Exact solutions of kinetic equation in the relaxation time approximation

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Motivation

- experimental data collected in ultra-relativistic heavy-ion collisions very well described by viscous hydrodynamics
- a lot of attention brought to the studies of kinetic coefficients, various methods lead to different forms of them
- studied systems are subject to rapid longitudinal expansion, large viscous corrections to the ideal energy-momentum tensor may cause unphysical results
- methods to improve early-time dynamics:
  - complete second-order treatments (Denicol, Niemi, Molnar, Rischke)
  - third-order treatments (El, Xu, Greiner, Jaiswal)
  - anisotropic hydrodynamics (Florkowski, Martinez, Ryblewski, Strickland, Bazow, Heinz, Tinti)
- anisotropic hydrodynamics has various appealing features
- it would be useful to have exactly solvable case with which we could compare various approximation schemes

Motivation

- idea: perform comparisons of exact solutions of simple kinetic equations with approximate hydrodynamic approaches
- goal: judge efficacy of various approximation methods (second-order viscous hydrodynamics, anisotropic hydrodynamics)
- references:
  - W. Florkowski, R. Ryblewski, M. Strickland, Phys. Rev. C88 (2013) 024903
  - W. Florkowski, R. Ryblewski, M. Strickland, Nucl. Phys. A916 (2013) 249
  - W. Florkowski, E. Maksymiuk, R. Ryblewski, M. Strickland, arXiv:1402.7348

Kinetic equation
General setup

- Boltzmann equation in the relaxation time approximation

$$p^\mu \partial_\mu f(x,p) = C[f(x,p)] \qquad C[f] = p^\mu u_\mu \, \frac{f^{\rm eq} - f}{\tau_{\rm eq}}$$

background distribution (Boltzmann statistics)

$$f^{\rm eq} = \frac{g_s}{(2\pi)^3} \exp\left(-\frac{p^\mu u_\mu}{T}\right)$$

- boost-invariant variables (Bialas, Czyz)

$$w = tp_{\parallel} - zE \qquad\qquad v = tE - zp_{\parallel}$$

- one-dimensional boost-invariant evolution

$$\frac{\partial f}{\partial \tau} = \frac{f^{\rm eq} - f}{\tau_{\rm eq}}$$

$$f^{\rm eq}(\tau, w, p_\perp) = \frac{g_s}{(2\pi)^3} \exp\left(-\frac{\sqrt{w^2 + (m^2 + p_\perp^2)\tau^2}}{T\tau}\right)$$

Kinetic equation
Thermodynamic variables

- particle density, energy density, transverse and longitudinal pressure

$$
\begin{aligned}
n(\tau) &= g_0 \int dP \, \frac{v}{\tau} \; f(\tau, w, p_\perp) \\
\mathcal{E}(\tau) &= g_0 \int dP \, \frac{v^2}{\tau^2} \, f(\tau, w, p_\perp) \\
\mathcal{P}_T(\tau) &= g_0 \int dP \, \frac{p_T^2}{2} \, f(\tau, w, p_\perp) \\
\mathcal{P}_L(\tau) &= g_0 \int dP \, \frac{w^2}{\tau^2} \, f(\tau, w, p_\perp)
\end{aligned}
$$

Kinetic equation
Landau matching

$$
\begin{aligned}
T^{\mu\nu} &= (\mathcal{E}+\mathcal{P}_T)u^\mu u^\nu - \mathcal{P}_T g^{\mu\nu} + (\mathcal{P}_L-\mathcal{P}_T)z^\mu z^\nu \\
T^{\mu\nu}_{\rm eq} &= (\mathcal{E}_{\rm eq}+\mathcal{P}_{\rm eq})u^\mu u^\nu - \mathcal{P}_{\rm eq} g^{\mu\nu}
\end{aligned}
$$

$$
T^{\mu\nu}_{\rm eq;LRF} = \begin{pmatrix} \mathcal{E}_{\rm eq} & 0 & 0 & 0 \\ 0 & \mathcal{P}_{\rm eq} & 0 & 0 \\ 0 & 0 & \mathcal{P}_{\rm eq} & 0 \\ 0 & 0 & 0 & \mathcal{P}_{\rm eq} \end{pmatrix}, \quad T^{\mu\nu}_{\rm LRF} = \begin{pmatrix} \mathcal{E} & 0 & 0 & 0 \\ 0 & \mathcal{P}_T & 0 & 0 \\ 0 & 0 & \mathcal{P}_T & 0 \\ 0 & 0 & 0 & \mathcal{P}_L \end{pmatrix}
$$

- determination of effective temperature

$$
\begin{aligned}
u_\mu T^{\mu\nu} &= u_\mu T^{\mu\nu}_{\rm eq} \\
\mathcal{E}(\tau) &= \mathcal{E}^{\rm eq}(\tau) \\
&= g_0 \int dP\, \frac{v^2}{\tau^2}\, f^{\rm eq}(\tau, w, p_\perp) \\
&= \frac{g_0 T m^2}{\pi^2}\left[3TK_2\left(\frac{m}{T}\right) + mK_1\left(\frac{m}{T}\right)\right]
\end{aligned}
$$

Kinetic equation
Formal solution

- formal solution

$$f(\tau, w, p_\perp) = D(\tau, \tau_0) f_0(w, p_\perp) + \int_{\tau_0}^{\tau} \frac{d\tau'}{\tau_{\rm eq}(\tau')} D(\tau, \tau') \, f^{\rm eq}(\tau', w, p_\perp)$$

$$D(\tau_2, \tau_1) = \exp\left[- \int_{\tau_1}^{\tau_2} \frac{d\tau''}{\tau_{\rm eq}(\tau'')} \right]$$

- initial condition (Romatschke-Strickland form)

$$f_0(w, p_\perp) = \frac{g_s}{(2\pi)^3} \exp\left[- \frac{\sqrt{(1+\xi_0) w^2 + (m^2 + p_\perp^2) \tau_0^2}}{\Lambda_0 \, \tau_0} \right]$$

$\xi_0 = \xi(\tau_0)$ - initial value of the anisotropy parameter
$\Lambda_0 = \Lambda(\tau_0)$ - initial transverse-momentum scale

Kinetic equation
Numerical method

$$Tm^2 \left[3TK_2\left(\frac{m}{T}\right) + mK_1\left(\frac{m}{T}\right)\right] = \frac{g_s}{4}\left[D(\tau,\tau_0)\Lambda_0^4\widetilde{\mathcal{H}}_2\left(\frac{\tau_0}{\tau\sqrt{1+\xi_0}}, \frac{m}{\Lambda_0}\right)\right.$$
$$\left. + \int_{\tau_0}^{\tau}\frac{d\tau'}{\tau_{\rm eq}(\tau')}D(\tau,\tau')T'^4\widetilde{\mathcal{H}}_2\left(\frac{\tau'}{\tau}, \frac{m}{T'}\right)\right]$$

- numerical (iterative method)
  1) use a trial function $T' = T(\tau')$ on the RHS of the dynamic equation
  2) the LHS of the dynamic equation determines the new $T = T(\tau)$
  3) use the new $T(\tau)$ as the trial one
  4) repeat steps 1-3 until the stable $T(\tau)$ is found

Massless particles

Viscous hydrodynamics

$$T^{\mu\nu}_{\mathrm{visc;LRF}} = \begin{pmatrix} \mathcal{E}_{\mathrm{eq}} & 0 & 0 & 0 \\ 0 & \mathcal{P}_{\mathrm{eq}} + \frac{\Pi}{2} & 0 & 0 \\ 0 & 0 & \mathcal{P}_{\mathrm{eq}} + \frac{\Pi}{2} & 0 \\ 0 & 0 & 0 & \mathcal{P}_{\mathrm{eq}} - \Pi \end{pmatrix}$$

- equations of viscous hydrodynamics

$$\begin{aligned} \partial_\tau \mathcal{E} &= -\frac{\mathcal{E} + \mathcal{P}_{\mathrm{eq}}}{\tau} + \frac{\Pi}{\tau} \\ \Pi &= \frac{4\eta}{3\tau} && \text{(1st order)} \\ \partial_\tau \Pi &= -\frac{\Pi}{\tau_\pi} + \frac{4}{3}\frac{\eta}{\tau_\pi \tau} - \beta\frac{\Pi}{\tau} && \text{(2nd order)} \end{aligned}$$

shear relaxation time $\tau_\pi = \frac{5\bar{\eta}}{T}$

- $\beta = 4/3$ (Israel-Stewart (IS))
- $\beta = 38/21$ (Denicol, Niemi, Molnar, Rischke (DNMR), Jaiswal)

Anisotropic hydrodynamics

- assumption : distribution function is always well approximated by RS form
- the RS form is defined by transverse momentum scale $\Lambda(\tau)$ and anisotropy parameter $\xi(\tau)$
- energy density and pressures are given by simple formulas

$$\mathcal{E} = \frac{6g_0\Lambda^4}{\pi^2}\mathcal{R}(\xi) \qquad \mathcal{P}_T = \frac{3g_0\Lambda^4}{\pi^2}\mathcal{R}_T(\xi) \qquad \mathcal{P}_L = \frac{3g_0\Lambda^4}{\pi^2}\mathcal{R}_L(\xi) \qquad n = \frac{2g_0\Lambda^3}{\pi^2\sqrt{1+\xi}}$$

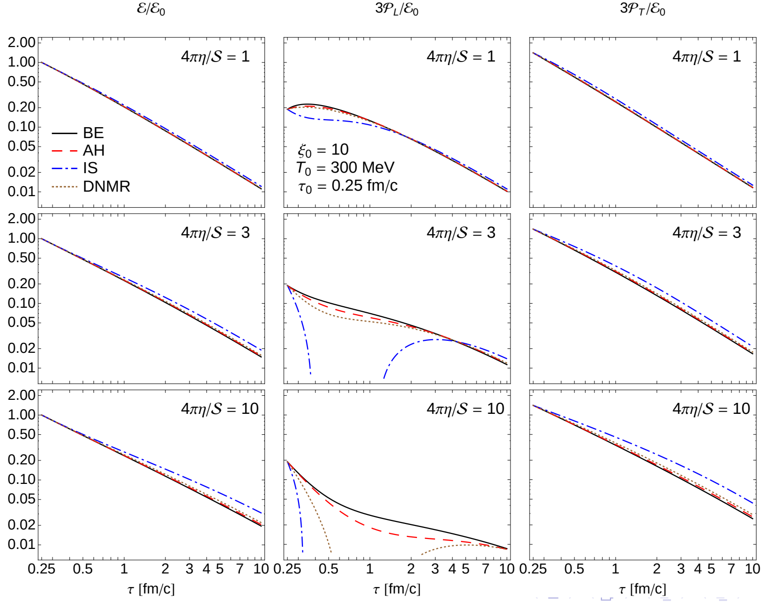
- the 0th and 1st moments of the BE in the RTA are evaluated

$$\frac{\partial_\tau \xi}{1+\xi} = \frac{2}{\tau} - \frac{4\mathcal{R}(\xi)}{\tau_{\rm eq}^{\rm AH}} \frac{\mathcal{R}^{3/4}(\xi)\sqrt{1+\xi}-1}{2\mathcal{R}(\xi)+3(1+\xi)\mathcal{R}'(\xi)}$$

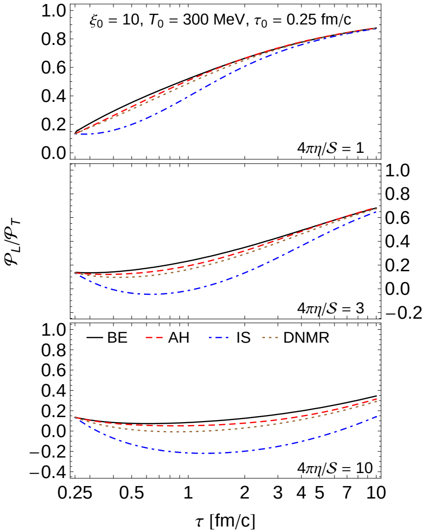
$$\frac{1}{1+\xi}\frac{\partial_\tau \Lambda}{\Lambda} = \frac{\mathcal{R}'(\xi)}{\tau_{\rm eq}^{\rm AH}} \frac{\mathcal{R}^{3/4}(\xi)\sqrt{1+\xi}-1}{2\mathcal{R}(\xi)+3(1+\xi)\mathcal{R}'(\xi)}$$

- relaxation time $\tau_{\rm eq}^{\rm AH} = \frac{5\bar{\eta}}{2\Lambda}$

Comparison with exact solutions

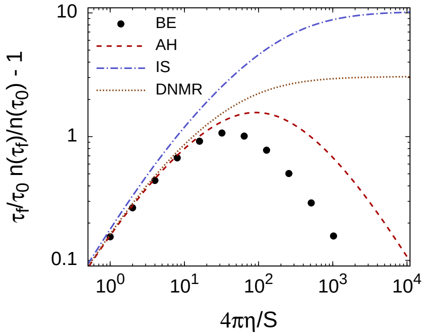


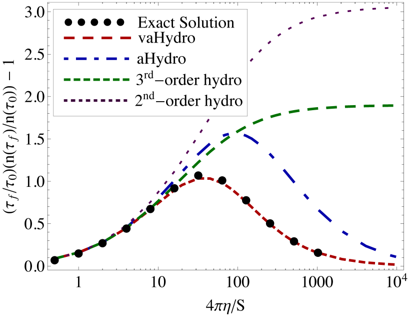
Comparison with exact solutions



Comparison with exact solutions

particle number production expected to vanish in two limits:
ideal hydrodynamical limit ($\eta/\mathcal{S} \to 0$) and free-streaming limit ($\eta/\mathcal{S} \to \infty$)





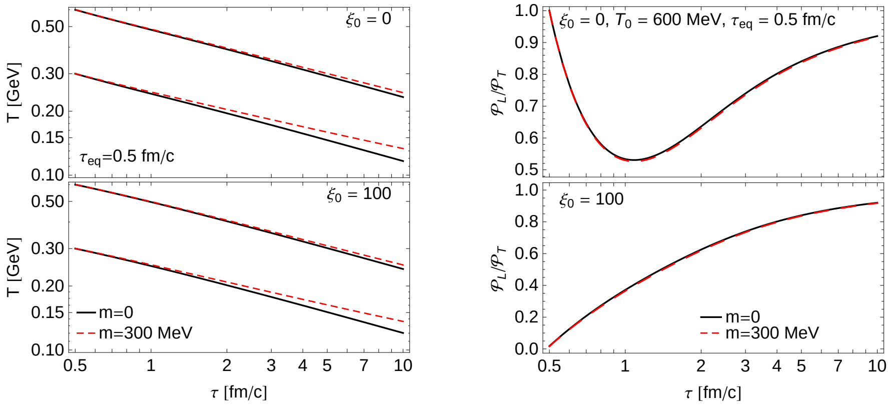
D.Bazow, U.Heinz, M.Strickland, arXiv:1311.6720

Massive particles

Exact solutions of kinetic equation

Thermodynamic variables

inclusion of finite parton masses weakly affects the thermalization



Extraction of shear η viscosity

- shear viscosity coefficient (Anderson, Witting)

$$\eta_{\rm hyd}(T) = \tau_{\rm eq} P_{\rm eq}(T) \frac{\mu^3}{15} \left(\frac{3G-\mu}{\mu^2} + \frac{K_1 - K_{i,1}}{K_2} \right)$$

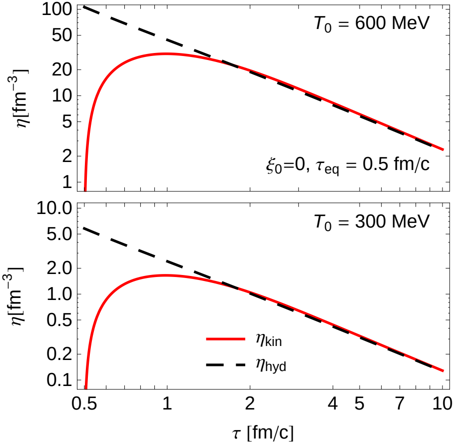
where

$$G = \frac{K_3}{K_2}, \qquad K_{i,1} = \frac{\pi}{2}\left[1 - \mu K_0 L_{-1} - \mu K_1 L_0\right], \qquad \mu = \frac{m}{T}$$

- shear viscosity extracted from kinetic equation

$$\eta_{\rm kin}(T) = \frac{(\mathcal{P}_T - \mathcal{P}_L)\,\tau}{2}$$

Extraction of shear η viscosity



Extraction of bulk ζ viscosity

- bulk viscosity coefficient (Redlich, Sasaki, Bozek, Romatschke)

$$\zeta_{\rm hyd}(T) = \tau_{\rm eq} P_{\rm eq}(T) \frac{\mu^3}{9} \left(-\frac{K_2}{3K_3 + \mu K_2} + \frac{K_1 - K_{i,1}}{K_2} \right)$$

- bulk pressure extracted from kinetic equation

$$\Pi_\zeta^{\rm kin} = \frac{1}{3} \left[\mathcal{P}_L + 2\mathcal{P}_T - 3\mathcal{P}_{\rm eq} \right]$$

Extraction of bulk ζ viscosity

1st order viscous hydrodynamic prediction:

$$\Pi_\zeta = -\frac{\zeta}{\tau}$$

2nd order viscous hydrodynamic predictions:

- Muronga, Heinz, Song, Chaudhuri

$$\tau_\Pi \dot{\Pi}_\zeta + \Pi_\zeta = -\frac{\zeta}{\tau} - \frac{1}{2}\tau_\Pi \Pi_\zeta \left[\frac{1}{\tau} - \left(\frac{\dot{\zeta}}{\zeta} + \frac{\dot{T}}{T}\right)\right] \qquad (a)$$

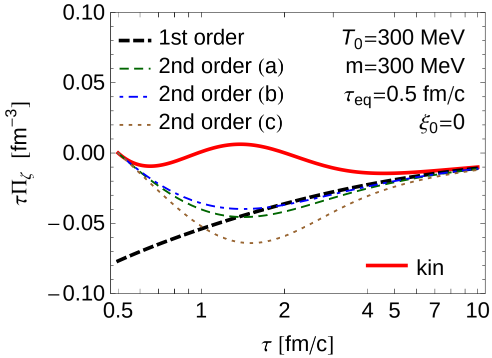
- Jaiswal, Bhalerao, Pal

$$\tau_\Pi \dot{\Pi}_\zeta + \Pi_\zeta = -\frac{\zeta}{\tau} - \frac{4}{3}\tau_\Pi \Pi_\zeta \frac{1}{\tau} \qquad (b)$$

- Heinz, Song, Chaudhuri (approximate formula)

$$\tau_\Pi \dot{\Pi}_\zeta + \Pi_\zeta = -\frac{\zeta}{\tau} \qquad (c)$$

Extraction of bulk ζ viscosity



Conclusions

- constructed exact solutions of the one-dimensional boost-invariant kinetic equation treated in the relaxation time approximation
- exact solutions can be used for testing various approximation methods
- established the correspondence between the late, near equilibrium evolution of the system described by the kinetic theory and by the viscous hydrodynamics
- interesting to include at least radial expansion in the exact solution of the kinetic equation

Thank You