Spin, Chirality and Kinetic Theory

M. Stephanov

Introduction and motivation

- Interesting applications of CME/CVE/SP in non-equilibrium conditions – such as heavy-ion collisions, or high-frequency response – beyond hydro.

- Kinetic theory: a non-equilibrium description.

  "Classical", weakly coupled, but . . .

- Important for understanding the mechanism of CME/CVE/SP and the connection of hydro to microscopic dynamics. Kinetic theory is a bridge between QFT and hydro.

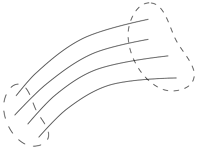
- This talk is about the features specific to kinetic theories with spin, such as chiral kinetic theory.

Puzzle 1: Anomaly

- Kin. regime: collisions are rare enough that motion is classical.

Each particle follows classical trajectory $\boldsymbol{x}(t)$, $\boldsymbol{p}(t)$. A "cloud" $f(\boldsymbol{x},\boldsymbol{p})$ evolves with time. In a comoving 6-volume, the number of particles can only be changed by collisions:

$$\frac{df}{dt} \equiv \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \boldsymbol{x}}\dot{\boldsymbol{x}} + \frac{\partial f}{\partial \boldsymbol{p}}\dot{\boldsymbol{p}} = C[f].$$

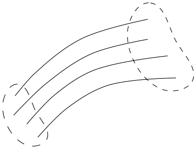


Puzzle 1: Anomaly

- Kin. regime: collisions are rare enough that motion is classical.

Each particle follows classical trajectory $\boldsymbol{x}(t)$, $\boldsymbol{p}(t)$. A "cloud" $f(\boldsymbol{x},\boldsymbol{p})$ evolves with time. In a comoving 6-volume, the number of particles can only be changed by collisions:

$$\frac{df}{dt} \equiv \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \boldsymbol{x}}\dot{\boldsymbol{x}} + \frac{\partial f}{\partial \boldsymbol{p}}\dot{\boldsymbol{p}} = C[f].$$



Ignore collisions for now.

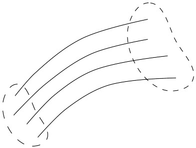
- The number of particles in the phase space *cannot change*?
- How can *classical* equation account for *quantum* anomaly?

Puzzle 1: Anomaly

- Kin. regime: collisions are rare enough that motion is classical.

Each particle follows classical trajectory $\boldsymbol{x}(t)$, $\boldsymbol{p}(t)$. A “cloud” $f(\boldsymbol{x}, \boldsymbol{p})$ evolves with time. In a comoving 6-volume, the number of particles can only be changed by collisions:

$$\frac{df}{dt} \equiv \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \boldsymbol{x}}\dot{\boldsymbol{x}} + \frac{\partial f}{\partial \boldsymbol{p}}\dot{\boldsymbol{p}} = C[f].$$



Ignore collisions for now.

- The number of particles in the phase space *cannot change*?
- How can *classical* equation account for *quantum* anomaly?

Spin/Chirality

Spin

- Spin for $m \neq 0$:

  In $\boldsymbol{p} = 0$ frame, spin vector $\boldsymbol{s}$ is a d.o.f. separate from $\boldsymbol{p}$.

  Obeys precession equation $\dfrac{d\boldsymbol{s}}{dt} = \dfrac{ge}{2m}\boldsymbol{s} \times \boldsymbol{B}$.

- Spin for $m = 0$:

  There is no $\boldsymbol{p} = 0$ frame.

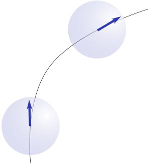
  Spin $\boldsymbol{s}$ is not an independent d.o.f., but rather $\boldsymbol{s} \parallel \boldsymbol{p}$.

  As $\boldsymbol{p}$ changes, there is a feedback from spin on e.o.m.s

  Kinetic theory of particles obeying these e.o.m.s – CKT.

Action, Berry phase and e.o.m.s

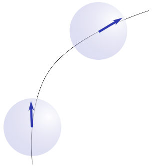
Change of momentum direction requires rotation in (quantum) spin space, which adds a phase to the action:



$$\mathcal{I} = \int (\boldsymbol{p} + \boldsymbol{A}) \cdot d\boldsymbol{x} - (\mathcal{E} + \Phi) dt \underbrace{- \boldsymbol{a_p} \cdot d\boldsymbol{p}}_{\text{Berry phase } \mathcal{O}(\hbar)}$$

Action, Berry phase and e.o.m.s

Change of momentum direction requires rotation in (quantum) spin space, which adds a phase to the action:



$$\mathcal{I} = \int (\boldsymbol{p} + \boldsymbol{A}) \cdot d\boldsymbol{x} - (\mathcal{E} + \Phi) dt \underbrace{- \ \boldsymbol{a_p} \cdot d\boldsymbol{p}}_{\text{Berry phase } \mathcal{O}(\hbar)}$$

Equations of motion by variation:

$$\dot{\boldsymbol{x}} - \boldsymbol{v} - \overbrace{\dot{\boldsymbol{p}} \times \boldsymbol{b}}^{\text{anom. velocity}} = 0;$$

$$\dot{\boldsymbol{p}} - \boldsymbol{E} - \dot{\boldsymbol{x}} \times \boldsymbol{B} = 0;$$

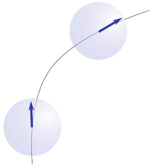
$$\boldsymbol{v} = \partial \mathcal{E} / \partial \boldsymbol{p}, \ \ \mathcal{E} \equiv |\boldsymbol{p}| - \frac{\hat{\boldsymbol{p}} \cdot \boldsymbol{B}}{2|\boldsymbol{p}|}.$$

Berry curvature:

$$\boldsymbol{b} \equiv \boldsymbol{\nabla_p} \times \boldsymbol{a_p} = \frac{\hat{\boldsymbol{p}}}{2|\boldsymbol{p}|^2}.$$

Action, Berry phase and e.o.m.s

Change of momentum direction requires rotation in (quantum) spin space, which adds a phase to the action:



$$\mathcal{I} = \int (\boldsymbol{p} + \boldsymbol{A}) \cdot d\boldsymbol{x} - (\mathcal{E} + \Phi) dt \underbrace{- \ \boldsymbol{a_p} \cdot d\boldsymbol{p}}_{\text{Berry phase } \mathcal{O}(\hbar)}$$

Equations of motion by variation:

$$\dot{\boldsymbol{x}} - \boldsymbol{v} - \overbrace{\dot{\boldsymbol{p}} \times \boldsymbol{b}}^{\text{anom. velocity}} = 0;$$

$$\dot{\boldsymbol{p}} - \boldsymbol{E} - \dot{\boldsymbol{x}} \times \boldsymbol{B} = 0;$$

$$\boldsymbol{v} = \partial \mathcal{E} / \partial \boldsymbol{p}, \ \ \mathcal{E} \equiv |\boldsymbol{p}| - \frac{\hat{\boldsymbol{p}} \cdot \boldsymbol{B}}{2|\boldsymbol{p}|}.$$

Berry curvature:

$$\boldsymbol{b} \equiv \boldsymbol{\nabla}_{\boldsymbol{p}} \times \boldsymbol{a_p} = \frac{\hat{\boldsymbol{p}}}{2|\boldsymbol{p}|^2}.$$

The invariant measure now is $\dfrac{d^3\boldsymbol{x}\, d^3\boldsymbol{p}}{(2\pi)^3} \sqrt{G}$, with $\sqrt{G} = 1 + \boldsymbol{b} \cdot \boldsymbol{B}$.

Chiral anomaly

Liouville eqn. (phase space conservation) now anomalous at $\boldsymbol{p} = 0$:

$$\frac{\partial}{\partial t}\sqrt{G} + \frac{\partial}{\partial \boldsymbol{x}}(\sqrt{G}\dot{\boldsymbol{x}}) + \frac{\partial}{\partial \boldsymbol{p}}(\sqrt{G}\dot{\boldsymbol{p}}) = (\boldsymbol{E}\cdot\boldsymbol{B})\,\underbrace{(\boldsymbol{\nabla}_{\boldsymbol{p}}\cdot\boldsymbol{b})}_{2\pi\delta^3(\boldsymbol{p})},$$

Chiral anomaly

Liouville eqn. (phase space conservation) now anomalous at $\boldsymbol{p} = 0$:

$$\frac{\partial}{\partial t}\sqrt{G} + \frac{\partial}{\partial \boldsymbol{x}}(\sqrt{G}\dot{\boldsymbol{x}}) + \frac{\partial}{\partial \boldsymbol{p}}(\sqrt{G}\dot{\boldsymbol{p}}) = (\boldsymbol{E}\cdot\boldsymbol{B})\ \underbrace{(\boldsymbol{\nabla}_{\boldsymbol{p}}\cdot\boldsymbol{b})}_{2\pi\delta^3(\boldsymbol{p})},$$

Thus current $\boldsymbol{J} = \int_{\boldsymbol{p}}\sqrt{G}f\dot{\boldsymbol{x}}$ is not conserved:

$$\frac{\partial n}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{J} = \frac{1}{4\pi^2}\boldsymbol{E}\cdot\boldsymbol{B}\ f|_{\boldsymbol{p}=0}\ ,$$

Chiral anomaly

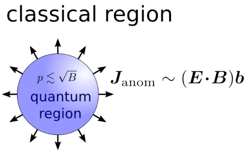
Liouville eqn. (phase space conservation) now anomalous at $\boldsymbol{p} = 0$:

$$\frac{\partial}{\partial t}\sqrt{G} + \frac{\partial}{\partial \boldsymbol{x}}(\sqrt{G}\dot{\boldsymbol{x}}) + \frac{\partial}{\partial \boldsymbol{p}}(\sqrt{G}\dot{\boldsymbol{p}}) = (\boldsymbol{E}\cdot\boldsymbol{B})\ \underbrace{(\boldsymbol{\nabla}_{\boldsymbol{p}}\cdot\boldsymbol{b})}_{2\pi\delta^3(\boldsymbol{p})},$$

Thus current $\boldsymbol{J} = \int_{\boldsymbol{p}}\sqrt{G}f\dot{\boldsymbol{x}}$ is not conserved:

$$\frac{\partial n}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{J} = \frac{1}{4\pi^2}\boldsymbol{E}\cdot\boldsymbol{B}\ f|_{\boldsymbol{p}=0}\ ,$$

Berry “monopole” at $\boldsymbol{p} = 0$ acts as source/sink of particle number current.
Region $p \lesssim \sqrt{B}$ is quantum.
“Level crossing” at $\boldsymbol{p} = 0$.



CME

Solving eoms

$$\dot{\boldsymbol{x}} - \boldsymbol{v} - \dot{\boldsymbol{p}} \times \boldsymbol{b} = 0;$$
$$\dot{\boldsymbol{p}} - \boldsymbol{E} - \dot{\boldsymbol{x}} \times \boldsymbol{B} = 0;$$

for $\dot{\boldsymbol{x}}$:

$$\boldsymbol{J} = \int_{\boldsymbol{p}} \sqrt{G} f \, \dot{\boldsymbol{x}} = \underbrace{\int_{\boldsymbol{p}} f \boldsymbol{v}}_{\text{normal current}} + \underbrace{\boldsymbol{E} \times \int_{\boldsymbol{p}} f \boldsymbol{b}}_{\text{anom. Hall current}} + \underbrace{\boldsymbol{B} \int_{\boldsymbol{p}} f (\boldsymbol{v} \cdot \boldsymbol{b})}_{\text{CME}}$$

CME

Solving eoms

$$\dot{\boldsymbol{x}} - \boldsymbol{v} - \dot{\boldsymbol{p}} \times \boldsymbol{b} = 0;$$
$$\dot{\boldsymbol{p}} - \boldsymbol{E} - \dot{\boldsymbol{x}} \times \boldsymbol{B} = 0;$$

for $\dot{\boldsymbol{x}}$:

$$\boldsymbol{J} = \int_{\boldsymbol{p}} \sqrt{G} f \, \dot{\boldsymbol{x}} = \underbrace{\int_{\boldsymbol{p}} f \boldsymbol{v}}_{\text{normal current}} + \underbrace{\boldsymbol{E} \times \int_{\boldsymbol{p}} f \boldsymbol{b}}_{\text{anom. Hall current}} + \underbrace{\boldsymbol{B} \int_{\boldsymbol{p}} f (\boldsymbol{v} \cdot \boldsymbol{b})}_{\text{CME}}$$

In equilibrium $f(\boldsymbol{p})$ is $f(\mathcal{E})$:

$$\boldsymbol{J}_{\text{CME}} = \boldsymbol{B} \frac{1}{4\pi^2} \int_0^\infty f(\mathcal{E}) d\mathcal{E}$$
$$= \frac{1}{4\pi^2} \mu \boldsymbol{B} \quad \text{for FD } T = 0.$$

CVE

- First, define CVE in kinetic theory: response to rotation.

CVE

- First, define CVE in kinetic theory: response to rotation.
- Replace Lorentz force with Coriolis force (MS, Yin, 2012):

$$\dot{\boldsymbol{p}} = 2\mathcal{E}\,\dot{\boldsymbol{x}} \times \boldsymbol{\omega} \qquad \text{i.e.,} \boxed{\;\boldsymbol{B} \to 2\mathcal{E}\boldsymbol{\omega}}.$$

- Then CME $\longrightarrow$ CVE

$$\boldsymbol{J}_{\rm CME} = \boldsymbol{B}\int_{\boldsymbol{p}} f\,\hat{\boldsymbol{p}}\cdot\boldsymbol{b} \quad\longrightarrow\quad \boldsymbol{J}_{\rm CVE} = \boldsymbol{\omega}\int_{\boldsymbol{p}} 2\mathcal{E}\,f\,\hat{\boldsymbol{p}}\cdot\boldsymbol{b}$$

For example, a distribution $f(\mathcal{E})$ gives

$$\boldsymbol{J}_{\rm CVE} = \frac{\boldsymbol{\omega}}{4\pi^2}\int_0^\infty f(\mathcal{E})\,2\mathcal{E}\,d\mathcal{E}$$

$$= \frac{1}{4\pi^2}\mu^2\boldsymbol{\omega} \quad \text{for FD } T = 0.$$

Puzzle 2: Where is Lorentz invariance?

$$\mathcal{I} = \int (\boldsymbol{p} + \boldsymbol{A}) \cdot d\boldsymbol{x} - (|\boldsymbol{p}| + \Phi)dt - \boldsymbol{a}_{\boldsymbol{p}} \cdot d\boldsymbol{p} + \frac{\hat{\boldsymbol{p}} \cdot \boldsymbol{B}}{2|\boldsymbol{p}|}dt$$

Puzzle 2: Where is Lorentz invariance?

$$\mathcal{I} = \int (\boldsymbol{p} + \boldsymbol{A}) \cdot d\boldsymbol{x} - (|\boldsymbol{p}| + \Phi)dt - \boldsymbol{a_p} \cdot d\boldsymbol{p} + \frac{\hat{\boldsymbol{p}} \cdot \boldsymbol{B}}{2|\boldsymbol{p}|}dt$$

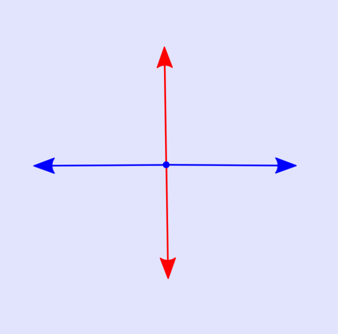
Modified Lorentz transfromation: PRL 113(2014)182302

$$\delta\boldsymbol{x} = \boldsymbol{\beta}t + \frac{\boldsymbol{\beta} \times \hat{\boldsymbol{p}}}{2|\boldsymbol{p}|}, \quad \delta\boldsymbol{p} = \boldsymbol{\beta}\mathcal{E} + \frac{\boldsymbol{\beta} \times \hat{\boldsymbol{p}}}{2|\boldsymbol{p}|} \times \boldsymbol{B}, \quad \delta t = \boldsymbol{\beta} \cdot \boldsymbol{x}.$$

- Side jump.
- Magnetic moment ($\boldsymbol{m} = \frac{\hat{\boldsymbol{p}}}{2|\boldsymbol{p}|}$) needed by Lorentz invariance

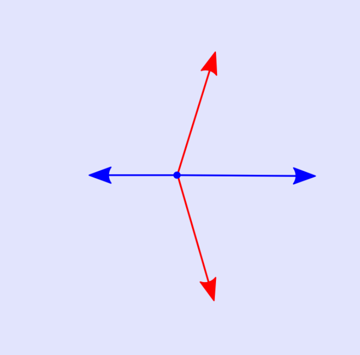
(Son-Yamamoto)

Boost, side jump and angular momentum conservation



+ boost =

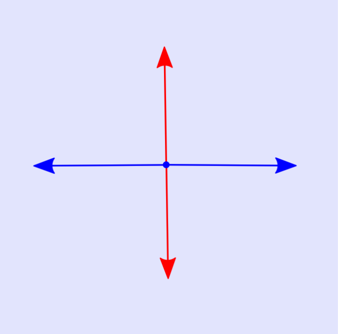
$\longrightarrow \boldsymbol{\beta}$



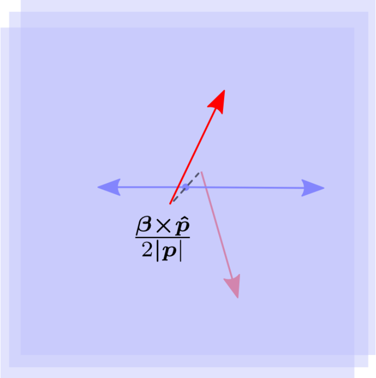
$\boldsymbol{P}_{\rm in} = \boldsymbol{P}_{\rm out} = 0$
$\boldsymbol{S}_{\rm in} = \boldsymbol{S}_{\rm out} = 0$
$\boldsymbol{L}_{\rm in} = \boldsymbol{L}_{\rm out} = 0$

$\boldsymbol{P}_{\rm in} = \boldsymbol{P}_{\rm out}$
$\boldsymbol{S}_{\rm in} = 0, \quad \boldsymbol{S}_{\rm out} = \rightarrow \mathcal{O}(\hbar)$
$\boldsymbol{L}_{\rm in} = 0, \quad \boldsymbol{L}_{\rm out} = 0???$

Boost, side jump and angular momentum conservation



+ boost = $\longrightarrow \boldsymbol{\beta}$



$$\boldsymbol{P}_{\rm in} = \boldsymbol{P}_{\rm out} = 0$$
$$\boldsymbol{S}_{\rm in} = \boldsymbol{S}_{\rm out} = 0$$
$$\boldsymbol{L}_{\rm in} = \boldsymbol{L}_{\rm out} = 0$$

$$\boldsymbol{P}_{\rm in} = \boldsymbol{P}_{\rm out}$$
$$\boldsymbol{S}_{\rm in} = 0, \quad \boldsymbol{S}_{\rm out} = \rightarrow \mathcal{O}(\hbar)$$
$$\boldsymbol{L}_{\rm in} = 0, \quad \boldsymbol{L}_{\rm out} = \leftarrow$$

“Side jump”

Collision kernel nonlocal

Magnetization current

Conservation defines current up to a trivially conserved term (curl).

Liouville current:

$$\boldsymbol{J}' = \int_{\boldsymbol{p}} \sqrt{G} f \, \dot{\boldsymbol{x}}$$

Magnetization current

Conservation defines current up to a trivially conserved term (curl).

Liouville current:

$$\boldsymbol{J}' = \int_{\boldsymbol{p}} \sqrt{G} f\, \dot{\boldsymbol{x}}$$

Noether current (by variation of action):

$$\boldsymbol{J} \equiv \int_{\boldsymbol{p}} \sqrt{G} f\, \frac{\delta \mathcal{I}}{\delta \boldsymbol{A}} = \boldsymbol{J}' + \underbrace{\boldsymbol{\nabla} \times \int_{\boldsymbol{p}} \sqrt{G} f\, \frac{\hat{\boldsymbol{p}}}{2|\boldsymbol{p}|}}_{\boldsymbol{\nabla} \times \boldsymbol{M}}$$

magnetization current

Lorentz covariant

Another way calculate CVE: rotating distribution

For a locally isotropic, but slowly rotating distribution ($\boldsymbol{\nabla} \times \boldsymbol{u} = 2\boldsymbol{\omega}$):

$$f(\mathcal{E}') = f(|\boldsymbol{p}| - \boldsymbol{p} \cdot \boldsymbol{u}(\boldsymbol{x}) - \frac{1}{2}\hat{\boldsymbol{p}} \cdot \boldsymbol{\omega})$$

In the (inertial) lab frame, the Noether current

$$\boldsymbol{J} = \underbrace{\int_{\boldsymbol{p}} f \hat{\boldsymbol{p}}}_{\text{normal current}} + \underbrace{\boldsymbol{\nabla} \times \int_{\boldsymbol{p}} f \frac{\hat{\boldsymbol{p}}}{2|\boldsymbol{p}|}}_{\text{magnetization current}}$$

equals to

$$\boldsymbol{J} = -\frac{\boldsymbol{\omega}}{2}\left(\frac{1}{3}+\frac{2}{3}\right)\int_{\boldsymbol{p}} \frac{\partial f(\mathcal{E})}{\partial \mathcal{E}} = \frac{\boldsymbol{\omega}}{4\pi^2}\int_0^\infty f\, 2p dp.$$

– the same as in the rotating frame.
As it should be, by Lorentz covariance of $\boldsymbol{J}$.

Collisions and Lorentz invariance

- To achieve equilibration we need collisions.

Collisions and Lorentz invariance

- To achieve equilibration we need collisions.
- Lorentz covariant form of kinetic equation for *scalar* particles:

$$\partial_\mu j^\mu = \int_{BCD} \underbrace{C_{ABCD}}_{W_{CD\to AB} - W_{AB\to CD}}$$

$$j^\mu = p^\mu f$$

Collisions and Lorentz invariance

- To achieve equilibration we need collisions.
- Lorentz covariant form of kinetic equation for *scalar* particles:

$$\partial_\mu j^\mu = \int_{BCD} \underbrace{C_{ABCD}}_{W_{CD\to AB}-W_{AB\to CD}}$$

$$j^\mu = p^\mu f$$

- Spin requires additional ingredients:
- f is not a Lorentz scalar (particle positions are frame-dependent).
- Side jump during collisions.

Position and spin in relativistic mechanics

CM of a spinning body has a known ambiguity in *classical* relativity.

Total angular momentum is well-defined, but not CM position and spin:

$$J^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu + S^{\mu\nu},$$

Ambiguity: shift $x \to x + \Delta$ and $S^{\mu\nu} \to S^{\mu\nu} + \Delta^\nu p^\mu - \Delta^\mu p^\nu$.

Position and spin in relativistic mechanics

CM of a spinning body has a known ambiguity in *classical* relativity.

Total angular momentum is well-defined, but not CM position and spin:

$$J^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu + S^{\mu\nu},$$

Ambiguity: shift $x \to x + \Delta$ and $S^{\mu\nu} \to S^{\mu\nu} + \Delta^\nu p^\mu - \Delta^\mu p^\nu$.

$p_\mu S^{\mu\nu} = 0$, i.e., $S^{0i} = 0$ in rest frame, is insufficient to fix it for $m = 0$.

Together with

$$n_\nu S^{\mu\nu} = 0$$

is sufficient, but requires choice of frame n (where $S^{0i} = 0$, and $\boldsymbol{s} \parallel \boldsymbol{p}$ then, by $p_\mu S^{\mu\nu} = 0$).

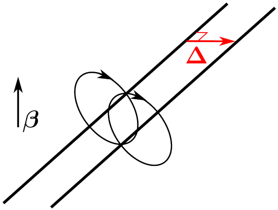
Spin and side jump

The two conditions $p_\mu S^{\mu\nu} = 0$ and $n_\nu S^{\mu\nu} = 0$ determine

$$S_n^{\mu\nu} = \lambda \, \frac{\epsilon^{\mu\nu\alpha\beta} p_\alpha n_\beta}{p \cdot n}$$

and the side jump of the position in frame n' relative to n:

$$\Delta^\mu_{nn'} = -\frac{S^{\mu\nu}_{n'} n_\nu}{p \cdot n} = \lambda \, \frac{\epsilon^{\mu\alpha\beta\gamma} p_\alpha n_\beta n'_\gamma}{(p \cdot n)(p \cdot n')} \, .$$



(this is finite boost generalization of side jump.

(Chen *et al* PRL 115(2015)021601)

Correspondingly: $f_n(x) = f_{n'}(x + \Delta_{nn'})$.

Collisionfull CKT

Chen *et al*

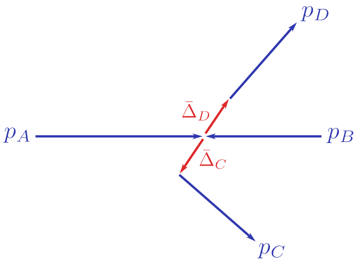
$$\partial_\mu j^\mu = \int_{BCD} \underbrace{C_{ABCD}}_{W_{CD\to AB} - W_{AB\to CD}}$$

$$j^\mu = \underbrace{p^\mu f}_{\text{normal current}} + \underbrace{S^{\mu\nu}\partial_\nu f}_{\text{magnetization current}} + \underbrace{\int_{BCD} C_{ABCD}\,\bar\Delta^\mu}_{\text{jump current}}$$

f, $S^{\mu\nu}$ and $\bar\Delta$ depend on n

but

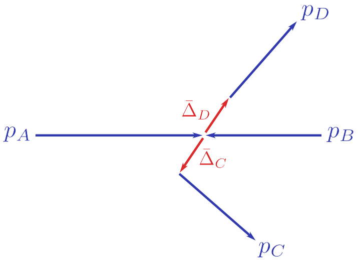
j^μ is frame-independent!



Side jump

$$\bar{\Delta} \equiv \Delta_{n\bar{n}} = \lambda \frac{p \wedge n \wedge \bar{n}}{(p \cdot n)(p \cdot \bar{n})}$$

is a side jump associated with boost from Lab (n) to CM frame ($\bar{n}$) of the collision.



Collision side jumps are essential for $\boldsymbol{L} + \boldsymbol{S}$ conservation (and Lorentz invariance) in the process of $\boldsymbol{L} \leftrightarrow \boldsymbol{S}$ transfer, i.e., spin relaxation/polarization phenomena.

Side jumps also present for $m \neq 0$, since they are required by $\boldsymbol{L} + \boldsymbol{S}$ conservation.

Recent progress

- Derivation of covariant CKT (also spin kinetics for $m \neq 0$) using Wigner function formalism (collisionless) and Kadanoff-Baym (including collisions):
  Hidaka-Pu-Yang, Huang-Shi-Jiang-Liao-Zhuang, Gao-Liang-Wang-Wang, Weickgenannt-Sheng-Speranza-Wang-Rischke, . . .
- In worldline formalism: *Mueler-Venugopalan*
- Derivation of hydrodynamics from CKT: *Shi-Gale-Jeon*
  Interesting to extend to $m \neq 0$, calculate relaxation rates/kinetic coefficients.
- Spin equilibration: *Li-Yee, Kapusta-Rrapaj-Rudaz, . . .*
- Shear-induced polarization (related to magnetization current) appears essential to understanding the Λ "sign puzzle":
  Liu-Yin, Becattini-Buzzegoli-Palermo, Fu-Liu-Pang-Song-Yin, . . .

H-theorem

Covariant entropy current:

$$\mathcal{H}^\mu = p^\mu \mathcal{H} + S^{\mu\nu}\partial_\nu \mathcal{H} \; + \int\limits_{BCD} C_{ABCD}\, \bar{\Delta}^\mu \,\frac{\partial \mathcal{H}}{\partial f}$$

$$\mathcal{H} = f \ln \frac{1}{f} + (1-f)\ln\frac{1}{1-f} \quad \Leftrightarrow \quad \partial_\mu \mathcal{H}^\mu \geq 0$$

H-theorem

Covariant entropy current:

$$\mathcal{H}^\mu = p^\mu \mathcal{H} + S^{\mu\nu}\partial_\nu \mathcal{H} \; + \int_{BCD} C_{ABCD}\, \bar{\Delta}^\mu \, \frac{\partial \mathcal{H}}{\partial f}$$

$$\mathcal{H} = f \ln \frac{1}{f} + (1-f)\ln\frac{1}{1-f} \quad \Leftrightarrow \quad \partial_\mu \mathcal{H}^\mu \geq 0$$

Equilibrium solution:

$$\partial_\mu \mathcal{H}^\mu = 0 \quad \Leftrightarrow \quad \ln\frac{1-f}{f} = \underbrace{\beta\left(p\cdot u + \frac{1}{2}S^{\alpha\beta}\omega_{\alpha\beta} - \mu\right)}_{\text{linear comb. of conserved quantities}}$$

This is uniformly rotating, locally FD, distribution.

CVE coefficients

- Charge current: $J_\mu = \sum_q \int_p q j^\mu = n_q u^\mu + \xi \omega^\mu$.

  At finite T, with a/p: $$\xi = \frac{\mu^2}{4\pi^2} + \frac{T^2}{12}.$$

CVE coefficients

- Charge current: $J_\mu = \sum_q \int_p q j^\mu = n_q u^\mu + \xi \omega^\mu$.

  At finite T, with a/p: $$\xi = \frac{\mu^2}{4\pi^2} + \frac{T^2}{12}.$$

- Energy-momentum (conserved and covariant):

$$T^{\mu\nu} = \sum_q \int_p \frac{1}{2}(p^\mu j^\nu + p^\nu j^\mu)$$

  Energy flow (momentum density) $\boldsymbol{P} = \xi_T \, \boldsymbol{\omega}$ with

$$\xi_T = \frac{\mu^3}{6\pi^2} + \frac{\mu T^2}{6}\,,$$ (2 $\times$ Vilenkin's ξ_T)

CVE coefficients

- Charge current: $J_\mu = \sum_q \int_p q j^\mu = n_q u^\mu + \xi\omega^\mu$.

  At finite T, with a/p: $\xi = \dfrac{\mu^2}{4\pi^2} + \dfrac{T^2}{12}$.

- Energy-momentum (conserved and covariant):

$$T^{\mu\nu} = \sum_q \int_p \frac{1}{2}(\, p^\mu j^\nu + p^\nu j^\mu \,)$$

  Energy flow (momentum density) $\boldsymbol{P} = \xi_T\, \boldsymbol{\omega}$ with

$$\xi_T = \frac{\mu^3}{6\pi^2} + \frac{\mu T^2}{6}\,, \qquad \text{(2 } \times \text{ Vilenkin's } \xi_T\text{)}$$

- Entropy: $\boldsymbol{H} = \xi_H\, \boldsymbol{\omega}$ with $\xi_H = \dfrac{\mu T}{6}$.

Flows and drag

MS-Yee, PRL116(2016)122302

Static obstacle (impurity) at rest in some frame U:

$$\partial \cdot j = \mathcal{C} + \mathcal{C}_U; \qquad \mathcal{C}_U = \int_{AB} C_{AB}$$

$$C_{AB} = W_{B\to A} - W_{A\to B}$$

Drag (momentum transfer from impurity):

$$F^\mu = \int_{AB} C_{AB}\,(p_A - p_B)^\mu$$

vanishes for the solution with $u = U$.

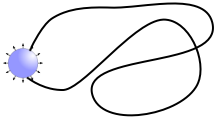
No drag, but there are flows of charge, energy (as in superfluid) but also entropy!

Summary/Conclusions

- Spin adds $\mathcal{O}(\hbar)$ terms to EOMs:
  Berry curvature and magnetic mom.

$$\dot{\boldsymbol{x}} = \frac{\partial \mathcal{E}}{\partial \boldsymbol{p}} + \dot{\boldsymbol{p}} \times \boldsymbol{b}$$

- Berry monopole accounts for CME
  and anomaly (source/sink at $\boldsymbol{p} = \boldsymbol{0}$).



$$\boldsymbol{B} \to 2\mathcal{E}\boldsymbol{\omega}$$

- CVE from CKT in two ways:
  rotating frame or rotating distribution

$$\boldsymbol{J} = \underset{1/3}{\langle \dot{\boldsymbol{x}} \rangle} + \underset{2/3}{\boldsymbol{\nabla} \times \boldsymbol{M}}$$

- Nontrivial Lorentz invariance:
  side jump to conserve $\boldsymbol{L} + \boldsymbol{S}$;
  and requires $\Delta\mathcal{E} = -\boldsymbol{m} \cdot \boldsymbol{B}$.

$$\delta \boldsymbol{x} = \boldsymbol{\beta} t + \frac{\boldsymbol{\beta} \times \hat{\boldsymbol{p}}}{2|\boldsymbol{p}|}$$

- Lorentz invariance requires
  side jumps in collision kernel
  and jump currents