Spin and chirality in hydrodynamics

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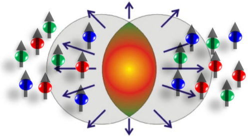

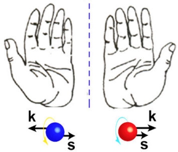

The 6th International Conference on Chirality, Vorticity and Magnetic Field in Heavy Ion Collisions

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Outline

Introduction

- ⋄ Spin polarization and spin transports
- ⋄ Chiral transport phenomena

Construction of spin hydrodynamics

- ⋄ Subtleties in constructing spin hydrodynamics
- ⋄ Spin hydrodynamic constitutive relations
- ⋄ Construction of chiral hydrodynamics

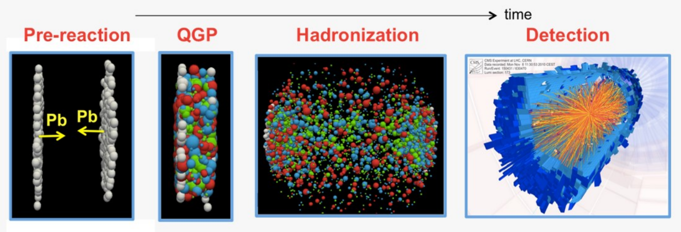
Spin Cooper-Frye formula

Summary

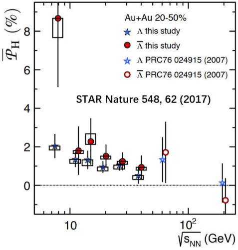
Introduction

Spin polarization/transport phenomena

- Spin polarization and spin transport phenomena are very common:
  - Heavy ion collisions (HICs) and quark gluon plasma (QGP)



  - Hyperon spin polarization

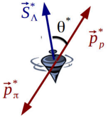


parity-violating decay of hyperons

In case of Λ's decay, daughter proton preferentially decays in the direction of Λ's spin (opposite for anti-Λ)

$$\frac{dN}{d\Omega^*} = \frac{1}{4\pi}(1 + \alpha \mathbf{P}_{\mathbf{\Lambda}} \cdot \mathbf{p}_{\mathbf{p}}^*)$$

α: Λ decay parameter (α_Λ=0.732)
P_Λ: Λ polarization
p_p^*: proton momentum in Λ rest frame

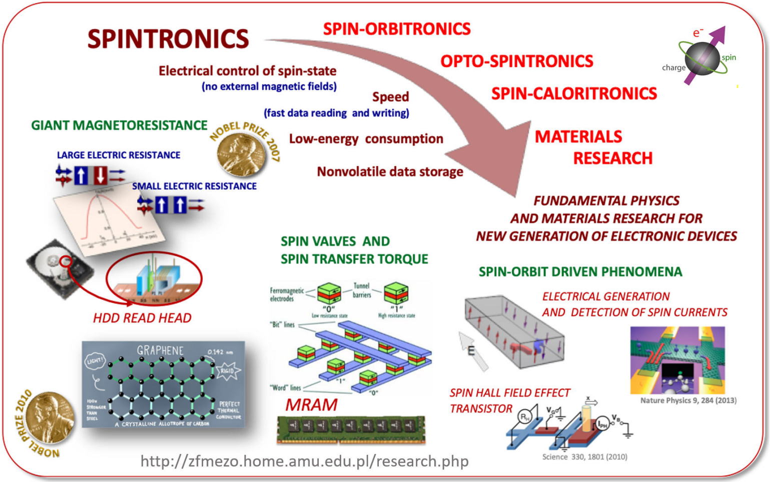


$$\Lambda \to p + \pi^+$$

(BR: 63.9%, cτ~7.9 cm)

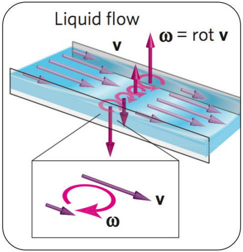
Spin polarization/transport phenomena

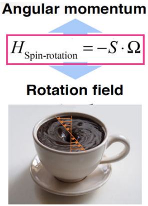
- Spin polarization and spin transport phenomena are very common:
  - Spintronics in solid materials



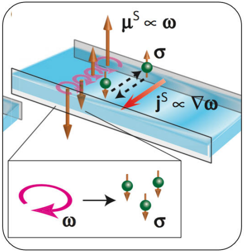
Spin polarization/transport phenomena

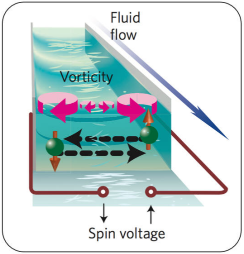
- Spin polarization and spin transport phenomena are very common:
  - Spintronics in liquid materials





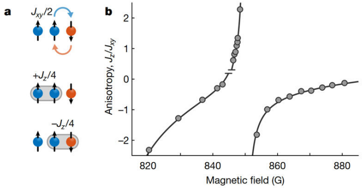
Takahashi et al. Nature Physics 12, 52 (2016)



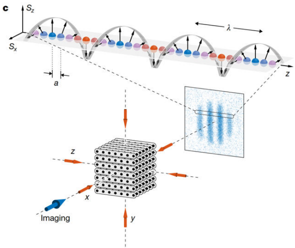


Spin polarization/transport phenomena

- Spin polarization and spin transport phenomena are very common:
  - Cold atoms on optic lattice



Jepsen et al. Nature 288, 403 (2020)



Heisenberg XXZ model by Li7 atoms

$$H=\sum_{\langle ii\rangle}\left[J_{xy}\left(S_i^xS_j^x+S_i^yS_j^y\right)+J_zS_i^zS_j^z\right]$$

Chiral transport phenomena

- Chiral magnetic effect and its cousins:
  - Chiral magnetic and separation effects:

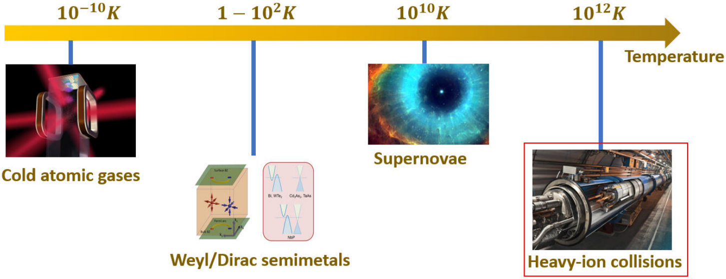
$$\boldsymbol{J} = \frac{\mu_5}{2\pi^2}\boldsymbol{B}, \quad \boldsymbol{J}_5 = \frac{\mu}{2\pi^2}\boldsymbol{B}$$

  - Chiral vortical effects:

$$\boldsymbol{J} = \frac{\mu\mu_5}{\pi^2}\boldsymbol{\omega}, \quad \boldsymbol{J}_5 = \left(\frac{\mu^2+\mu_5^2}{2\pi^2} + \frac{T^2}{6}\right)\boldsymbol{\omega}$$

  - ......

- Chiral transports are common:



Construction of spin hydrodynamics

Hydrodynamics

- Long-time large-distance effective theory of conserved quantities (hydrodynamic modes).
  - Non-hydro modes relax at a finite time scale $\tau = 1/\Gamma$. Hydro modes relax at $\tau_{\rm hydro} = 1/\omega_{\rm hydro}(k) \to \infty$ when $k \to 0$.
  - Hydrodynamics is constructed using derivative expansion.
  - Typical hydro modes: energy density, momentum density, baryon charge density, $\cdots$.
- If spin is conserved, the hydro equations would be given by:

$$\text{Charge conservation}: \quad \partial_\mu J^\mu(x) = 0,$$
$$\text{Energy} - \text{momentum conservation}: \quad \partial_\mu \Theta^{\mu\nu}(x) = 0,$$
$$\text{Spin conservation}: \quad \partial_\mu \Sigma^{\mu\nu\rho}(x) = 0,$$

with J^μ, $\Theta^{\mu\nu}$, and $\Sigma^{\mu\nu\rho}$ expanded order by order in gradient giving the constitutive relations, e.g.,

$$J^\mu = n u^\mu + O(\partial),$$
$$\Theta^{\mu\nu} = (\epsilon + p) u^\mu u^\nu + p\eta^{\mu\nu} + O(\partial),$$
$$\Sigma^{\mu\nu\rho} = \sigma^{\nu\rho} u^\mu + O(\partial)$$

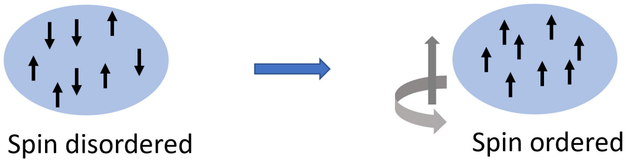
where $O(1)$ terms usually correspond to ideal hydrodynamics.

Can spin be a true hydro mode?

- But, spin is not conserved, only the total angular momentum is:

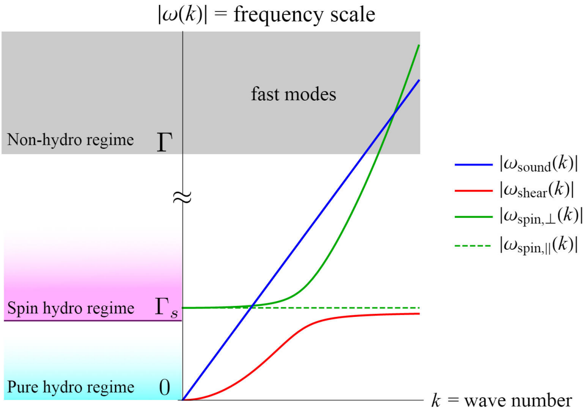
$$\partial_\mu J^{\mu\nu\rho} = 0,$$
$$J^{\mu\nu\rho} = x^\nu \Theta^{\mu\rho} - x^\rho \Theta^{\mu\nu} + \Sigma^{\mu\nu\rho}$$
$$\Rightarrow \qquad \partial_\mu \Sigma^{\mu\nu\rho}(x) = \Theta^{\rho\nu} - \Theta^{\nu\rho}$$

- Thus spin is a true hydro mode (conserved quantity) only when $\Theta^{\mu\nu}$ is symmetric.
  - In general, not possible. The anti-symmetric part of $\Theta^{\mu\nu}$ is a torque acting on spin.
  - Such torque is spin-orbit coupling (SOC). For Dirac fermions, SOC $\propto 1/m$ and thus vanishes at heavy fermion limit. $\Rightarrow$ Spin can be true hydro mode at non-relativistic limit.
- The transfer of AM between spin part and orbital part is generally dissipative.



The spin hydro regime

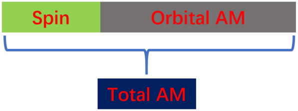
- When spin relaxation rate $\Gamma_s \ll$ relaxation rate of other heavy modes Γ, we can formulate an extended hydro framework for true hydro modes and spin modes.
  $\Rightarrow$ Relativistic dissipative spin hydrodynamics



Hongo, XGH, Kaminski, Stephanov, and Yee 2107.14231

Ambiguity in defining spin current

- The definition of spin current $\Sigma^{\mu\nu\rho}$ is ambiguous.



- Pseudo-gauge transformation: Transformations that preserve total AM and the conservation laws (Becattini, Florkowski, and Speranza 1807.10994, Florkowski, Ryblewski, and Kumar 1811.04409)

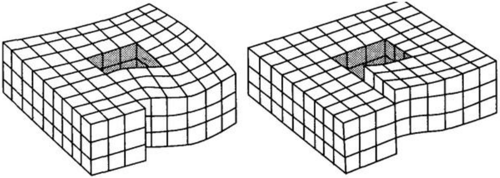
$$\Sigma^{\mu\nu\rho} \to \Sigma^{\mu\nu\rho} - \Phi^{\mu\nu\rho},$$

$$\Theta^{\mu\nu} \to \Theta^{\mu\nu} + \frac{1}{2}\partial_\lambda \left(\Phi^{\lambda\mu\nu} - \Phi^{\mu\lambda\nu} - \Phi^{\nu\lambda\mu}\right)$$

- Formulation of spin hydro depends on the choice of pseudo-gauge:
  - Non-anti-symmetric gauge, $\Sigma^{\mu\nu\rho} = u^\mu \sigma^{\nu\rho} + \cdots$ (Florkowski *et al.* 1705.00587, Montenegro *et al.* 1701.08263, Hattori *et al.* 1901.06615, Gallegos *et al.* 2101.04759, She *et al.* 2105.04060, $\cdots$)
  - Anti-symmetric gauge $\Sigma^{\mu\nu\rho} = \varepsilon^{\mu\nu\rho\gamma}\sigma_\gamma$ (Hongo *et al.* 2107.14231, Bhadury *et al.* 2002.03937, $\cdots$)
  - Relation between different pseudo-gauges (Li, Stephanov, and Yee 2011.12318, Fukushima and Pu 2010.01608)

Ambiguity in defining spin current

- Let us fix the pseudo-gauge by torsion. The spin current and energy-momentum tensor defined as gauge currents.
  - The currents are gauge invariant, diffeomorphism and local Lorentz covariant.
  - Conservation laws become the Ward-Takahashi identities of diffeomorphism and local Lorentz invariance of the theory.
  - Torsion may be realized in some solid materials as lattice defects.



edge dislocation   screw dislocation

- Reminder: If a global symmetry G of $S[\varphi]$ can be gauged into $S[\varphi, A]$, the gauge current of G is

$$J^{\mu} = \frac{\delta S[\varphi, A]}{\delta A_{\mu}}$$

Spin current and energy-momentum tensor

- For Poincare group with translation P^a and Lorentz transformation M^{ab}, it can be gauged by promoting

$$\partial_\mu \to D_\mu = \partial_\mu - i c_\mu{}^a P_a - \frac{i}{2}\omega_\mu{}^{ab} M_{ab}$$

$$\Rightarrow \qquad [D_\mu, D_\nu] = -i T^a{}_{\mu\nu} P_a - \frac{i}{2} R^{ab}{}_{\mu\nu} M_{ab}$$

- The geometric interpretation of $e_\mu{}^a = \delta_\mu^a + c_\mu{}^a, \omega_\mu{}^{ab}, T^a{}_{\mu\nu}, R^{ab}{}_{\mu\nu}$ are vierbein, spin connection, torsion, and curvature tensor of Einstein-Cartan spacetime (Sciama 1962, Kibble 1961, Hehl etal 1976)
- Energy-momentum tensor and spin current:

$$\Theta^\mu{}_a(x) \equiv \frac{1}{e(x)} \left.\frac{\delta S}{\delta e_\mu{}^a(x)}\right|_\omega, \quad \Sigma^\mu{}_{ab}(x) \equiv -\frac{2}{e(x)} \left.\frac{\delta S}{\delta \omega_\mu{}^{ab}(x)}\right|_e$$

- For QCD

$$\Theta^\mu{}_a = \frac{1}{2}\bar{q}\big(\gamma^\mu \overrightarrow{D}_a - \overleftarrow{D}_a \gamma^\mu\big) q + 2\mathrm{tr}\left(G^{\mu\rho} G_{a\rho}\right) + \mathcal{L}_{\mathrm{QCD}} e_a{}^\mu,$$

$$\Sigma^\mu{}_{ab} = -\frac{i}{2}\bar{q} e^\mu{}_c \{\gamma^c, \Sigma_{ab}\} q$$

Spin current and energy-momentum tensor

- This spin current is totally anti-symmetric so it contains 3 independent variables corresponding to 3 spin orientations. It is also determined by axial current

$$\Sigma^{\mu}{}_{ab} = -\frac{1}{2}\varepsilon^{\mu}{}_{abc}J_5^c$$

- Diffeomorphism and local Lorentz invanriance give Ward-Takahashi identities ($G_\mu = T^{\nu}{}_{\nu\mu}$)

$$(D_\mu - G_\mu)\Theta^{\mu}{}_{a} = -\Theta^{\mu}{}_{b}T^{b}{}_{\mu a} + \frac{1}{2}\Sigma^{\mu}{}_{b}{}^{c}R^{b}{}_{c\mu a} + F_{a\mu}J^{\mu},$$
$$(D_\mu - G_\mu)\Sigma^{\mu}{}_{ab} = -(\Theta_{ab} - \Theta_{ba})$$

- The equations of motion for vector and axial charges

$$(D_\mu - G_\mu)J^{\mu} = 0,$$
$$(D_\mu - G_\mu)J_5^{\mu} = m\mathcal{P} + \mathcal{A}$$

- Two interesting limits: heavy quark limit $\Rightarrow$ Spin hydro. Light quark limit $\Rightarrow$ chiral hydro.

Derivation of spin hydro

- Step 1: Identify (quasi-)hydro modes
  - Eight (quasi-)hydro variables: $\epsilon, n, u^a, \sigma_{ab}$ (or $\sigma_a = \varepsilon^{abcd} u_b \sigma_{cd}/2$) with constraints $u^2 = -1$, $\sigma^a u_a = \sigma_{ab} u^b = 0$.
  - Local first law of thermodynamics: $s = \beta(\epsilon + P - \mu n - \mu_{ab}\sigma^{ab}/2)$ and $Tds = d\epsilon - \mu dn - \mu^{ab} d\sigma_{ab}/2$.
  - Conjugate variables: inverse temperature $\beta \equiv \frac{\partial s}{\partial \epsilon}$, chemical potentials $\mu = \frac{\partial s}{\partial n}$, $\mu^{ab} = -\frac{T}{2}\frac{\partial s}{\partial \sigma_{ab}}$.
  - Power counting scheme

$$\{\beta, n, u^a, e_\mu{}^a\} = O(\partial^0) \quad \text{and} \quad \{\mu^{ab}, \sigma_{ab}, \omega_\mu{}^{ab}\} = O(\partial)$$

- Step 2: Tensor decomposition

$$\Theta^\mu{}_a = \epsilon u^\mu u_a + p\Delta^\mu_a + u^\mu \delta q_a - \delta q^\mu u_a + \delta\Theta^\mu{}_a,$$
$$\Sigma^\mu{}_{ab} = \varepsilon^\mu{}_{abc}(\sigma^c + \delta\sigma u^c)$$

- Step 3: Calculate divergence of entropy current

$$(\nabla_\mu - G_\mu)s^\mu = (\nabla_\mu - G_\mu)(\delta s^\mu + \beta\mu\delta J^\mu) - \delta\Theta^\mu{}_a\big|_{(s)}(D_\mu\beta^a - T^a_{\mu b}\beta^b)$$
$$- \delta\Theta^\mu{}_a\big|_{(a)}(D_\mu\beta^a - T^a_{\mu b}\beta^b - \beta\mu_\mu{}^a) - \delta J^\mu[\nabla_\mu(\beta\mu) - F_{\mu\nu}\beta^\nu] + O(\partial^3)$$

Derivation of spin hydro

- Step 4: Second law of local thermodynamics $(\nabla_\mu - G_\mu)s^\mu \geq 0$

$$\delta\Theta^\mu{}_a\big|_{(s)} = -\eta^{\mu}{}_a{}^\nu{}_b(D_\nu u^b - T^b{}_{\nu c}u^c),$$

$$\delta\Theta^\mu{}_a\big|_{(a)} = -(\eta_s)^{\mu}{}_a{}^\nu{}_b(D_\nu u^b - T^b{}_{\nu c}u^c - \mu_\nu{}^b)$$

$$\eta^{\mu}{}_a{}^\nu{}_b = 2\eta\left(\frac{1}{2}(\Delta^{\mu\nu}\Delta_{ab} + \Delta^\mu_b\Delta^\nu_a) - \frac{1}{3}\Delta^\mu_a\Delta^\nu_b\right) + \zeta\Delta^\mu_a\Delta^\nu_b,$$

$$(\eta_s)^{\mu}{}_a{}^\nu{}_b = \frac{1}{2}\eta_s(\Delta^{\mu\nu}\Delta_{ab} - \Delta^\mu_b\Delta^\nu_a).$$

  with $\eta \geq 0$ shear, $\zeta \geq 0$ bulk, and $\eta_s \geq 0$ rotational viscosities.
- This completes the derivation of first order dissipative spin hydro.
- In order to solve the spin hydrodynamic equations, an equation of state $p = p(\epsilon, n, \sigma_{ab})$ should be given.
- Interesting Kubo formula:

$$\begin{aligned}\eta_s &= 2\lim_{\Gamma_s \ll \omega \ll \Gamma} \frac{1}{\omega}\mathrm{Im}\tilde{G}_{\mathrm{R}}^{\Theta^x{}_y|_{(a)},\Theta^x{}_y|_{(a)}}(\omega, \boldsymbol{k} = \mathbf{0}) \\ &= \frac{\chi_s^2}{2}\left[\lim_{\omega\to 0}\frac{1}{\omega}\mathrm{Im}\tilde{G}_{\mathrm{R}}^{\Sigma^{0x}{}_y,\Sigma^{0x}{}_y}(\omega, \boldsymbol{k} = \mathbf{0})\right]^{-1}\end{aligned}$$

Hongo, XGH, Kaminski, Stephanov, and Yee 2107.14231

Chiral hydrodynamics

- The chiral hydro can be similarly obtained with dynamics of J_5^μ turned on and spin dynamics turned off (Son and Surowka 0906.5044, $\cdots$)

$$(\nabla_\mu - G_\mu)s^\mu = (\nabla_\mu - G_\mu)(\delta s^\mu + \beta\mu\delta J^\mu + \beta\mu_5\delta J_5^\mu) - \delta\Theta^\mu{}_a\Big|_{(s)}(D_\mu\beta^a - T^a{}_{\mu b}\beta^b) - \delta J^\mu[\nabla_\mu(\beta\mu) - F_{\mu\nu}\beta^\nu] - \delta J_5^\mu\nabla_\mu(\beta\mu_5) + \beta\mu_5\mathcal{A} + O(\partial^3)$$

- It is not positive definite with

$$\mathcal{A} = \frac{C_F}{4}F_{\mu\nu}\tilde{F}^{\mu\nu} + C_T\varepsilon^{\mu\nu\rho\sigma}\left[\tilde{R}_{\mu\nu\rho\sigma} + \frac{1}{2}T^\lambda{}_{\mu\nu}T_{\lambda\rho\sigma}\right] + O(\partial^3)$$

- The striking observation by Son and Surowka is that second law is recovered with δJ^μ, δJ_5^μ, and δs^μ allowing parity violating contributions, e.g.

$$\delta J_5^\mu|_{\text{anom}} = \lambda_{5B}B^\mu + \lambda_{5\omega}\omega^\mu + \lambda_{5T}\varepsilon^{\mu\nu\rho\sigma}T_{\nu\rho\sigma}$$

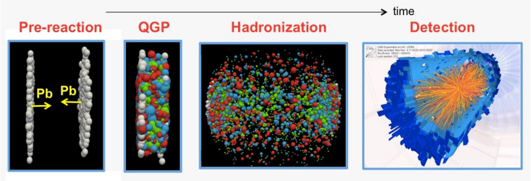
- The coefficients are also fixed by second law of thermodynamics

$$\lambda_{5B} = C_F\mu, \;\; \lambda_{5\omega} = \left[C_F(\mu^2 + \mu_5^2) + \frac{T^2}{6}\right], \;\; \lambda_{5T} = C_T$$

Spin Cooper-Frye formula

Freeze-out of particle number

- Hydrodynamics describes the bulk evolution of the hot medium.

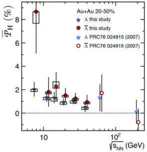


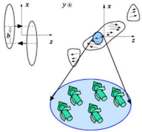
- Cooper-Frye type formula converts hydro outcomes to momentum space distributions.

$$N(p) = \int d\Xi_\mu \frac{p^\mu}{E_p} f(T(x), u^\mu(x), \mu(x))$$

- We need a similar formula to connect spin hydro with obervables.

$$\bar{S}^\mu(p) \qquad \Leftarrow \qquad T(x), u^\alpha(x), \mu(x), \mu_{\alpha\beta}(x)$$





Freeze-out of spin polarization

- Such a formula can be obtained via e.g. kinetic theory or local Gibbs density operator with same type of pseudo-gauge as spin hydro

$$\hat{\rho}_{\rm LE} = \frac{1}{Z_{\rm LE}} \exp\left\{ - \int d\Xi_\mu(y) \big[\hat{\Theta}^{\mu\nu}(y)\beta_\nu(y) + \hat{J}^\mu(y)\alpha(y) - \frac{1}{2}\hat{\Sigma}^{\mu\rho\sigma}(y)\mu_{\rho\sigma}(y) \big] \right\}$$

- Spin Cooper-Frye formula for Dirac fermions (Buzzegoli 2109.12084, Liu and XGH 2109.15301)

$$\bar{S}_\mu(p) = \bar{S}_{5\mu}(p) - \frac{1}{8\int d\Xi\cdot p \;\; n_F} \int d\Xi\cdot p \frac{n_F(1-n_F)}{E_p} \times \left\{ \epsilon_{\mu\nu\alpha\beta} p^\nu \mu^{\alpha\beta} + 2\frac{\varepsilon_{\mu\nu\rho\sigma}p^\rho n^\sigma}{p\cdot n} \left[p_\lambda(\xi^{\nu\lambda} + \Delta\mu^{\nu\lambda}) + \partial^\nu\alpha \right] \right\}$$

  - Here, $\xi_{\mu\nu} = \partial_{(\mu}\beta_{\nu)}$ is thermal shear and $\Delta\mu_{\alpha\beta} = \mu_{\alpha\beta} + \partial_{[\mu}\beta_{\nu]}$ is the difference between spin chemical potential and thermal vorticity.

Freeze-out of spin polarization

- Spin Cooper-Frye formula for Dirac fermions

$$\bar{S}_\mu(p) = \bar{S}_{5\mu}(p) - \frac{1}{8\int d\Xi \cdot p \;\; n_F} \int d\Xi \cdot p \frac{n_F(1-n_F)}{E_p}$$
$$\times \left\{ \epsilon_{\mu\nu\alpha\beta} p^\nu \mu^{\alpha\beta} + 2\frac{\varepsilon_{\mu\nu\rho\sigma} p^\rho n^\sigma}{p \cdot n} \left[p_\lambda (\xi^{\nu\lambda} + \Delta\mu^{\nu\lambda}) + \partial^\nu \alpha \right] \right\}$$

  - $\bar{S}_5^\mu$ is the polarization induced by finite chirality (Liu *et al.* 2002.03753, Shi *et al.* 2008.08618, Buzzegoli *et al.* 2009.13449, Gao 2105.08293)
  - When $\Delta\mu_{\alpha\beta} = 0$, namely, when spin chemical potential is given by thermal vorticity. It goes to previous results (Liu and Yin 2103.09200, Becattini, Buzzegoli, and Palermo 2103.10917)
  - When global equilibrium is reached$\Delta\mu_{\alpha\beta} = 0 = \xi_{\alpha\beta}$, it goes to previous results (Becattini *et al.* 1303.3431, Fang *et al.* 1604.04036, Liu *et al.* 2002.03753)
  - It is accurate at $O(\partial)$.
  - n^μ is a unit frame vector to specify helicity.
- With this formula, we can convert spin hydro into momentum space spin polarization.

Summary

Summary

- Spin polarization of hyperons are measured in heavy ion collision experiments.
- It is possible to formulate a hydrodynamic theory for spin and chirality transports.
- The first-order dissipative spin hydrodynamics has been constructed.
- The Cooper-Frye type spin polarization formula is known.

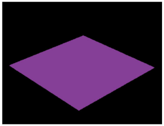
- Numerical spin hydrodynamics.
- Higher-order and causal spin hydrodynamics.
- Anomalous spin hydrodynamics.
- Calculation of rotational viscosity.
- ......

Thank you!

Backups

Linearized spin hydrodynamics

- Perturbation about global static thermal equilibrium

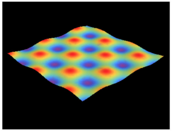


$\epsilon = \epsilon_0$

$u^\mu = (1, \mathbf{0})$

$\sigma^a = 0$

perturb $\Rightarrow$



$\epsilon = \epsilon_0 + \delta\epsilon$

$u^\mu = (1, \mathbf{0}) + \delta u^\mu$

$\sigma^a = 0 + \delta\sigma^a$

$$0 = \partial_0 \delta\epsilon + \partial_i \delta\pi^i,$$

$$0 = \partial_0 \delta\pi_i + c_s^2 \partial_i \delta\epsilon - \gamma_\parallel \partial_i \partial^j \delta\pi_j - (\gamma_\perp + \gamma_s)(\delta_i^j \nabla^2 - \partial_i \partial^j)\delta\pi_j + \frac{1}{2}\Gamma_s \varepsilon_{0ijk} \partial^j \delta\sigma^k,$$

$$0 = \partial_0 \delta\sigma_i + \Gamma_s \delta\sigma_i + 2\gamma_s \varepsilon_{0ijk} \partial^j \delta\pi^k$$

where we introduced a set of static/kinetic coefficients as

$$c_s^2 \equiv \frac{\partial p}{\partial \epsilon}, \quad \gamma_\parallel \equiv \frac{1}{\epsilon_0 + p_0}\left(\zeta + \frac{4}{3}\eta\right), \quad \gamma_\perp \equiv \frac{\eta}{\epsilon_0 + p_0},$$

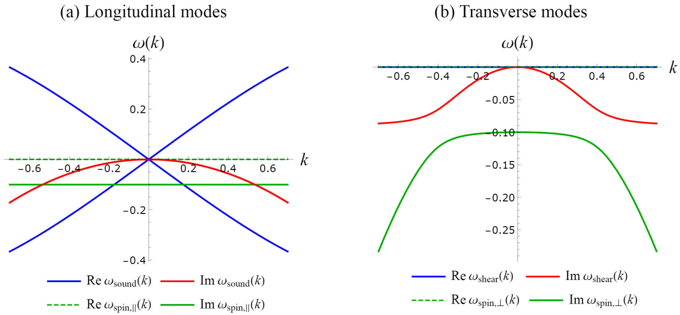
$$\chi_s \delta_{ij} \equiv \frac{\partial \sigma_i}{\partial \mu^j}, \quad \gamma_s \equiv \frac{\eta_s}{2(\epsilon_0 + p_0)}, \quad \Gamma_s \equiv \frac{2\eta_s}{\chi_s}$$

- One can obtain the dispersion relation of (quasi-)hydro modes and hydrodynamic correlation functions.

Linearized spin hydrodynamics

- Dispersion relations

$$\begin{cases} \bullet \text{ One pair of sound modes}: \ \omega_{\text{sound}}(\boldsymbol{k}) = \pm c_s|\boldsymbol{k}| - \frac{i}{2}\gamma_{\|}\boldsymbol{k}^2 + O(\boldsymbol{k}^3), \\ \bullet \text{ One longitudinal spin mode}: \ \omega_{\text{spin},\|}(\boldsymbol{k}) = -i\Gamma_s, \\ \bullet \text{ Two shear modes}: \ \omega_{\text{shear}}(\boldsymbol{k}) = -i\gamma_\perp\boldsymbol{k}^2 + O(\boldsymbol{k}^4), \\ \bullet \text{ Two transverse spin modes}: \ \omega_{\text{spin},\perp}(\boldsymbol{k}) = -i\Gamma_s - i\gamma_s\boldsymbol{k}^2 + O(\boldsymbol{k}^4). \end{cases}$$



- Mode mixing between shear and transverse spin mode: One gradient can affect two modes.

Green's functions

- Spin hydrodynamic retarded spin-spin correlator

$$\tilde{G}_{\mathrm{R}}^{\sigma^i\sigma^j}(\omega,\boldsymbol{k}) = \frac{i\chi_s\Gamma_s + \cdots}{\omega + i\Gamma_s + O(\boldsymbol{k}^2)}\delta^{ij},$$

$$\lim_{\boldsymbol{k}\to 0}\tilde{G}_{\mathrm{R}}^{\sigma_i\sigma_j}(\omega=0,\boldsymbol{k}) = \chi_s\delta^{ij}$$

- Recall the scale separation condition:

$$\delta\Theta^{\mu}{}_{a}\Big|_{(a)} = \begin{cases} -(\eta_s)^{\mu}{}_{a}{}^{\nu}{}_{b}(\tilde{D}_\nu u^b - u^c K_{c\nu}{}^{b} - \mu_\nu{}^{b}) & \text{when} \quad \Gamma_s \ll \omega \ll \Gamma, \\ 0 & \text{when} \quad \omega \ll \Gamma_s \end{cases}$$

  The spin hydrodynamic spin-spin correlator gives:

$$\omega\tilde{G}_{\mathrm{R}}^{\sigma^i\sigma^j}(\omega,\boldsymbol{k}=0) = \frac{i\chi_s\omega\Gamma_s + O(\omega^2)}{\omega + i\Gamma_s}\delta^{ij} \xrightarrow{\Gamma_s\ll\omega\ll\Gamma} \frac{i\chi_s\omega\Gamma_s}{\omega} = 2i\eta_s$$

- Field theoretical retarded spin-spin correlator: Kubo formula for rotational viscosity

$$\eta_s = \frac{1}{2}\lim_{\Gamma_s\ll\omega\ll\Gamma}\lim_{\boldsymbol{k}\to 0}\omega\mathrm{Im}\tilde{G}_{\mathrm{R}}^{\sigma^z\sigma^z}(\omega,\boldsymbol{k})$$

  It agrees with the linear-mode analysis.