-Gauge Sym. in Energy-Momentum Tensor

Fukushima-Pu (2020)

$$\mathcal{T}^{\mu\nu} = \Theta^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu} \quad \text{(conserved current redefined)}$$

$$K^{\lambda\mu\nu} = \frac{1}{2}\left(\Sigma^{\lambda\mu\nu} - \Sigma^{\mu\lambda\nu} + \Sigma^{\nu\mu\lambda}\right)$$

$$\mathcal{T}^{\mu\nu} = \Theta^{\mu\nu} + \frac{1}{2}\partial_\lambda\left(u^\lambda S^{\mu\nu} - u^\mu S^{\lambda\nu} + u^\nu S^{\mu\lambda}\right)$$

$$= \Theta^{\mu\nu}_{(\mathrm{s})} + \frac{1}{2}\left[\partial_\lambda\left(u^\mu S^{\nu\lambda} + u^\nu S^{\mu\lambda}\right)\right]$$

Only the symmetric form is gauge inv. in gauge theories!
But, there is no spin source…

Symmetric Form

Pseudo-Gauge Sym. in Energy-Momentum Tensor

Fukushima-Pu (2020)

$$\mathcal{T}^{\mu\nu} = \Theta^{\mu\nu} + \frac{1}{2}\partial_\lambda(u^\lambda S^{\mu\nu} - u^\mu S^{\lambda\nu} + u^\nu S^{\mu\lambda})$$

$$= \Theta^{\mu\nu}_{(\mathrm{s})} + \frac{1}{2}\left[\partial_\lambda(u^\mu S^{\nu\lambda} + u^\nu S^{\mu\lambda})\right]$$

Spin induced terms are "renormalized" in conv. quantities

$$2h^{(\mu}u^{\nu)} + \pi^{\mu\nu} + \frac{1}{2}\left[\partial_\lambda(u^\mu S^{\nu\lambda} + u^\nu S^{\mu\lambda})\right]$$

$$= \delta e u^\mu u^\nu + 2\left(h^{(\mu} + \delta h^{(\mu}\right)u^{\nu)} + \pi^{\mu\nu} + \delta\pi^{\mu\nu}$$

Symmetric Form

Pseudo-Gauge Sym. in Energy-Momentum Tensor

Fukushima-Pu (2020)

$$\delta e = u_\rho \partial_\sigma S^{\rho\sigma}$$

$$\delta h^\mu = \frac{1}{2}\left[\Delta^\mu_\sigma \partial_\lambda S^{\sigma\lambda} + u_\rho S^{\rho\lambda}\partial_\lambda u^\mu\right]$$

$$\delta\pi^{\mu\nu} = \partial_\lambda (u^{<\mu} S^{\nu>\lambda}) + \delta\Pi\Delta^{\mu\nu}$$

$$\delta\Pi = \frac{1}{3}\partial_\lambda (u^\sigma S^{\rho\lambda})\Delta_{\rho\sigma}$$

An electric current $\boldsymbol{j} \propto \nabla \times \boldsymbol{S}$ is implied… **Spin Vorticity Effect**

Symmetric Form

Pseudo-Gauge Sym. in Energy-Momentum Tensor

Fukushima-Pu (2020)

$$\partial_\mu \mathcal{S}^\mu = \cdots + \frac{1}{2}\big[\partial_\lambda(u^\mu S^{\nu\lambda} + u^\nu S^{\mu\lambda})\big]\partial_\mu \frac{u_\nu}{T} - \frac{\omega_{\rho\sigma}}{T}\partial_\lambda(u^\lambda S^{\rho\sigma})$$

$$= \frac{1}{2}\partial_\mu\left[\partial_\lambda(u^\lambda S^{\mu\nu} + u^\mu S^{\nu\lambda} + u^\nu S^{\mu\lambda})\frac{u_\nu}{T}\right]$$

Total derivative

$$-\frac{1}{2}\big[\partial_\lambda(u^\lambda S^{\mu\nu})\big]\partial_\mu \frac{u_\nu}{T} - \frac{\omega_{\rho\sigma}}{T}\partial_\lambda(u^\lambda S^{\rho\sigma})$$

Canonical results

$$\mathcal{S}^\mu \;\to\; \mathcal{S}'^\mu$$

Absorbed in the entropy,
then it is just canonical!

Symmetric Form

Contrary to some claims, it is possible to formulate the spin hydrodynamics with the symmetric form of the energy-momentum tensor.

Spin induced terms are renormalized by conventional physical quantities (energy density, heat current, etc) which could be measurable corrections.

Once renormalized, the formulation looks like the non-spin hydrodynamics — more clarified in S. Li, M. Stephanov, H.-U. Yee (2020)

This reminds me of...

Floquet Theory

$$\hat{H}(t+T) = \hat{H}(t)$$

Periodically driven system
(e.g. rotating fields)

Time evolution can be decomposed into

$$\hat{U}(t_2, t_1) = e^{-i\hat{K}(t_2)} e^{-i\hat{H}_F(t_1 - t_2)} e^{i\hat{K}(t_1)}$$

Static System

Kick Operators

Transforming to the rotating frame, and a uniform time evolution, and then transforming back.

This reminds me of…

Floquet Theory

Decomposition is not unique… (gauge freedom)

Some terms are induced in the static effective Hamiltonian.

Rotating time-dependent H = Static H with induced terms

Chiral anomaly retained? Yes, not through H, but K

Fukushima-Hidaka-Shimazaki-Taya (2021)

Further Questions

- **Phase Diagram with *B* and *J* not explored yet.**
  - □ Technical difficulties (unremovable divergences)
  - □ Physics setup not well defined…
- **Hydrodynamics with *E*, *B* and *J* not revealed yet.**
  - □ Spin MHD should be developed with anomalous couplings.
- **Floquet theory with quantum anomaly from the kick operator not discussed yet.**
  - □ Magnus expansion is a complementary approach to understand the anomaly induced phenomena.