Hydrodynamic fluctuations near and away from the critical point

Gökçe Başar

University of North Carolina, Chapel Hill

with Xin An, Mikhail Stephanov and Ho-Ung Yee

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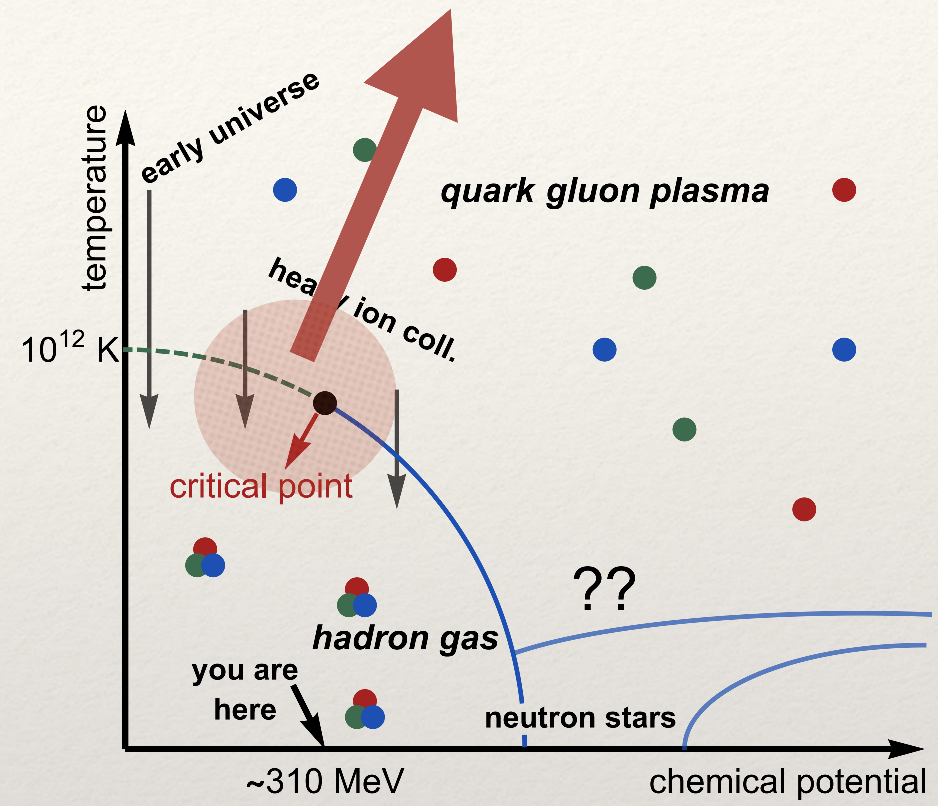
Motivations

- Fluctuations are large in the critical region

Non-equilibrium effects on the critical point

Dynamical framework including *non-critical fluctuations*

$\xi \to \infty \Rightarrow$ fluctuations $\to \infty$



- Small systems

Hydrodynamics with noise

fluctuation dissipation theorem $\Rightarrow$ thermal fluctuations $\propto \eta, \zeta, \lambda$

conserved quantity $\psi = (T^{00}, T^{0j}, J^0)$ flux: $\mathbf{J} = (T^{i0}, T^{ij}, J^i)$

hydrodynamics:

$$\partial_t \psi = -\nabla \cdot \mathbf{J}[\psi]$$

hydrodynamics with noise: [Kapusta, Muller, Stephanov, '12; Young '14; Singh, Shen, McDonald, Jeon, Gale '18,…]

stochastic variable $\psi \equiv \langle \breve{\psi} \rangle$ *noise*

$$\partial_t \breve{\psi} = -\nabla \cdot \left(\mathbf{J}[\breve{\psi}] + \xi \right)$$

$$\langle \xi(x) \xi(x') \rangle = 2TD\, \delta^{(4)}(x - x')$$

η, ζ, λ transport coef.

Deterministic approach

in addition to $\langle\breve{\psi}\rangle$ include the mode $G = \langle\breve{\psi}\breve{\psi}\rangle - \langle\breve{\psi}\rangle\langle\breve{\psi}\rangle$

$$\partial_t \psi = -\nabla \cdot \mathbf{J}[\psi, G]$$

$$\partial_t G = -\Gamma\left(G - G^{eq}[\psi]\right) + \mathscr{F}[G]$$

relaxation: $G \to G^{eq}[\psi]$

background gradients

(derived from Langevin-hydro eqn.)

Akamatsu, Mazeliauskas, Teaney '16: *homogenous boost invariant plasma*

Martinez, Schaefer '18: *homogenous boost invariant plasma with U(1) current*

An, GB, Stephanov, Yee, '19 -'20: *general formalism, arbitrary background, U(1) current*

Andreev '78: *non-relativistic fluids*

Deterministic approach

$$\breve{T}^{\mu\nu} = w(\breve{m}, \breve{\epsilon})\breve{u}^{\mu}\breve{u}^{\nu} + \breve{p}g^{\mu\nu} + \breve{T}^{\mu\nu}_{vis.}$$

nonlinearities

$$\langle \breve{T}^{\mu\nu} \rangle = T^{\mu\nu}[\langle \breve{\psi} \rangle] + \langle \delta\psi\delta\psi \rangle + \mathcal{O}(\langle \delta\psi^3 \rangle)$$

$\delta\psi = \breve{\psi} - \langle \breve{\psi} \rangle$ $\breve{\psi} \equiv (\breve{m}, \breve{p}, \breve{u}^{\mu})$ *m=n/s: entropy per baryon*

$$\breve{J}^{\mu\nu} = n(\breve{m}, \breve{p})u^{\mu} + \nu^{\mu}$$

$$\langle \breve{J}^{\mu} \rangle = J^{\mu}[\langle \breve{\psi} \rangle] + \langle \delta\psi\delta\psi \rangle + \mathcal{O}(\langle \delta\psi^3 \rangle)$$

Wigner function

Phase space distribution of fluctuations

x ~ background gradient scale $\qquad$ q ~ fluctuation wave vector

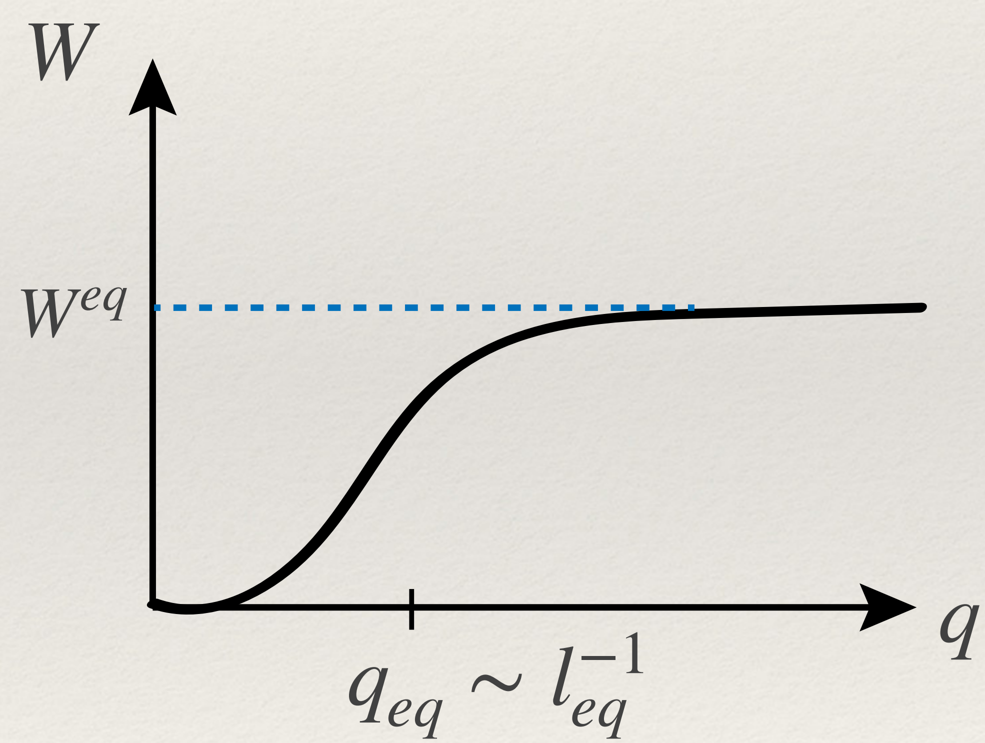
$$W(t, \mathbf{x}; \mathbf{q}) \overset{!}{=} \int d^3y e^{-i\mathbf{y}\cdot\mathbf{q}} \langle \delta\psi(t, \mathbf{x} + \mathbf{y}/2) \delta\psi(t, \mathbf{x} - \mathbf{y}/2) \rangle$$

``Equal time" two point correlation function of fluctuations

- Describes relaxation to equilibrium
- Lorentz covariant definition (confluent derivatives / correlators) in [An, GB, Stephanov, Yee, '19 PRC 100 024910,1912.13456]

Physics of the Wigner function

- Equilibrium: $W = W^{eq}$
- Reaching equilibrium takes time: $l_{eq} \sim \sqrt{\gamma_\eta L / c_s}$
- Fluctuations with wavelengths $\lambda \lesssim l_{eq}$ deviate from their equilibrium value



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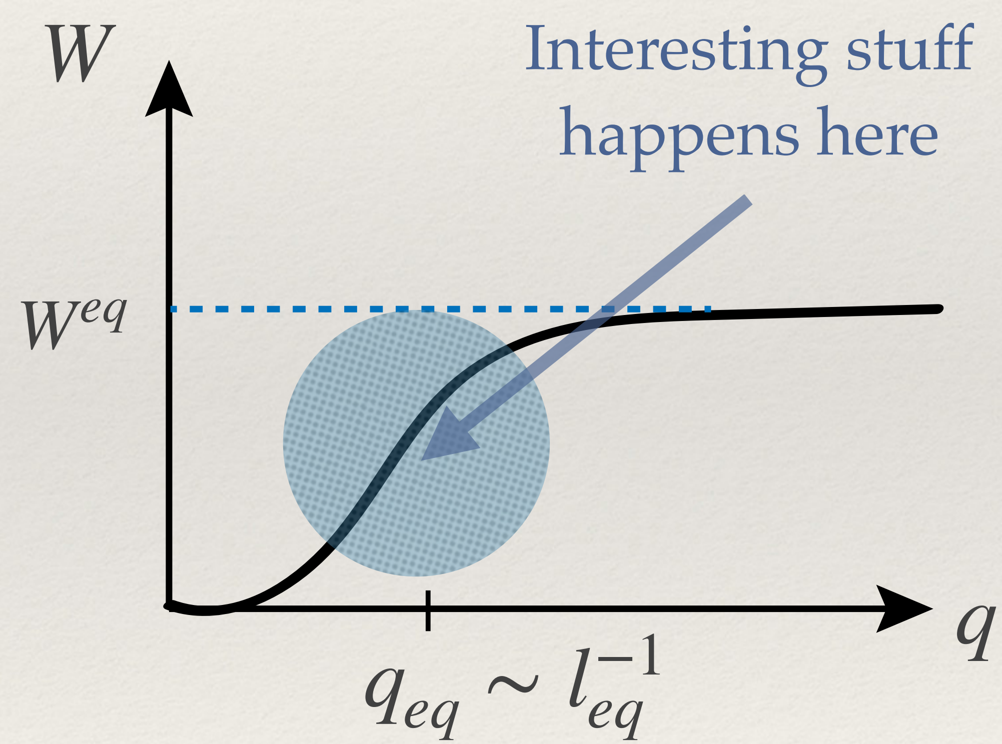


Fluctuation evolution equations: sound

- Modes of W: 1 *sound*, 1 *diffusive*, 3 *shear*, 2 *shear/trans.*
- Evolution equation for W = kinetic equation for phonons

sound mode:

$N_s = W_L / c_s |q| w$

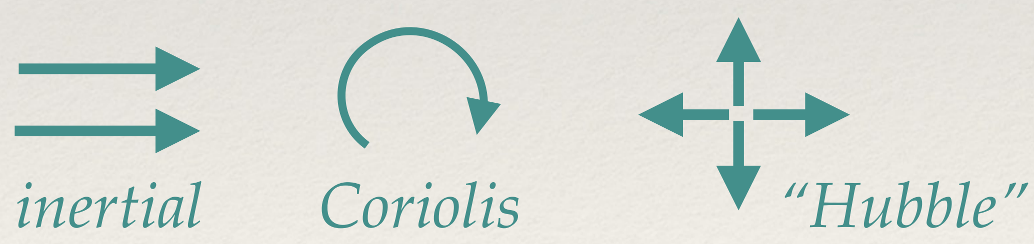
sound damping: $\zeta/w + 4/3\eta/w + \kappa/c_p$

$$\mathscr{L}[N_s] \equiv [(u+v)\cdot\bar{\nabla} + F\cdot\frac{\partial}{\partial q}]N_s = -\gamma_L q^2 [N_s - T/E]$$

phonon energy = $c_s |q|$

Liouville operator
In an effective metric
$g^{\mu\nu}_{eff}(x) = -c_s^2 u^\mu u^\nu + \Delta^{\mu\nu}$

forces:



phonon velocity = $c_s \hat{q}$

[An, GB, Stephanov, Yee, '19, 1912.13456]

Fluctuation evolution equations

remaining modes: $N_{mm}, N_{m(i)}, N_{(ij)} \propto W_{AB}$

*m=n/s: entropy per baryon, i,j ∈{1,2}: transverse to **q***

$$\mathscr{L}[N_{mm}] = -2\gamma_\lambda q^2(N_{mm} - N^{eq}_{mm}) + \mathscr{F}_1[N_{m(i)}]$$

$$\mathscr{L}[N_{m(i)}] = -(\gamma_\eta + \gamma_\lambda)q^2(N_{m(i)} - N^{eq}_{m(i)}) + \mathscr{F}_2[N_{mm}, N_{m(i)}, N_{(ij)}]$$

$$\mathscr{L}[N_{(ij)}] = -2\gamma_\eta q^2(N_{(ij)} - N^{eq}_{(ij)}) + \mathscr{F}_3[N_{(ij)}, N_{m(i)}]$$

[An, GB, Stephanov, Yee, ’19, 1912.13456]

Renormalization

Fluctuation contributions to $\langle \breve{T}^{\mu\nu} \rangle, \langle \breve{J}^{\mu} \rangle$

$$\langle \delta\psi(x)\delta\psi(x)\rangle = G(x,0) = \int d^3q W(x,q) \sim \Lambda^3 W^{eq} + \Lambda \partial u + finite$$

$G(x,y) \sim \delta^3(y)$ *equilibrium* *renormalize* η, ζ

- Renormalization can be carried out analytically
- No divergence due to "infinite noise" in numerics

Cutoff independent coupled system of equations:

$$\partial_\mu \langle \breve{T}^{\mu\nu} \rangle = 0, \quad \partial_\mu \langle \breve{J}^{\mu} \rangle = 0, \quad u \cdot \bar{\nabla} W = \text{Relaxation}[W \to W^{eq}]$$

feedback of fluctuations

dynamics of fluctuations in an arbitrary background

Long time tails

$$\langle \delta\psi(x)\delta\psi(x)\rangle = \int d^3q W(x,q) \sim \Lambda^3 W^{eq} + \Lambda \partial u + \mathit{finite}$$

- Physics is contained in the finite part: *"long time tails"*

$$\int d^3q \tilde{W}(x,q) \sim q_{eq}^3 \sim k^{3/2}, \omega^{3/2} \qquad \Rightarrow \tilde{W}(t) \sim t^{-3/2}$$

- Gradient expansion revisited

$$\langle \breve{T}^{\mu\nu}\rangle \sim T^{\mu\nu}_{ideal} + c_1 k + c_{3/2} k^{3/2} + c_2 k^2 + \dots$$

viscous ↗ $c_1 k$; *fluctuations* ↑ $c_{3/2} k^{3/2}$; second order ↖ $c_2 k^2$

$$\langle \breve{J}^{\mu}\rangle \sim n u^{\mu} - \lambda k + c_{3/2} k^{3/2} + c_2 k^2 + \dots$$

Long time tails

$$\lambda(\omega) = \lambda - \omega^{1/2} \frac{T^2 n^2}{w^2} \frac{(1-i)}{6\sqrt{2}\pi} \left(\frac{c_p T}{(\gamma_\eta + \gamma_\lambda)^{3/2} w} + \frac{c_s^2}{2\gamma_L^{3/2}} \right)$$

$$\eta(\omega) = \eta - \omega^{1/2} T \frac{(1-i)}{60\sqrt{2}\pi} \left(\frac{1}{\gamma_L^{3/2}} + \frac{7}{(2\gamma_\eta)^{3/2}} \right)$$

$$\zeta(\omega) = \zeta - \omega^{1/2} T \frac{(1-i)}{36\sqrt{2}\pi} \left(\frac{1}{\gamma_L^{3/2}} \left(1 - 3\dot{T} + 3\dot{c}_s\right)^2 + \frac{4}{(2\gamma_\eta)^{3/2}} \left(1 - \frac{3}{2}(\dot{T} + c_s^2)\right)^2 + \frac{9}{2(2\gamma_\lambda)^{3/2}} \left(1 - \dot{c}_p\right)^2 \right)$$

e.g. Bjorken expansion: [Akamatsu, Mazeliauskas, Teaney ’16]

$$\tau^2 T^{\eta\eta} = p - \frac{c_1}{\tau} + \frac{c_{3/2}}{\tau^{3/2}} + \frac{c_2}{\tau^2} + \dots$$

[see also loop calculations, Moore, Kovtun, …]

[An, GB, Stephanov, Yee, ’19, 1912.13456]

Approach to critical point

In the vicinity of the critical point various modes relax with different rates

$$\mathscr{L}[N_{mm}] = -2\gamma_\lambda q^2(N_{mm} - N_{mm}^{eq}) + \mathscr{F}_1[N_{m(i)}]$$

$$\mathscr{L}[N_{m(i)}] = -(\gamma_\lambda + \gamma_\eta)q^2(N_{m(i)} - N_{m(i)}^{eq}) + \mathscr{F}_2[N_{mm}, N_{m(i)}, N_{(ij)}]$$

$$\mathscr{L}[N_{(ij)}] = -2\gamma_\eta q^2(N_{(ij)} - N_{(ij)}^{eq}) + \mathscr{F}_3[N_{(ij)}, N_{m(i)}]$$

$$\Gamma_\lambda \sim \xi^{-z} \simeq \xi^{-3} \gg \Gamma_\eta \sim \xi^{z+d-8} \simeq \xi^{-2} \quad (\Gamma \sim \gamma q^2 : \text{relaxation rate})$$

- N_{mm} relaxes the slowest
- hydrodynamics can only describe modes with $\omega < \gamma_\lambda^{-1} < < \xi^{-1}$

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- N_{mm} relaxes the slowest
- hydrodynamics can only describe modes with $\omega < \Gamma_\lambda^{-1} < < \xi^{-1}$
- "hydro +" [Stephanov, Yin, '17] : hydro + N_{mm}, good for $\omega < \Gamma_\eta^{-1} < < \xi^{-1}$
- "hydro ++": hydro + all $N_{AB}s$ + background flow, good for $\omega < \xi^{-1}$

Hydro ++

- Near the critical point fluctuations are dominated by $q \sim \xi^{-1}$
- Non-local effects are important
- No exact treatment of the non-localities exists at the moment
- 1-loop approximation a la [Kawasaki '70]
- Two modifications to the fluctuation evolution equations:

$$N_{mm}^{eq} = \frac{c_p}{n} \to \frac{c_p/n}{1+(q\xi)^2}$$

→ *finite correlation length*

$$2\gamma_\lambda q^2 \to \Gamma_\lambda(q) \equiv 2\gamma_\lambda(q) q^2 \equiv 2\left(\frac{\kappa_{n.c.}}{c_p} + \frac{T}{6\pi\eta\xi} K(q\xi)\right) q^2$$

$$K(x) = \frac{3}{4x^2}\left(1 + x^2 + (x^3 - x^{-1}\arctan x)\right)$$

critical contribution to relax. rate

[An, GB, Stephanov, Yee, '19, 1912.13456]

Hydro ++

$$\mathscr{L}[N_{mm}] = -2\gamma_\lambda(q)q^2(N_{mm} - N^{eq}_{mm}(q)) - \frac{n}{w}t^{(i)} \cdot \partial m \left(N_{(i)m} + N_{m(i)}\right)$$

$$\mathscr{L}[N_{m(i)}] = -(\gamma_\eta + \gamma_\lambda(q))q^2(N_{m(i)} - N^{eq}_{m(i)}) - \partial^\nu u^\mu t^{(i)}_\mu t^{(j)}_\nu N_{m(j)} - \frac{n}{w}t^{(j)} \cdot \partial m N_{(ij)}$$
$$+Tn\left(\frac{1}{c_p(q)}t^{(i)} \cdot \partial m + \frac{T}{w}t^{(i)} \cdot \partial\alpha\right)N_{mm}$$

$$\mathscr{L}[N_{(ij)}] = -2\gamma_\eta q^2(N_{(ij)} - N^{eq}_{(ij)}) - \partial^\nu u^\mu \left(t^{(i)}_\mu t^{(j)}_\nu N_{(kj)} + t^{(j)}_\mu t^{(k)}_\nu N_{(ik)}\right)$$
$$+\frac{\alpha_p T^2 n}{w}\partial^\mu \left(t^{(i)}_\mu N_{m(j)} + t^{(j)}_\mu t^{(j)}_\mu N_{(i)m}\right)$$

$t^{(i)} \cdot t^{(j)} = \delta^{ij}, \quad t^{(i)} \cdot q = t^{(i)} \cdot u(x) = 0$

[An, GB, Stephanov, Yee, '19, 1912.13456]

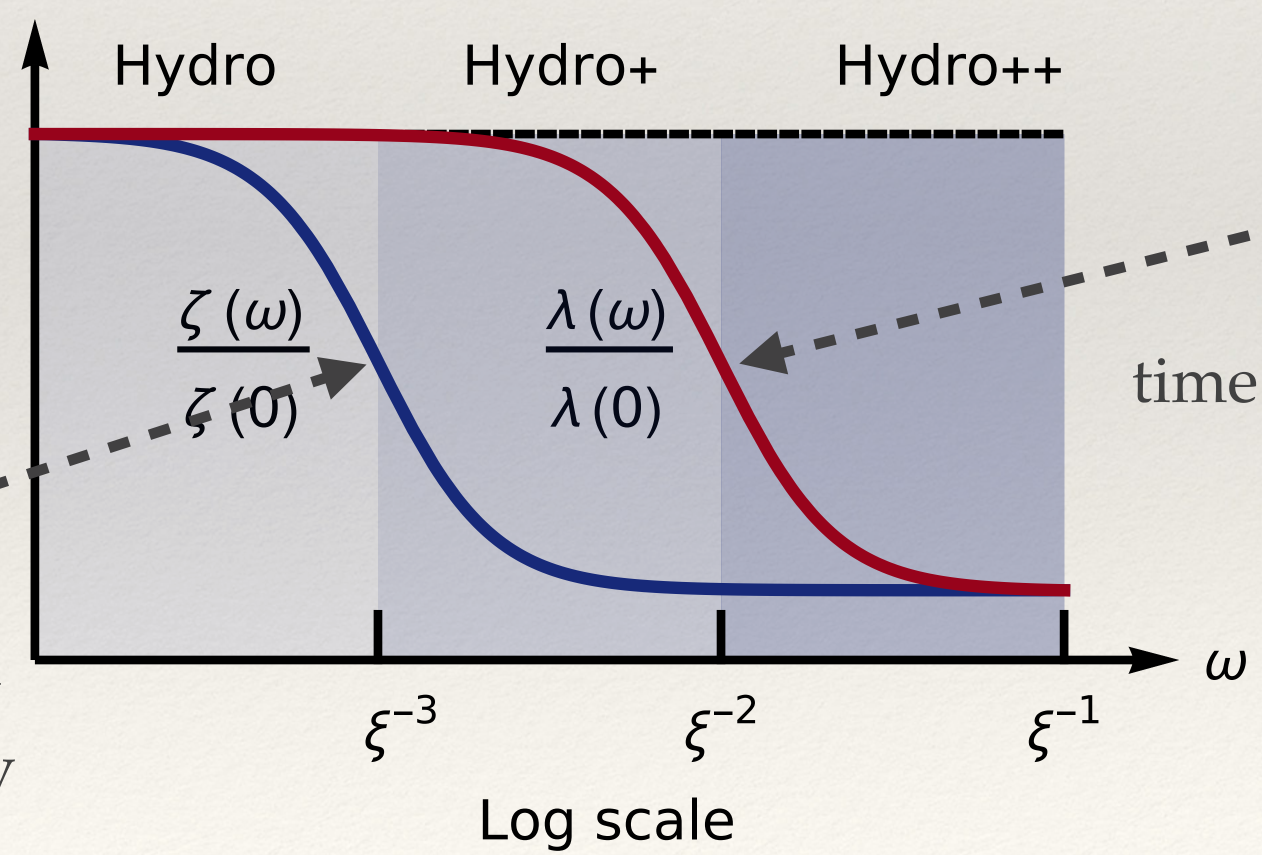
Hydro ++

- Critical contribution to conductivity:

  (generated dynamically: $N_{m(i)} \propto c_p(q\xi) \times \xi^{-1} \times \partial u$)

$$\lambda = \lambda_0 + \left(\frac{Tn}{w}\right)^2 \frac{c_p T}{6\pi\eta\xi} \sim \xi \qquad (c_p \sim \xi^2)$$

- Frequency dependence of bulk viscosity and conductivity



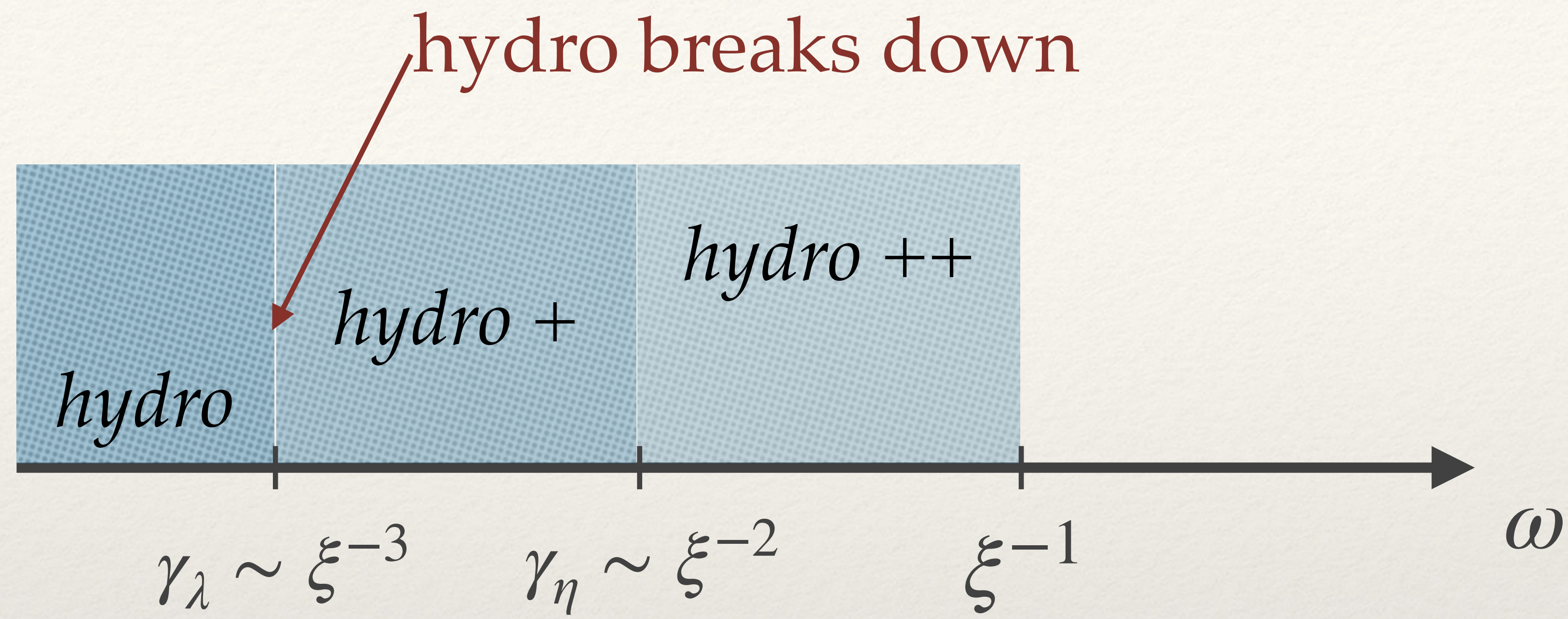
$N_{m(i)}$ falls out of equilibrium

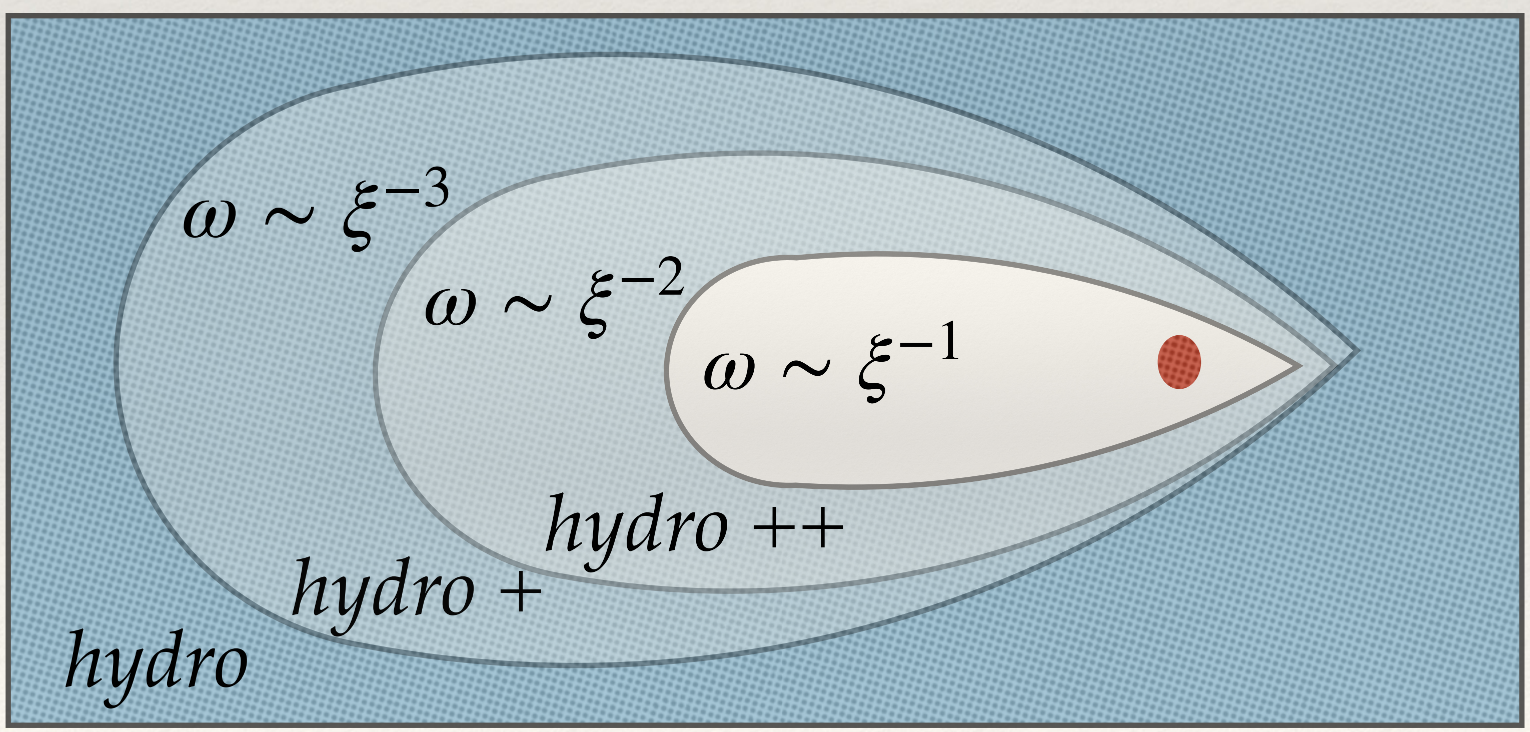
time lag of conductivity

N_{mm} falls out of equilibrium

time lag of bulk viscosity

Hydro++


hydro breaks down
hydro
hydro +
hydro ++
$\gamma_\lambda \sim \xi^{-3}$
$\gamma_\eta \sim \xi^{-2}$
ξ^{-1}
ω


$\omega \sim \xi^{-3}$
$\omega \sim \xi^{-2}$
$\omega \sim \xi^{-1}$
hydro ++
hydro +
hydro

Conclusions

- Deterministic formalism for charged fluids with arbitrary background
- Renormalization done analytically, finite, cutoff independent, coupled set of equations that can be solved numerically
- Dynamical evolution equation for fluctuations can be solved separately
- Hydro ++: a practical way of simulating near critical dynamics

In progress:

- Higher point correlators (cumulants)

  [see also Muhkerjee, Venugoplalan, Yin'15 ; Nahrgang, Bluhm, Schäfer, Bass, '18,…]

- Freeze-out dynamics?

  [Oliiynchenko, Koch '19]

- First order transition?