DARK MATTER HEATING NEUTRON STARS

THE "ELECTRONIC" FRONTIER

Flip Tanedo
UC Riverside Particle Theory

WORK WITH
ANIKET JOGLEKAR
NIRMAL RAJ
HAI-BO YU

10 FEB 2020

PHYSICS & ASTRONOMY

BROOKHAVEN NATIONAL LABORATORY

This is a neutron star

$$M_\star = 1.5 \text{ M}_\odot$$

$$R_\star = 12.6 \text{ km}$$

The Little Prince, A. St.-Exupery via *The Atlantic*

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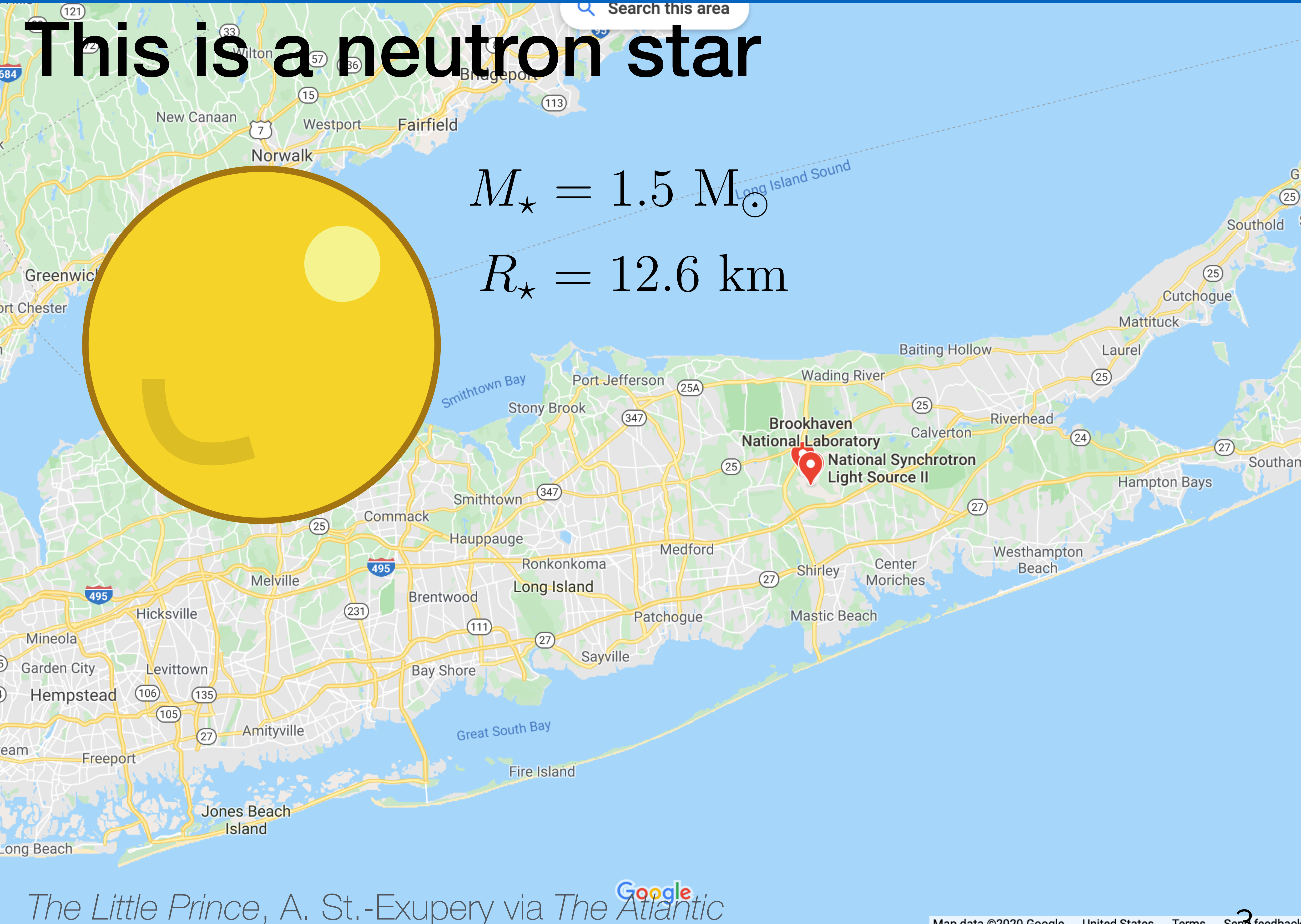


The Little Prince, A. St.-Exupery via *The Atlantic*

This is a neutron star

$$M_\star = 1.5\ \mathrm{M}_\odot$$

$$R_\star = 12.6\ \mathrm{km}$$

Pretty big, pretty dense.
Full of neutrons.
Also electrons. (and p, μ)

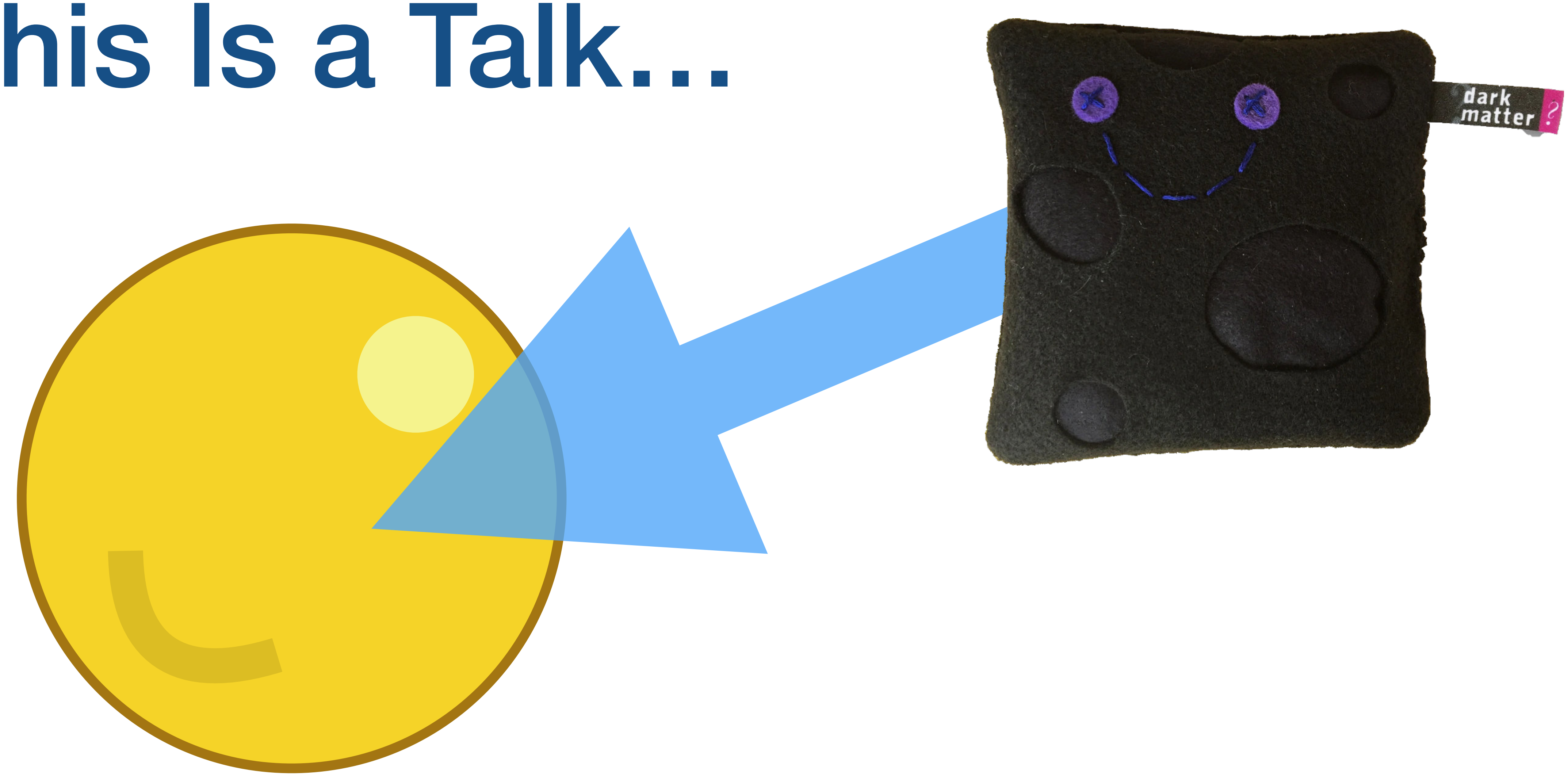
Today: neutron stars as a laboratory for particle physics.

Julie Peasley, particlezoo.net

This is dark matter

Julie Peasley, particlezoo.net

This Is a Talk…



kinetic heating: neutrons

New results & formalism

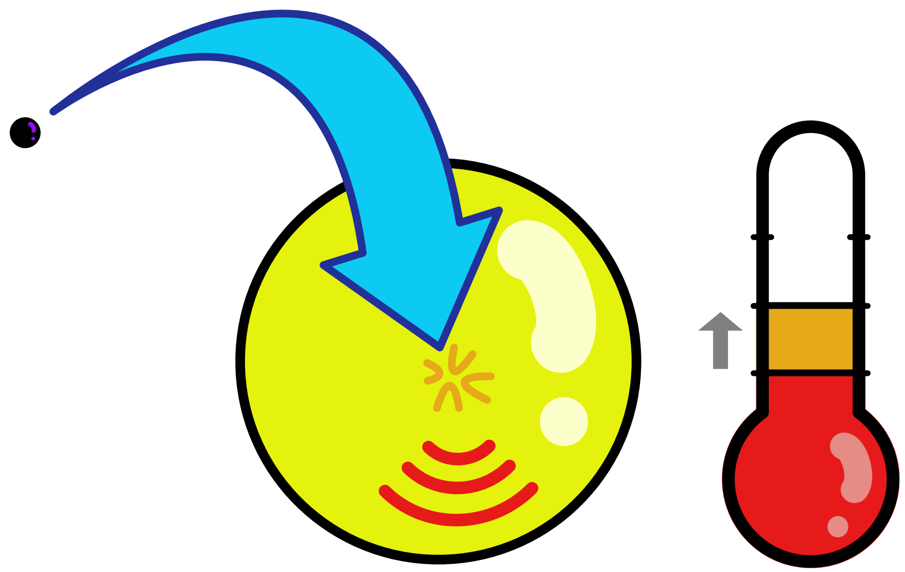
pre-history: Goldman & Nussinov '89, Gould et al '90

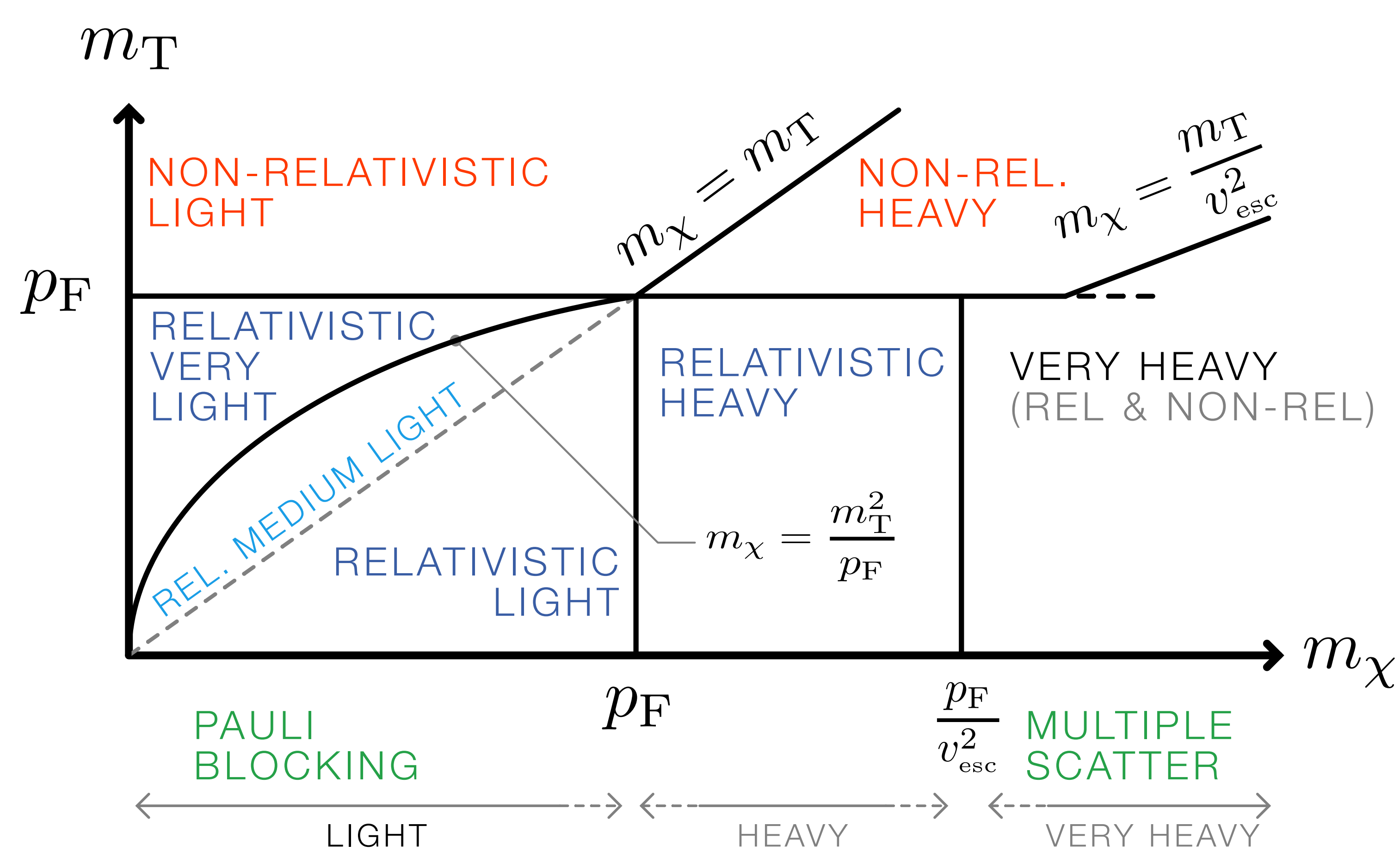
Outline


dark matter


dark matter
neutron


dark matter
electron




m_T
p_F
NON-RELATIVISTIC LIGHT
$m_\chi = m_T$
NON-REL. HEAVY
$m_\chi = \frac{m_T}{v_{esc}^2}$
RELATIVISTIC VERY LIGHT
REL. MEDIUM LIGHT
RELATIVISTIC LIGHT
RELATIVISTIC HEAVY
$m_\chi = \frac{m_T^2}{p_F}$
VERY HEAVY
(REL & NON-REL)
m_χ
p_F
$\frac{p_F}{v_{esc}^2}$
PAULI BLOCKING
MULTIPLE SCATTER
LIGHT
HEAVY
VERY HEAVY

What is dark matter?

Most likely a particle.
Definitely beyond the SM.

ASSUMPTION:

interacts with ordinary matter. See tomorrow's colloquium.

We know: local density, what it *is not*.
We have: robust, complementary search program.
We want: more robust, more complementary, cheap

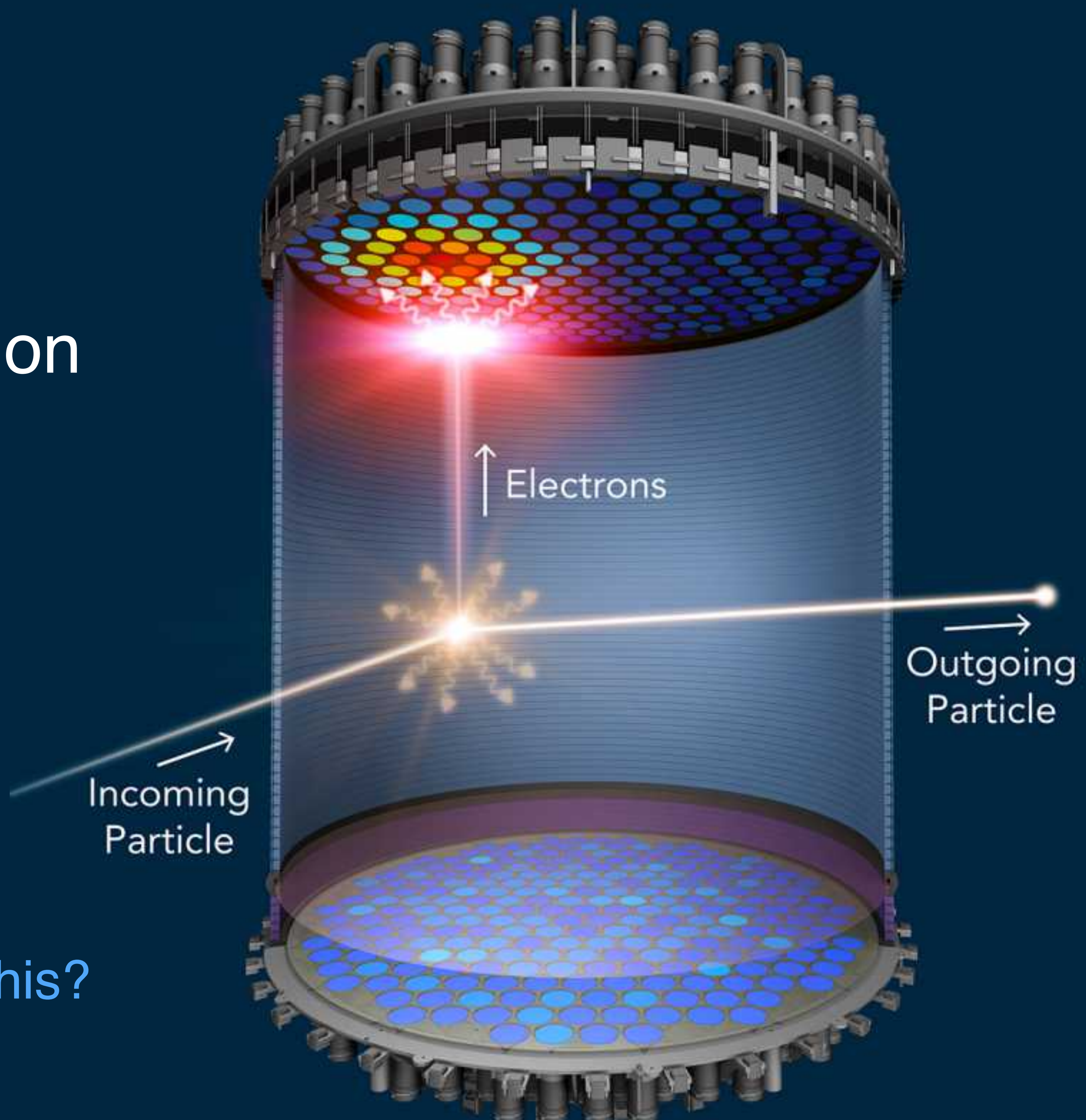
e.g. someone else is already paying for it.

Dark Matter in the Lab

Look for the recoil energy of dark matter on ordinary matter.
e.g. Xenon nucleus

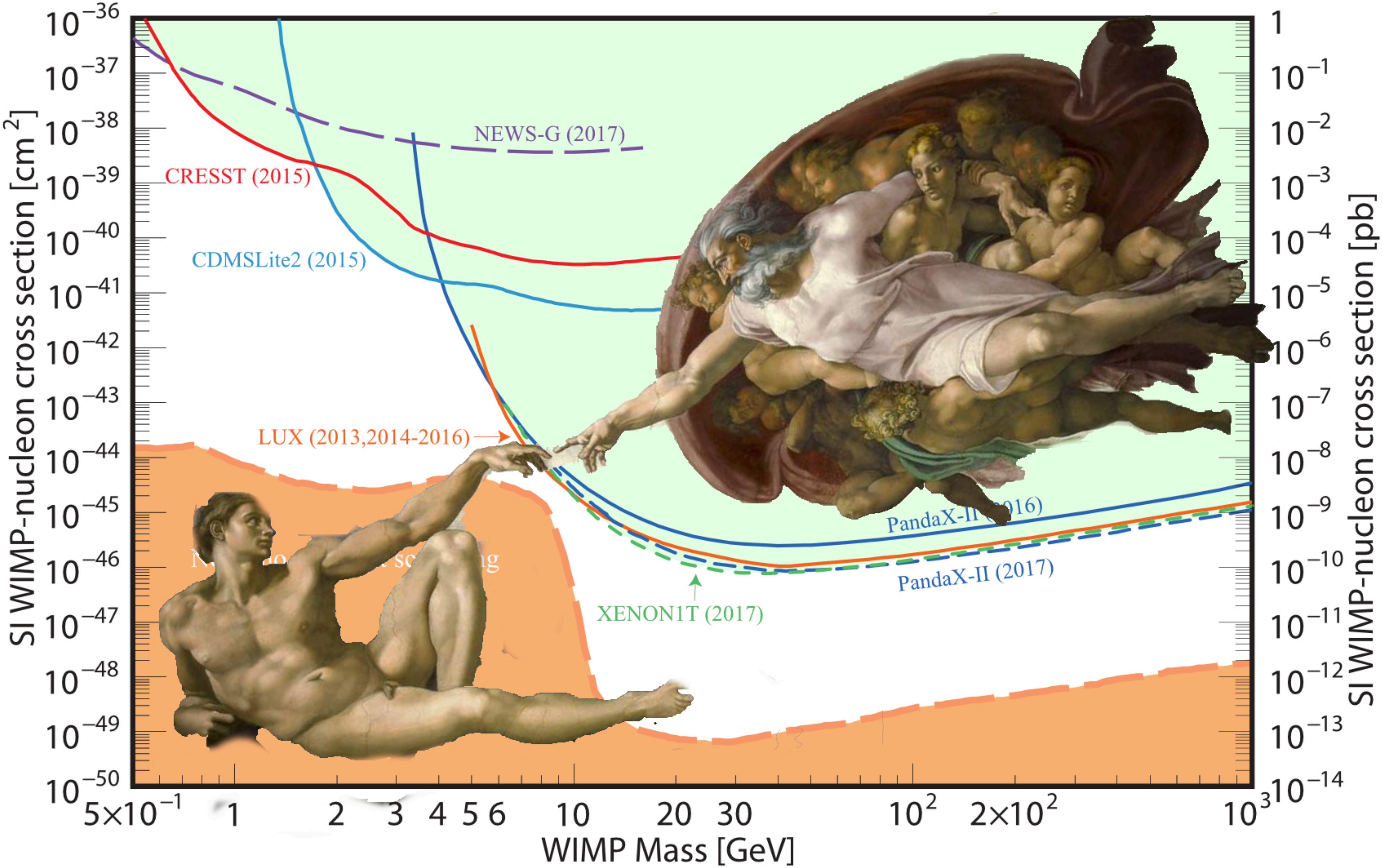
Use large volume to compensate for rare events.

Why *should* dark matter do this?
See tomorrow's colloquium.



via LUX-LZ (kipac.stanford.edu/research/topics/direct-dark-matter-detection)

Direct Detection



PDG 2018 (& apologies to Michelangelo)

Limitations

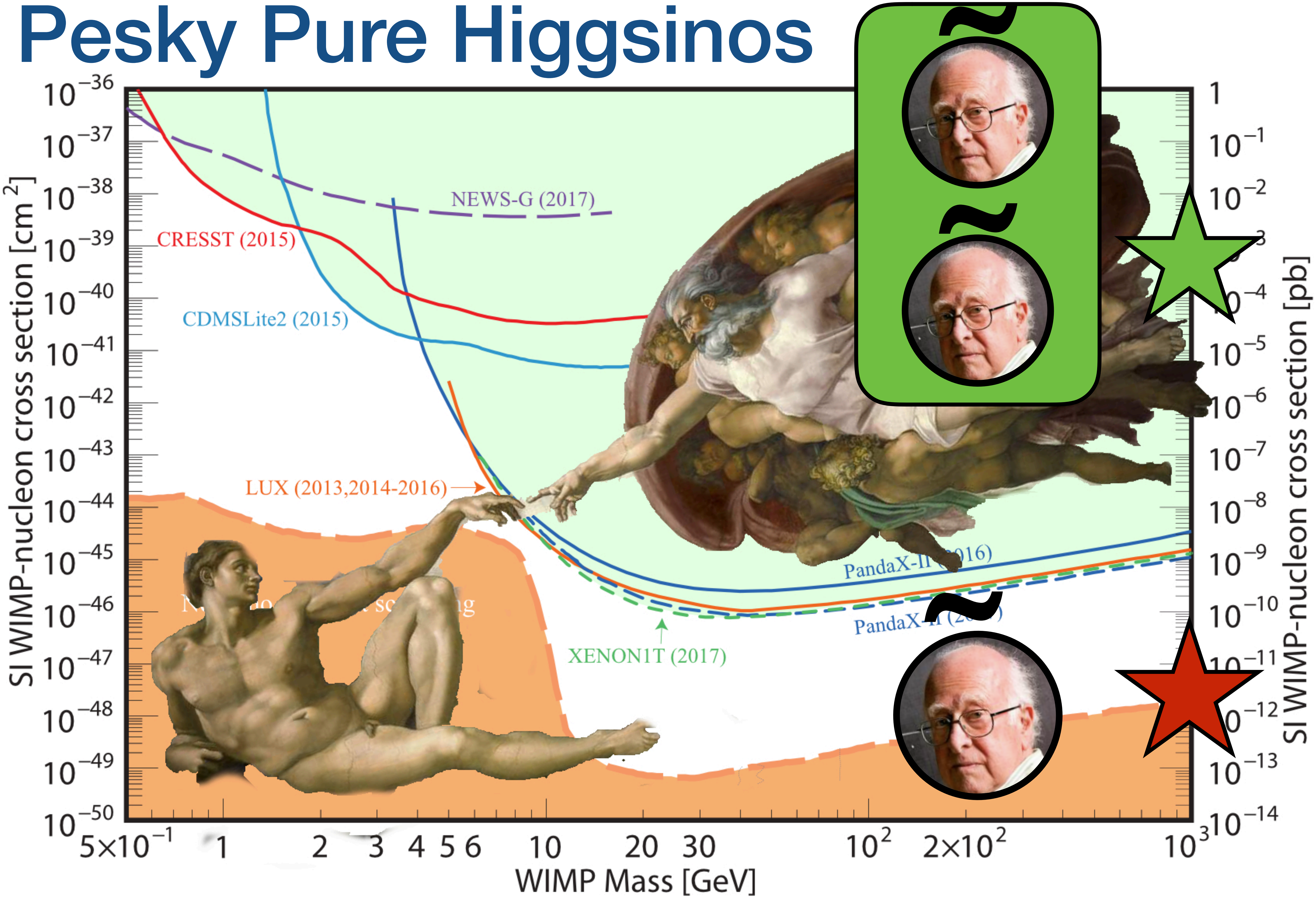
Volumes can only be so large.

Dark matter is slow, $q \sim 100$ keV
Can hide with momentum/spin-dependent interactions, inelastic scattering

Neutrino background.

Exotic: *too* strong interactions (ceiling), very light/heavy, leptophilic dark matter

Pesky Pure Higgsinos



e.g. Krall and Reece 1705.04843, Hill and Solon 1309.4092; *Science News*

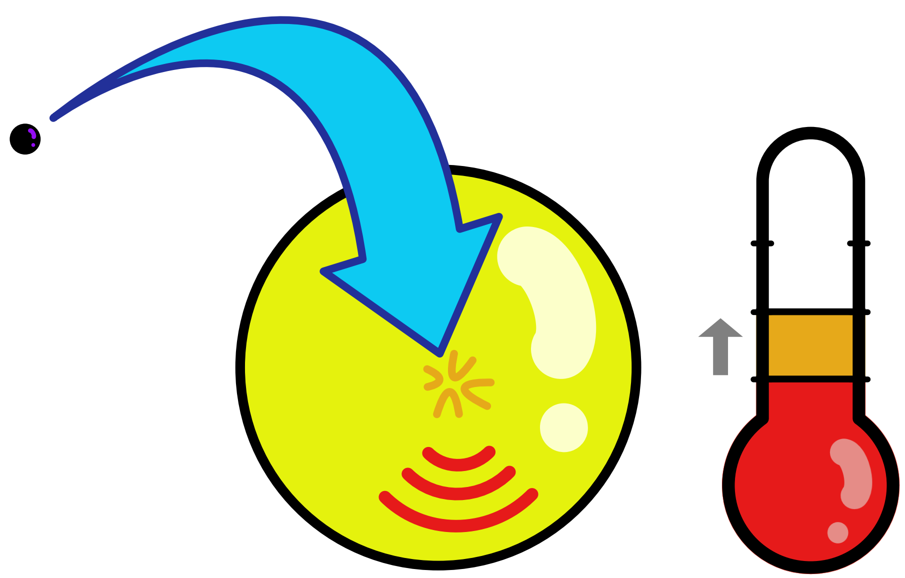
Direct Detection in Space

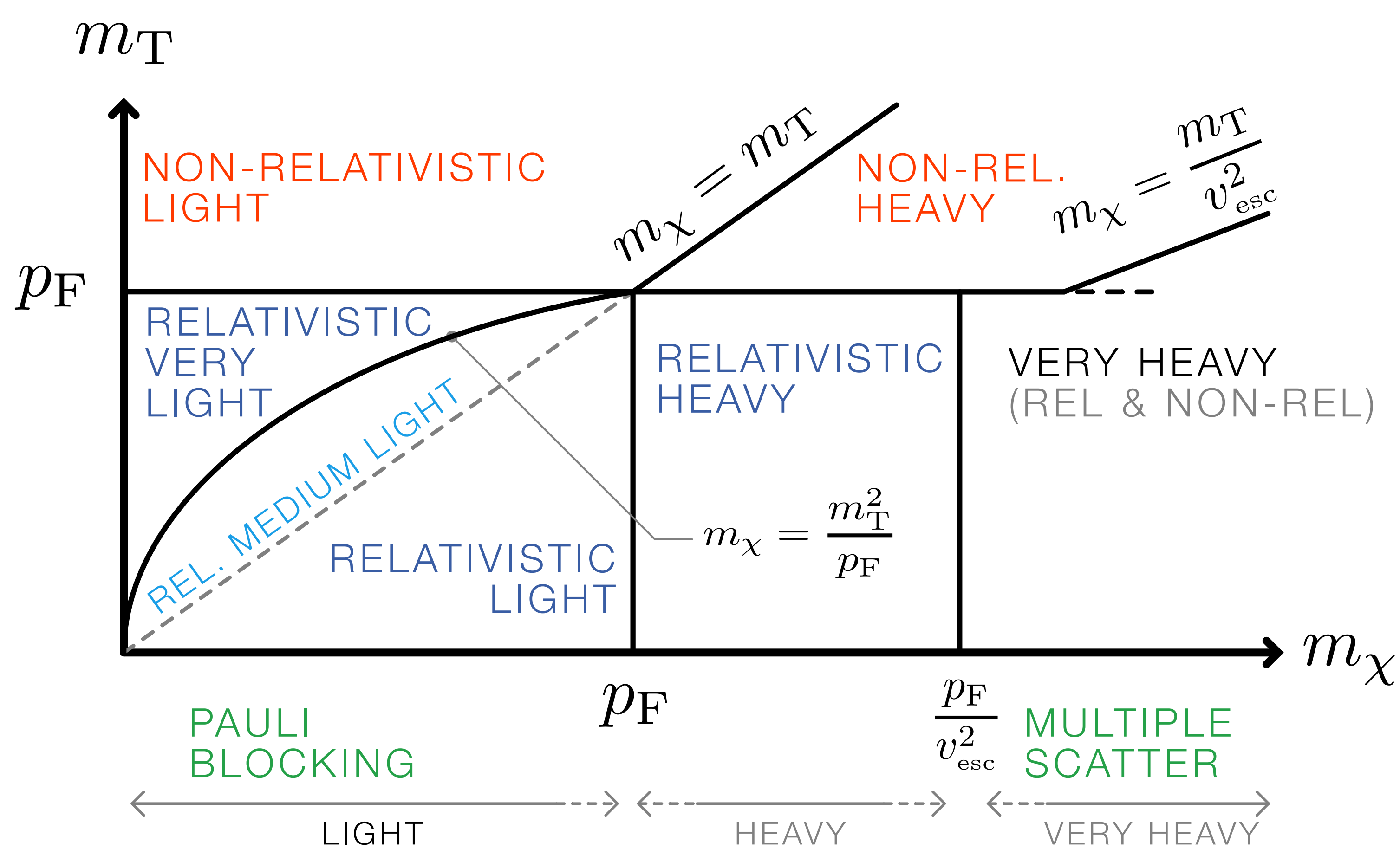
Can we have *huge* detector?
Maybe something in space?
Really dense; *accelerates* dark matter?

problems

1. Need a standard “laboratory”
2. Need really long cables to transmit data

Outline





Julie Peasley, particlezoo.net

Dark Kinetic Heating of Neutron Stars and An Infrared Window On WIMPs, SIMPs, and Pure Higgsinos

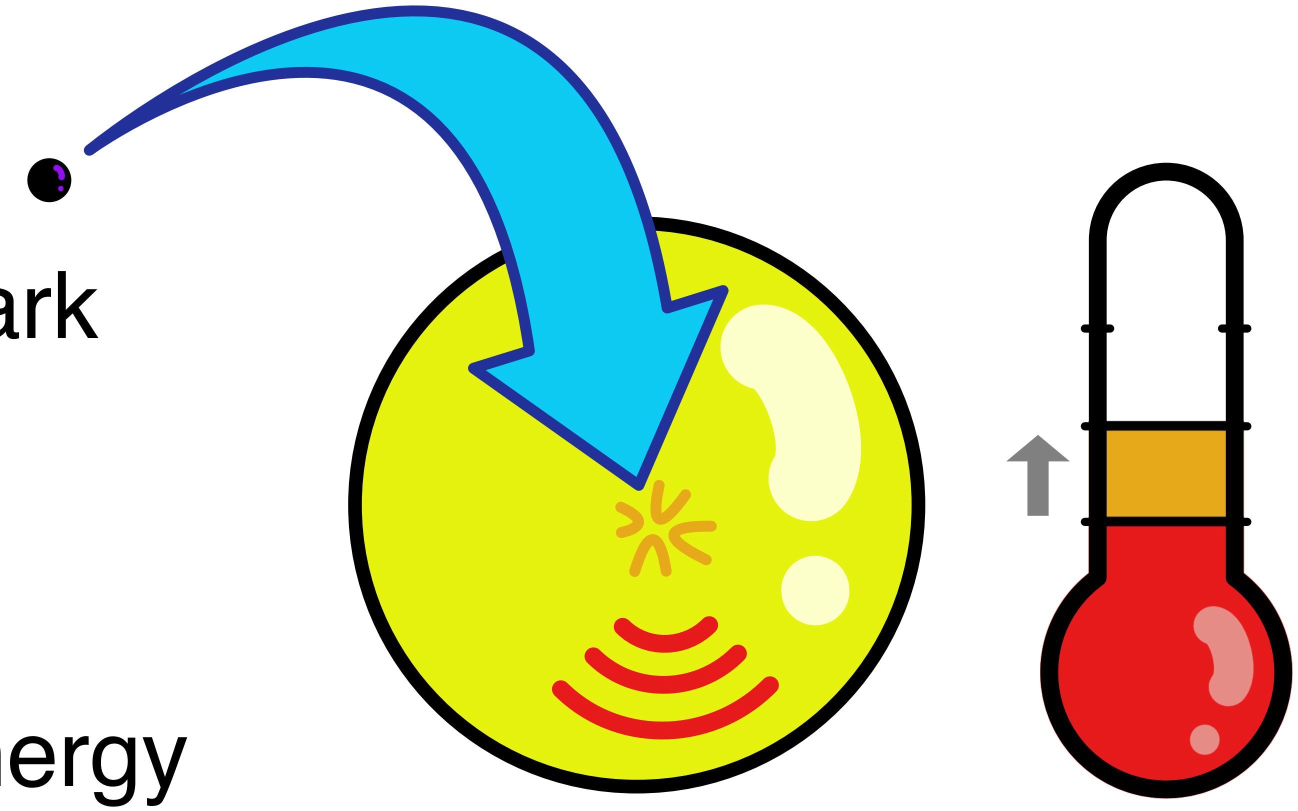
Masha Baryakhtar,[1] Joseph Bramante,[1] Shirley Weishi Li,[2] Tim Linden,[2] and Nirmal Raj[3]

[1] *Perimeter Institute for Theoretical Physics, Waterloo, Ontario, N2L 2Y5, Canada*
[2] *CCAPP and Department of Physics, The Ohio State University, Columbus, OH, 43210, USA*
[3] *Department of Physics, University of Notre Dame, Notre Dame, IN, 46556, USA*

We identify a largely model-independent signature of dark matter interactions with nucleons and electrons. Dark matter in the local galactic halo, gravitationally accelerated to over half the speed of light, scatters against and deposits kinetic energy into neutron stars, heating them to infrared blackbody temperatures. The resulting radiation could potentially be detected by the James Webb Space Telescope, the Thirty Meter Telescope, or the European Extremely Large Telescope. This mechanism also produces optical emission from neutron stars in the galactic bulge, and X-ray emission near the galactic center, because dark matter is denser in these regions. For GeV - PeV mass dark matter, dark kinetic heating would initially unmask any spin-independent or spin-dependent dark matter-nucleon cross-sections exceeding 2×10^{-45} cm^2, with improved sensitivity after more telescope exposure. For lighter-than-GeV dark matter, cross-section sensitivity scales inversely with dark matter mass because of Pauli blocking; for heavier-than-PeV dark matter, it scales linearly with mass as a result of needing multiple scatters for capture. Future observations of dark sector-warmed neutron stars could determine whether dark matter annihilates in or only kinetically heats neutron stars. Because inelastic inter-state transitions of up to a few GeV would occur in relativistic scattering against nucleons, elusive inelastic dark matter like pure Higgsinos can also be discovered.

1704.01577

How it works



There's a continuous flux of dark matter incident on star.

Dark matter scatters off target **neutrons**, imparting kinetic energy to *heat* the star.

Sufficient: dark matter is **captured** in the star; it loses its asymptotic kinetic energy.

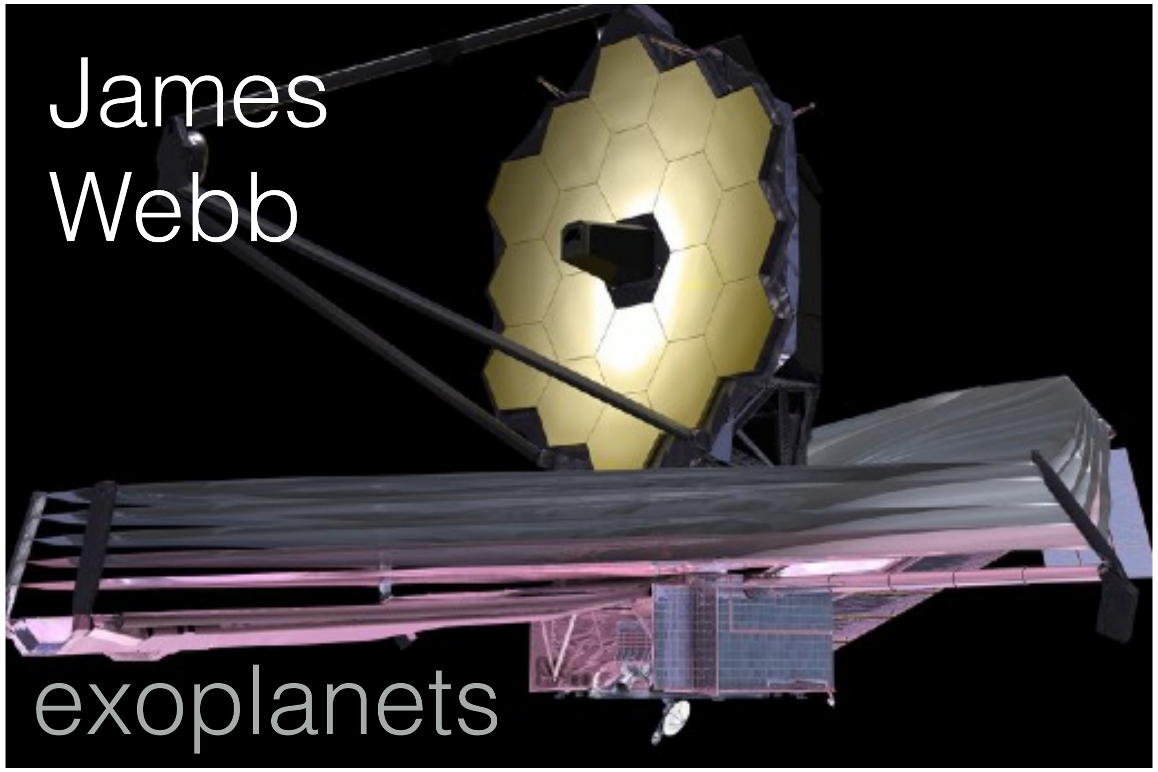
Detectable?

* protons too

Detectability

Detect radio pulses to identify nearby, *old* neutron stars.

Expect: ~100 w/in 50 pc



Measure temperature with infrared telescopes.

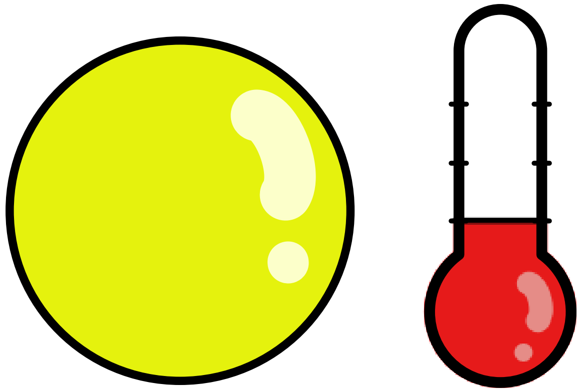
For 2σ:

$$10^5 \text{ sec} \left(\frac{d}{10\text{pc}}\right)^4$$

… and you may only need one.

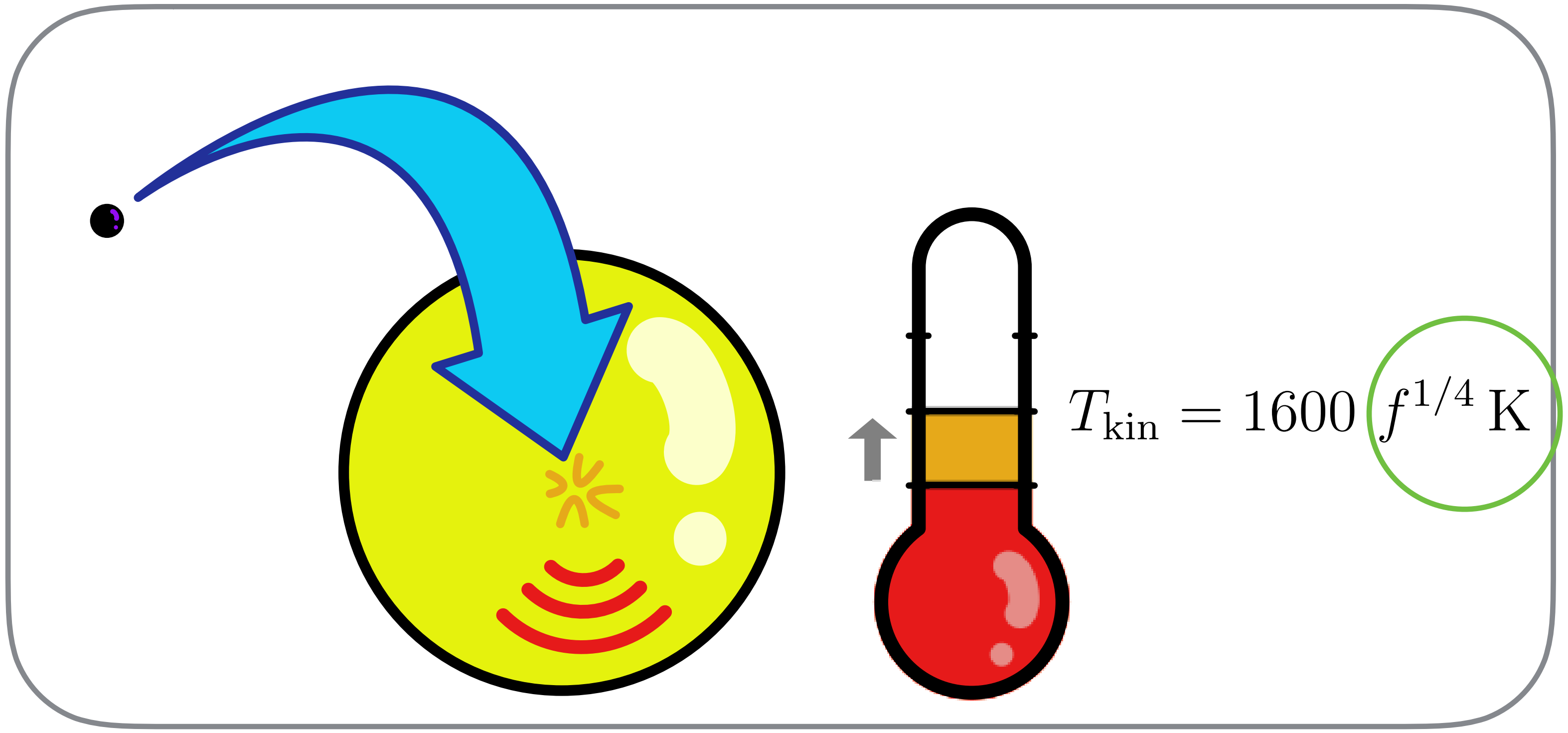
Image credits: FAST, JWST

How it works

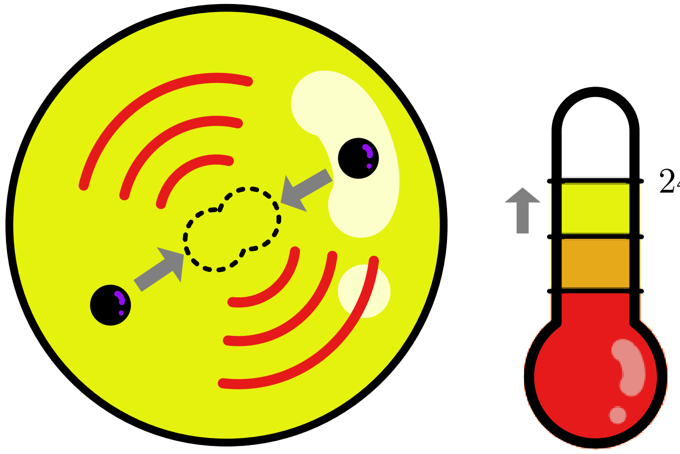


1000 K (20 Myr)   (old and cold)

100 K (Gyr)



$T_{\rm kin} = 1600\, f^{1/4}\,{\rm K}$



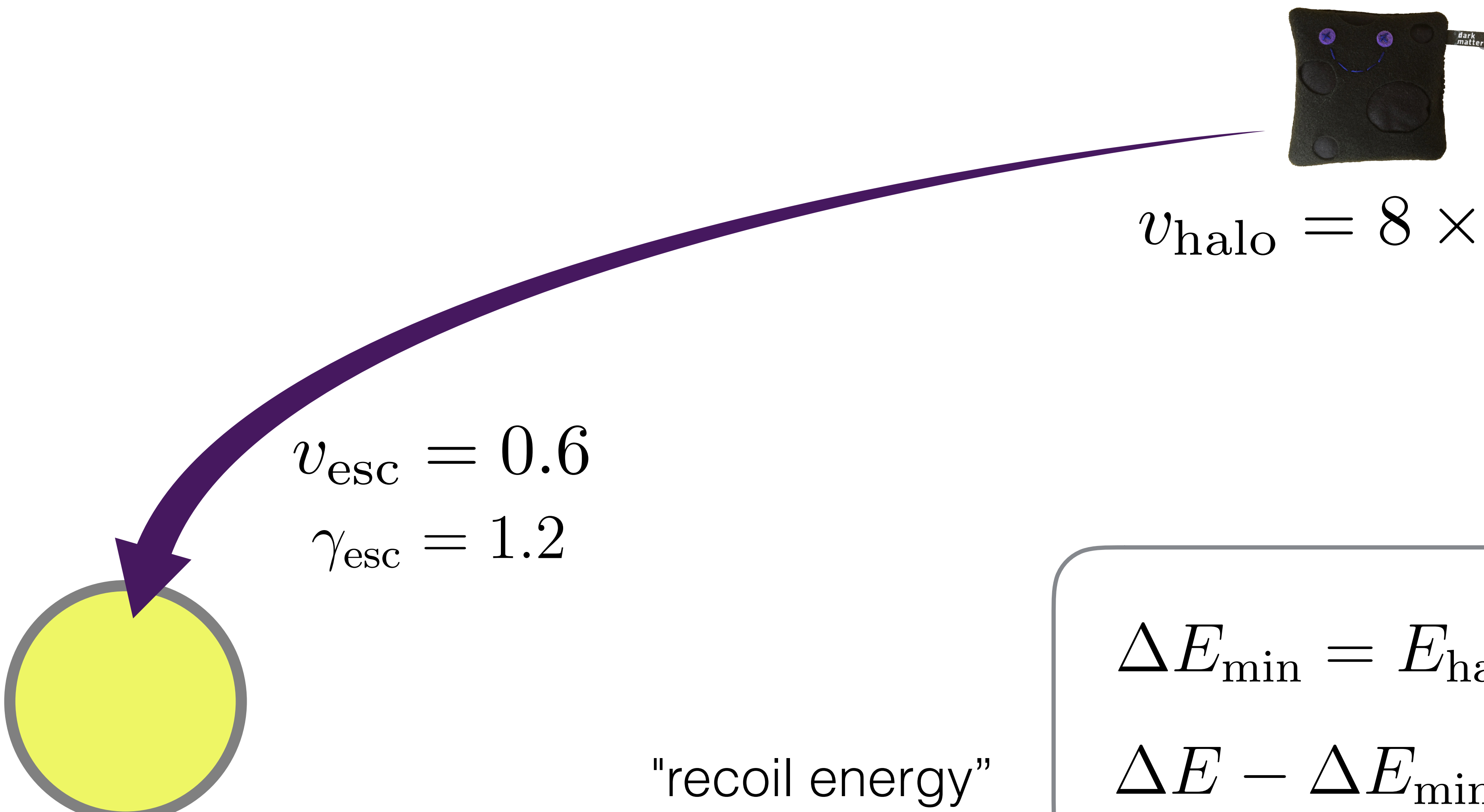
$2480\,{\rm K}\, f^{1/4}$

extra credit
not in this talk

Conditions for Capture & Heating

Dark matter must lose its asymptotic kinetic energy.

Velocity must be less than the escape velocity.



$$v_{\text{halo}} = 8 \times 10^{-4}$$

$$v_{\text{esc}} = 0.6$$

$$\gamma_{\text{esc}} = 1.2$$

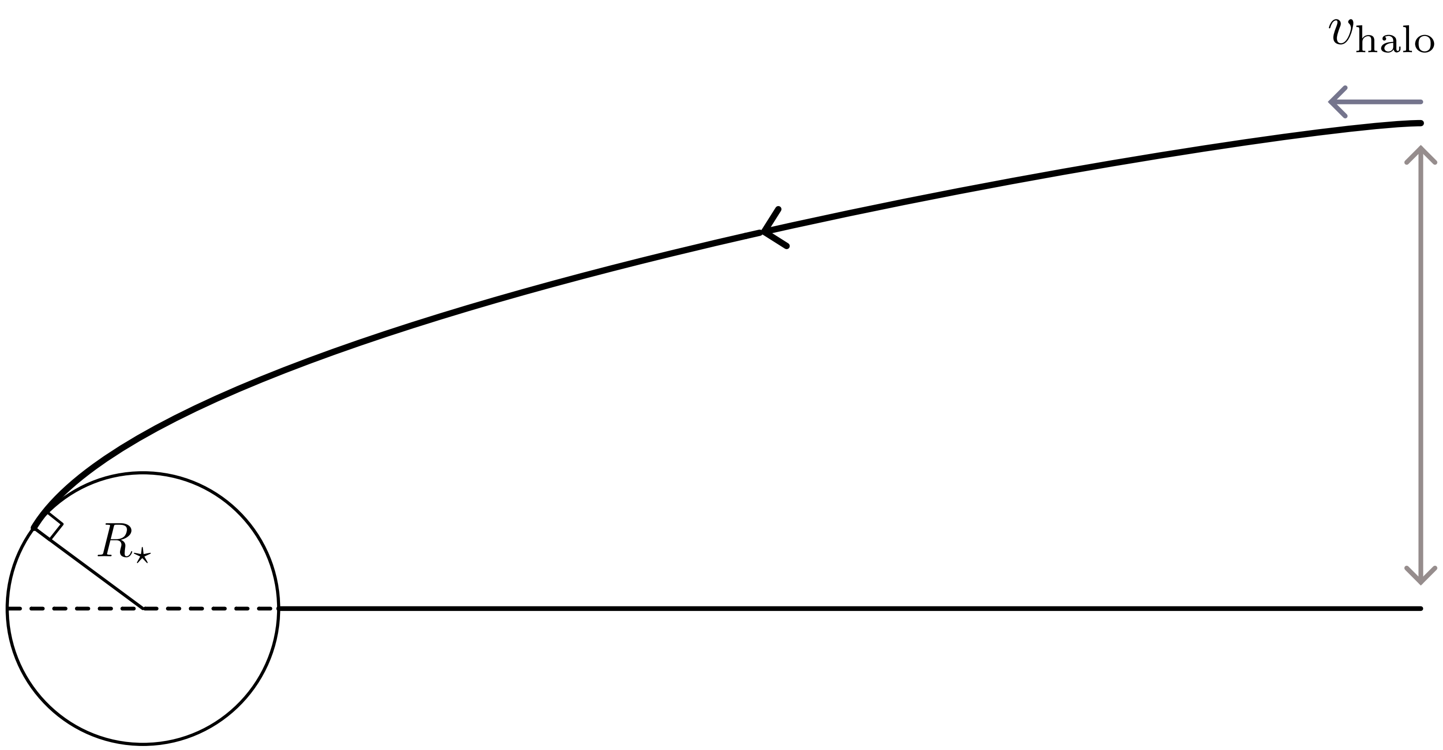
"recoil energy"

$$\Delta E_{\text{min}} = E_{\text{halo}} = \frac{1}{2} m_\chi v_{\text{halo}}^2$$

$$\Delta E - \Delta E_{\text{min}} > 0$$

Dark matter capture: see, e.g. Gould (1987, '88, …)

How much heating?



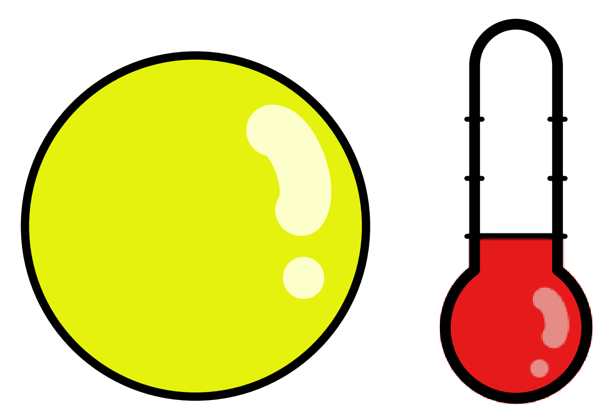
$$b_{\text{max}} = R_\star \frac{v_{\text{esc}}}{v_{\text{halo}}} \gamma_{\text{esc}}$$

$$\dot{M}_\chi = \pi b_{\text{max}}^2 v_{\text{esc}} \rho_\chi \approx 3.1 \times 10^{25} \, \frac{\text{GeV}}{\text{s}} \approx 55 \, \frac{\text{g}}{\text{s}} \, .$$

$$\dot{K} = (\gamma_{\text{esc}} - 1) \, \dot{M} f \approx 6.5 \times 10^{24} \text{ GeV s}^{-1} \, .$$

$$T_{\text{kin}} = 1600 \, f^{1/4} \, \text{K}$$

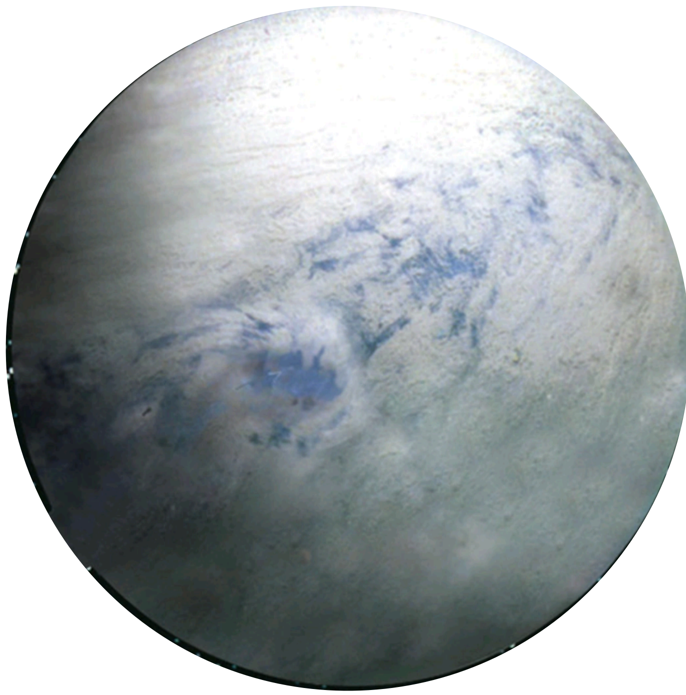
vs.



No heating:
100 K (Gyr)

b_{max}: see your favorite GR text no DM heating **Hoth**

DM kinetic heating **Mustafar**

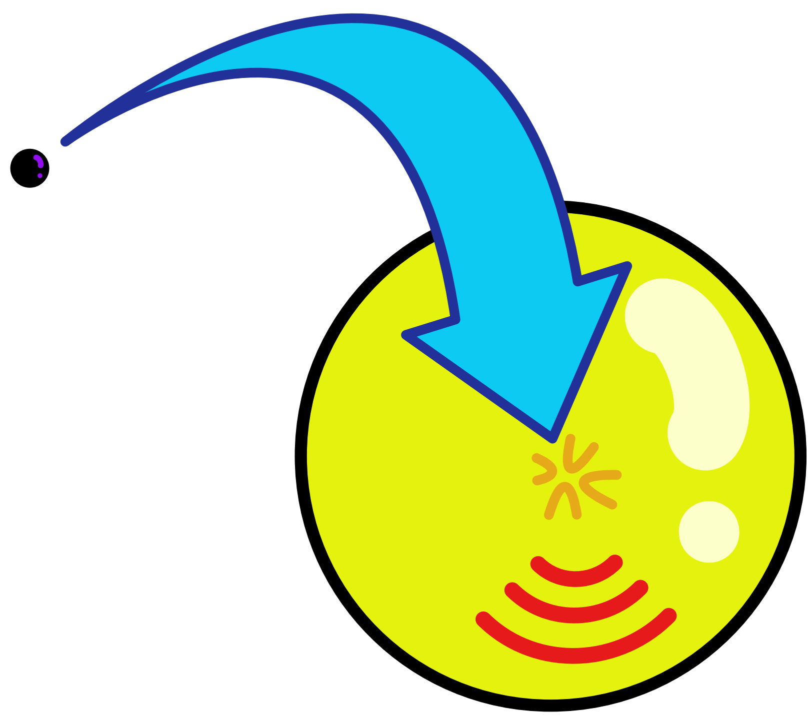


T ~ 100 K

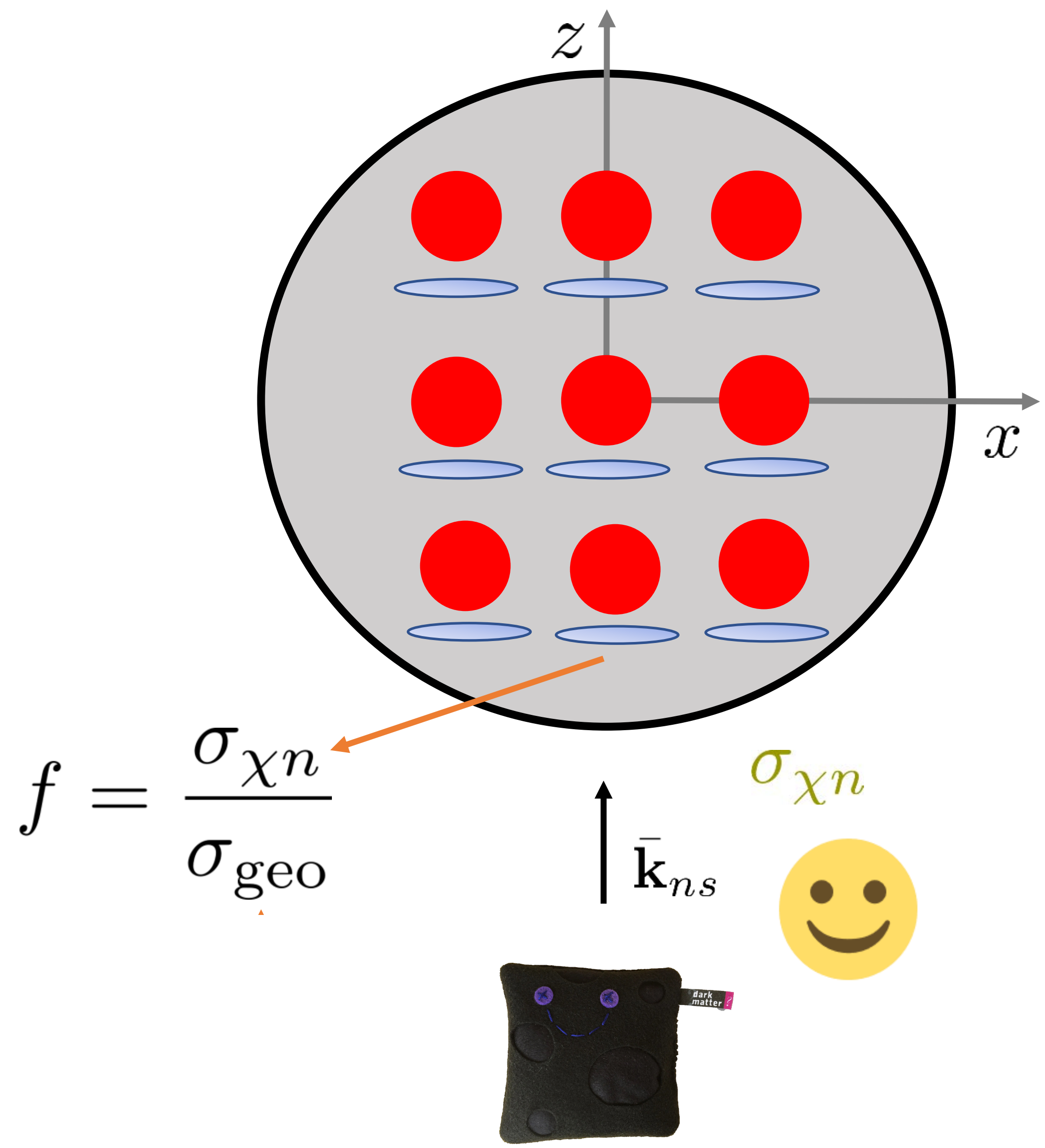
T ~ 1600 K

Images: Wookiepedia, *The Little Prince*

Geometric Cross Section



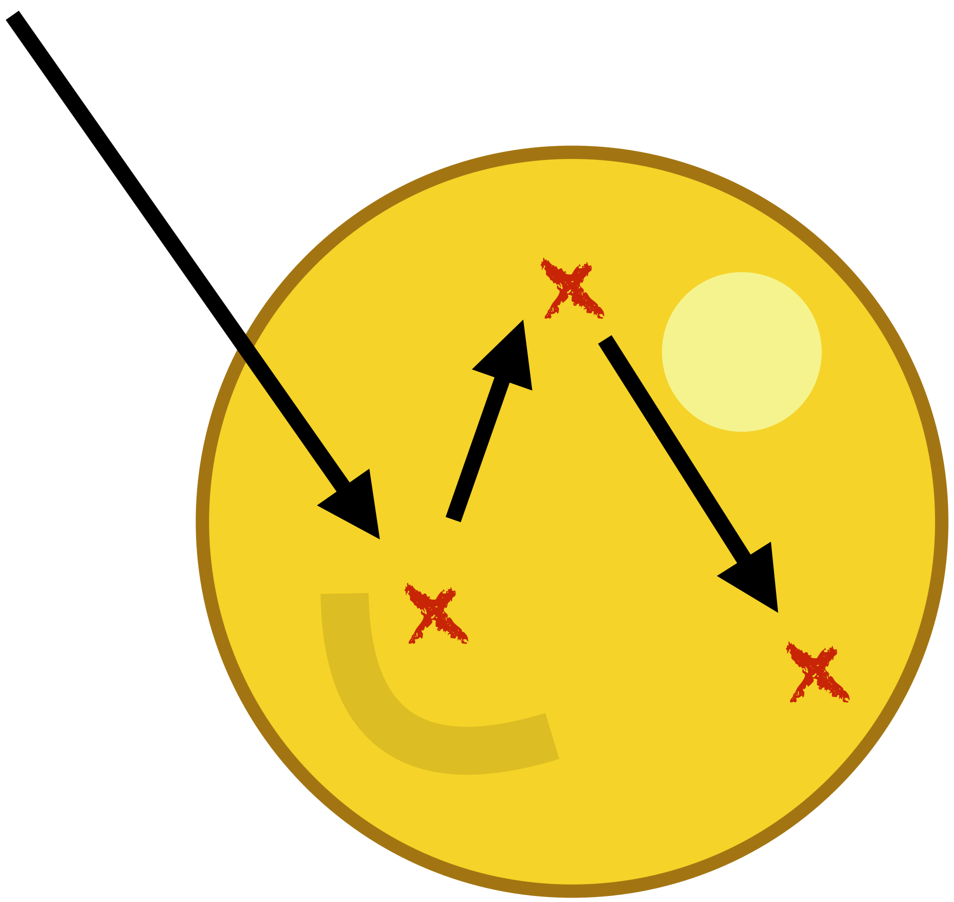
$$T_{\text{kin}} = 1600\, f^{1/4}\ \text{K}$$



Breaks down for extreme masses

Right: Aniket Joglekar

Breakdown 1: not enough ΔE



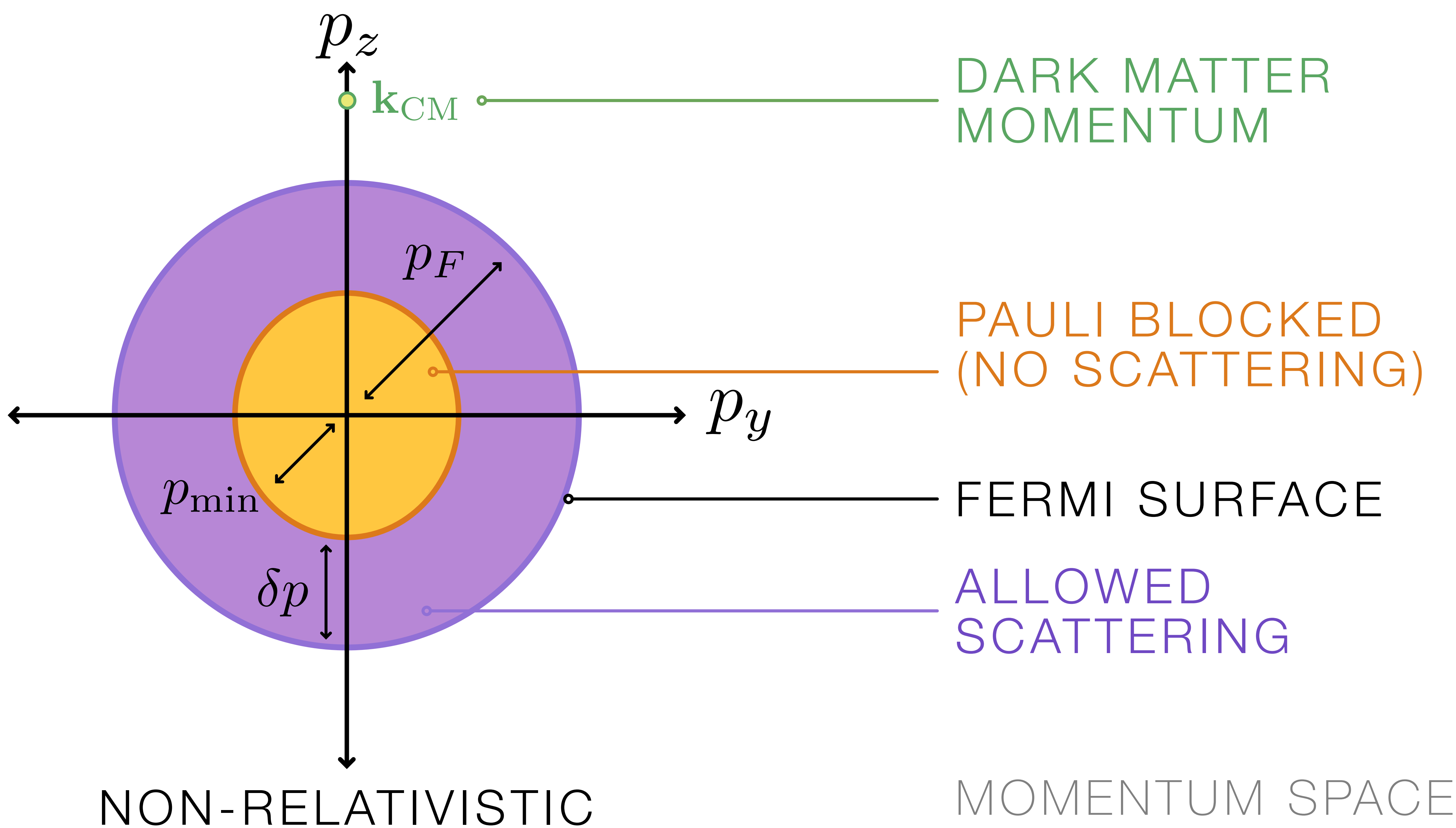
Heavy dark matter does not transfer enough energy.
(c.f. why we like Xenon)

Multiple scatters required to capture in the star.

Non-relativistic limit:

$$\Delta E = \frac{m_{\rm T} m_\chi^2}{m_\chi^2 + m_{\rm T}^2 + 2\gamma_{\rm esc} m_\chi m_{\rm T}} \frac{v_{\rm esc}^2}{1 - v_{\rm esc}^2} (1 - \cos\psi)$$

Breakdown 2: Pauli Blocking



Light DM can’t overcome Pauli blocking

Breakdown of Geometric σ

m_χ

TOO HEAVY

$$\sigma^{\text{multi}}_{\text{thres.}} \approx \frac{m_\chi}{10^6\ \text{GeV}} \sigma_{\text{thres.}}$$

not enough energy transfer

10^6 GeV

GOLDILOCKS

$$\sigma_{\text{thres.}} = \frac{\text{geometric cross section}}{\text{number of targets}} = \pi R_\star^2 \frac{m_{\text{T}}}{M_\star}$$

GeV

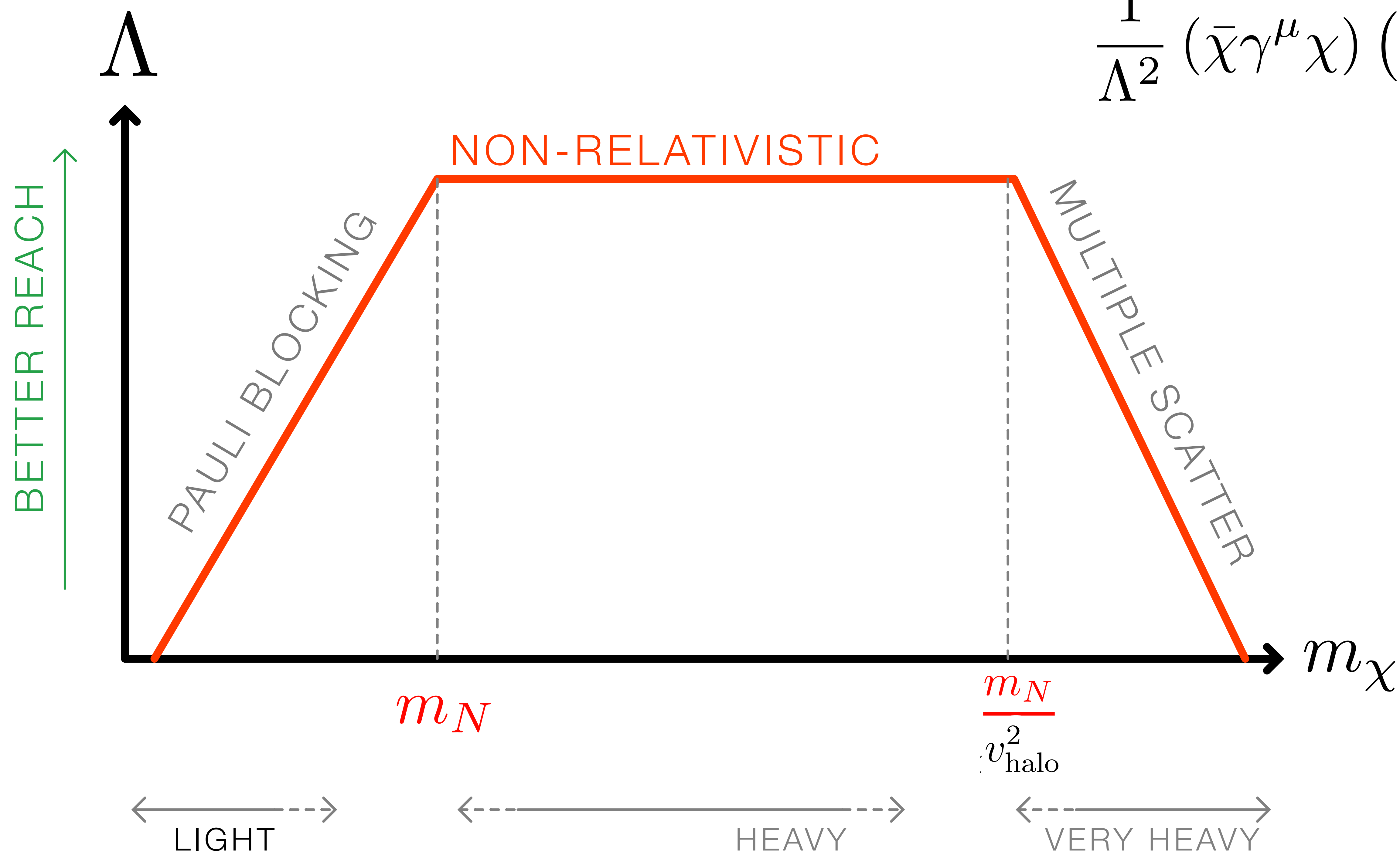
TOO LIGHT

$$\sigma^{\text{Pauli}}_{\text{thres.}} = \frac{1}{3}\frac{p_F}{\delta p}\sigma_{\text{thres.}} \approx \frac{\text{GeV}}{m_\chi}\sigma_{\text{thres.}}$$

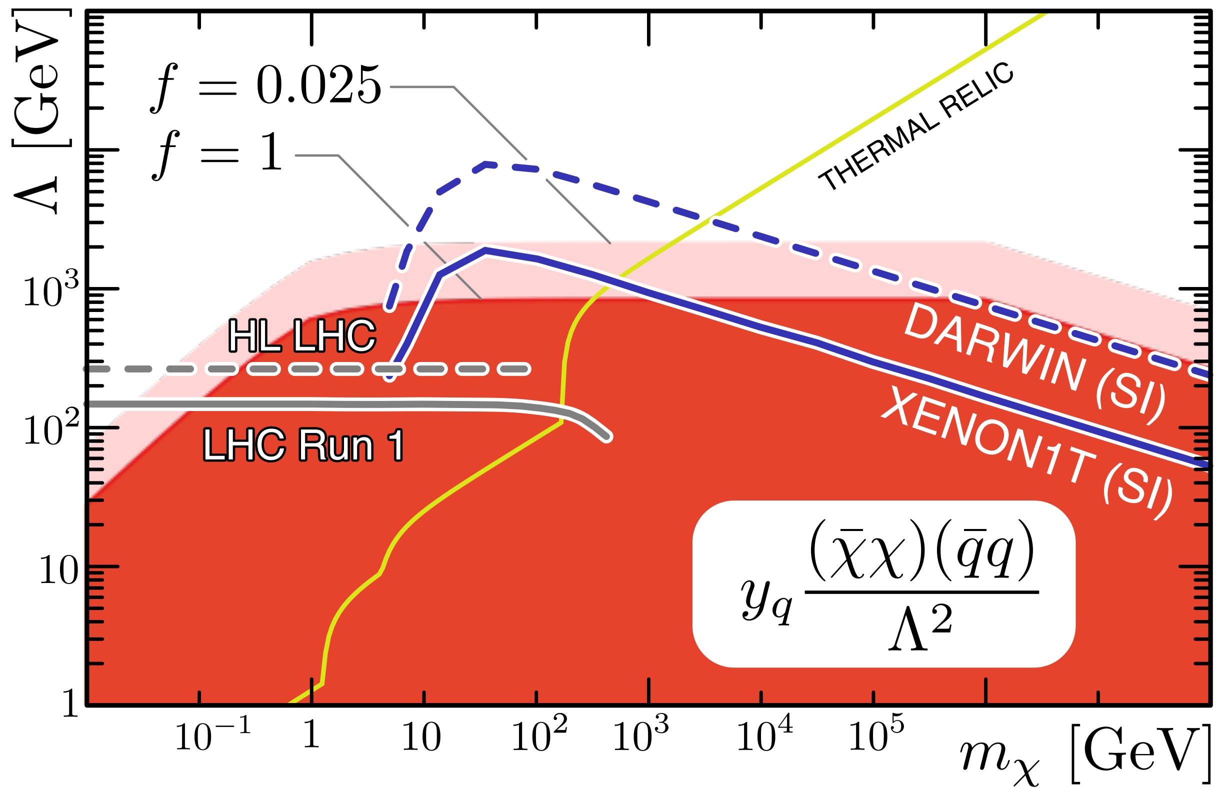
not enough phase space

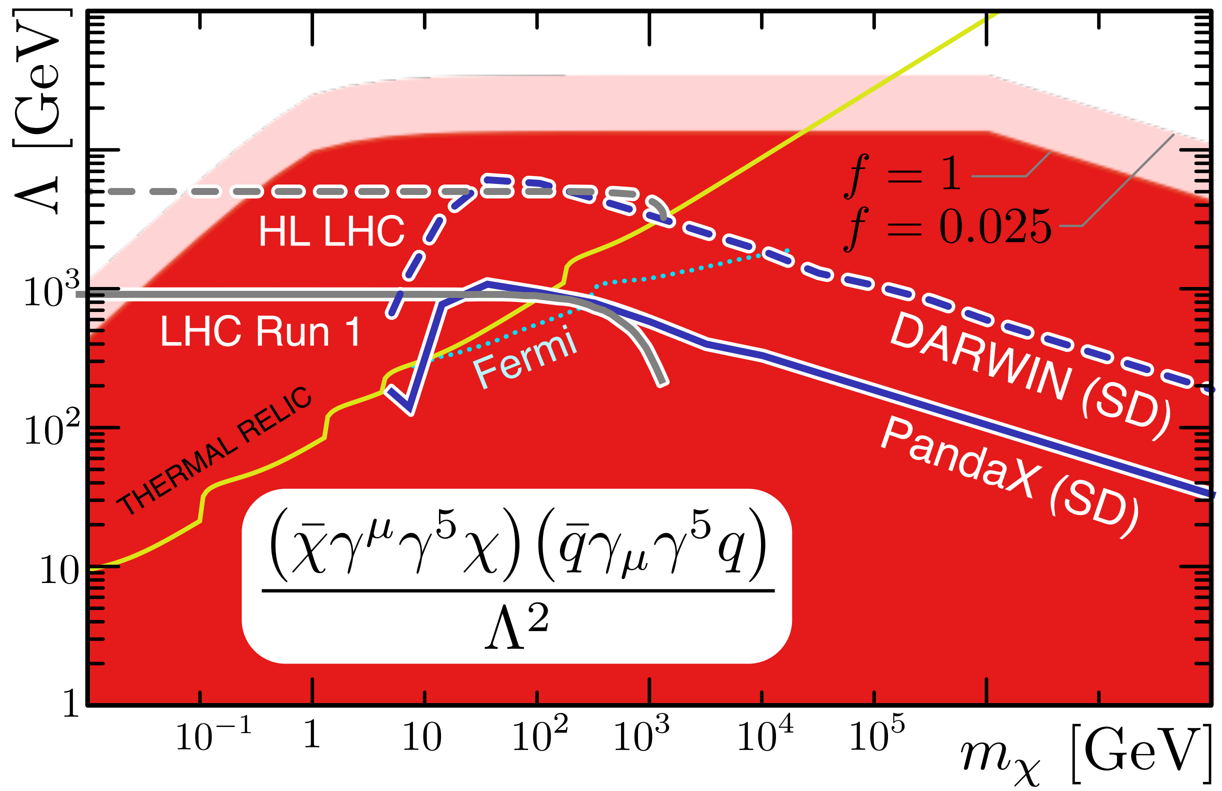
Result: reach of kinetic heating

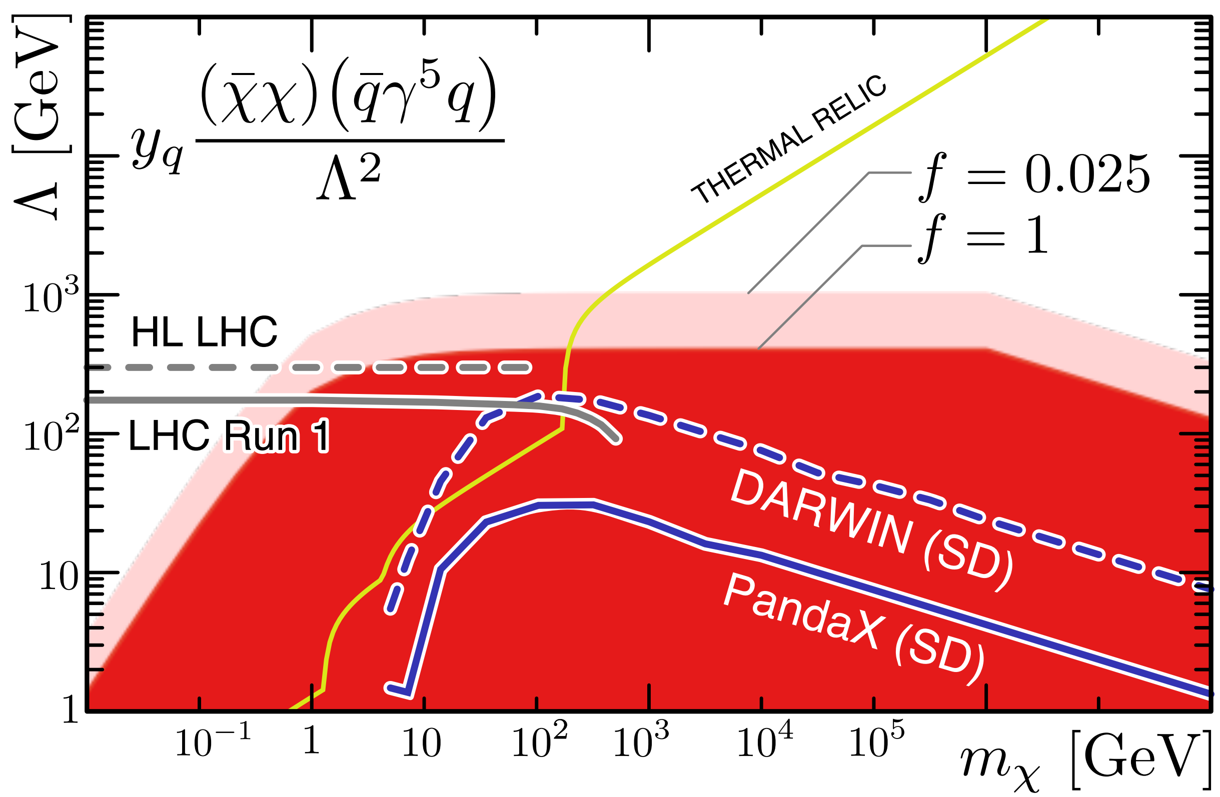
$$\frac{1}{\Lambda^2}\left(\bar{\chi}\gamma^{\mu}\chi\right)\left(\bar{\xi}\gamma_{\mu}\xi\right)$$

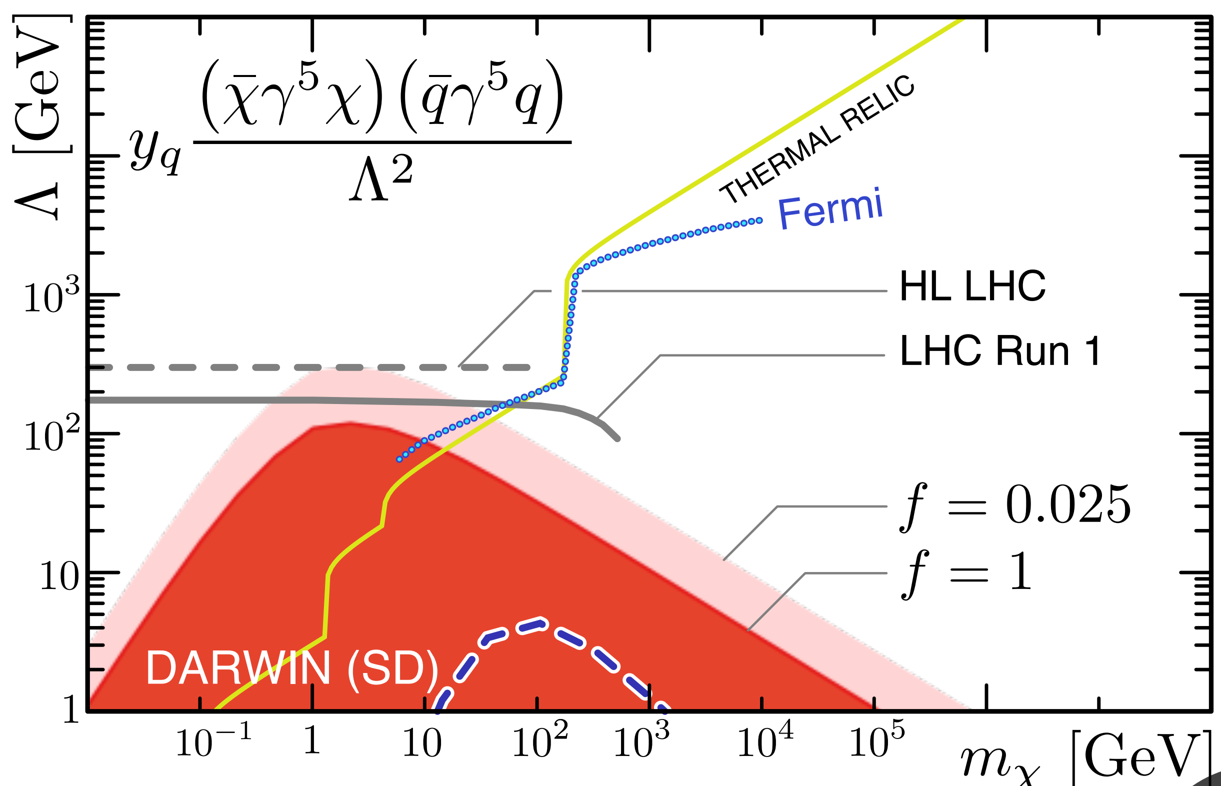


Comparing to Direct Detection



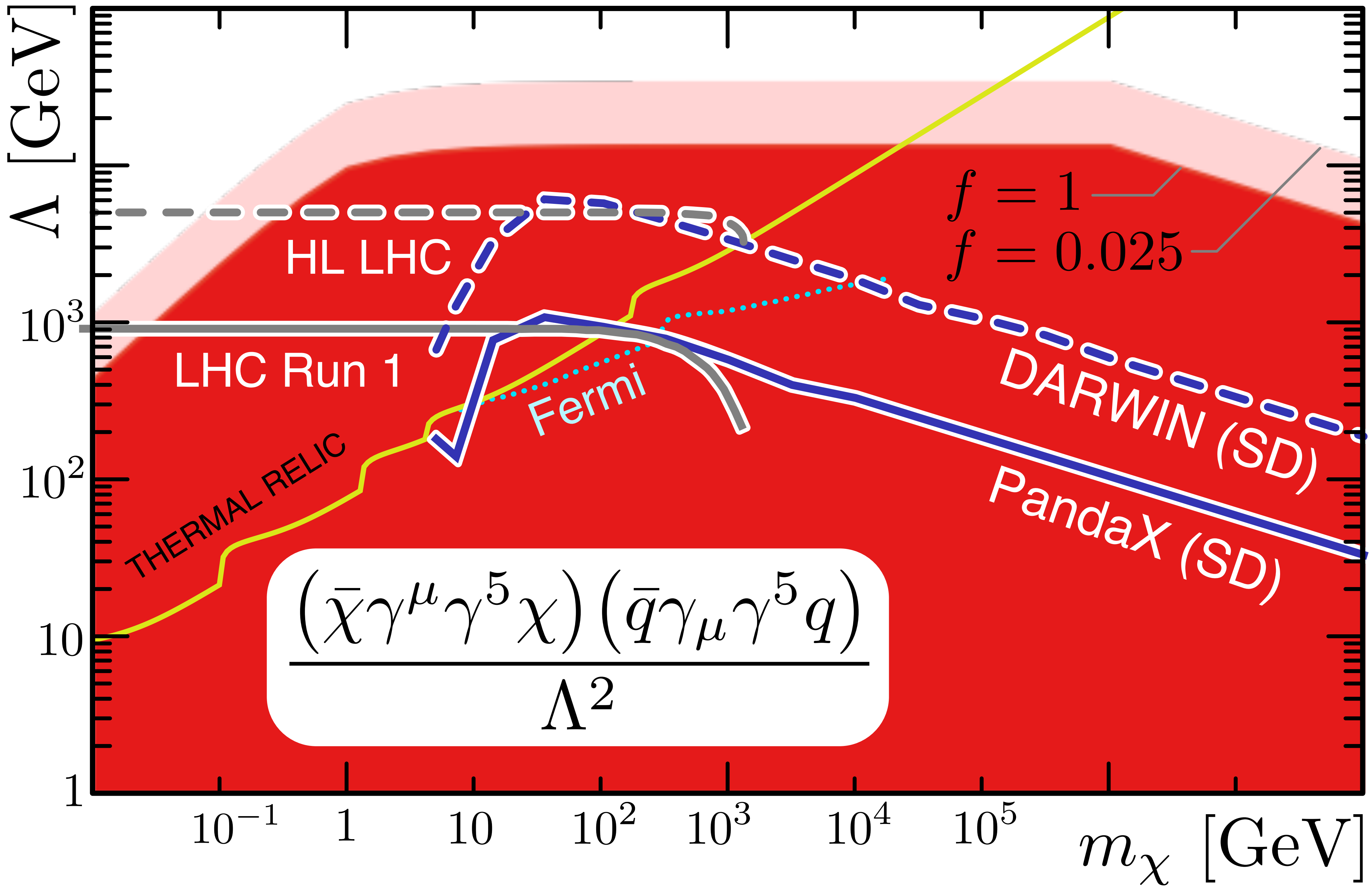






Raj, FT, Yu, 1707.09442

Typical Comparison



Raj, FT, Yu, 1707.09442

How we win (vs direct detection)
how we complement existing program

Large volume, high density

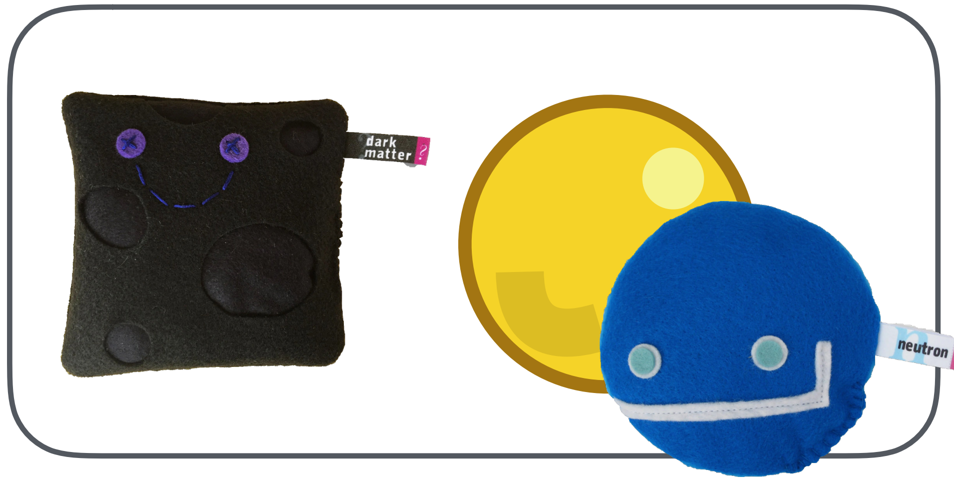
Dark matter is accelerated
Better reach for momentum-suppressed interactions, inelastic scattering (up to 200 MeV)

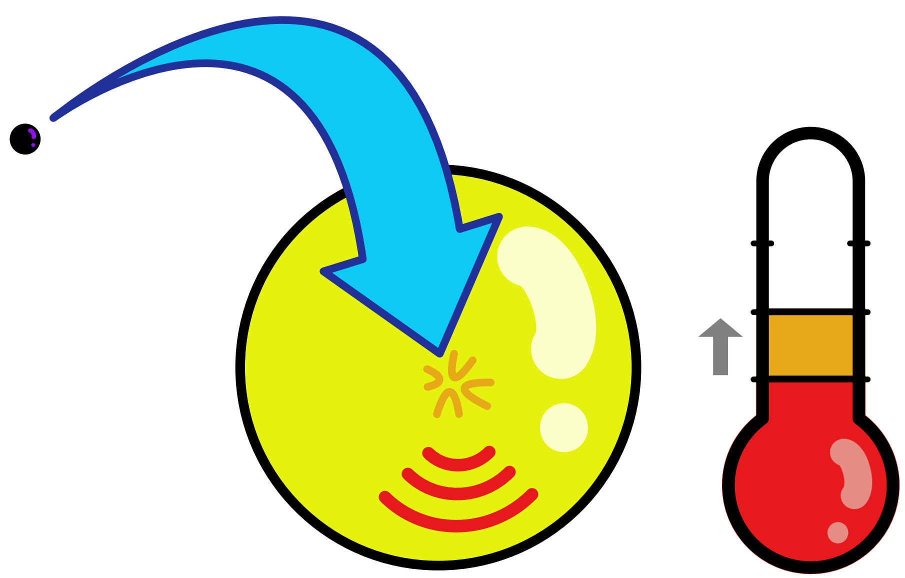
No ceiling (strong int) **or floors** (neutrino BG)**.**

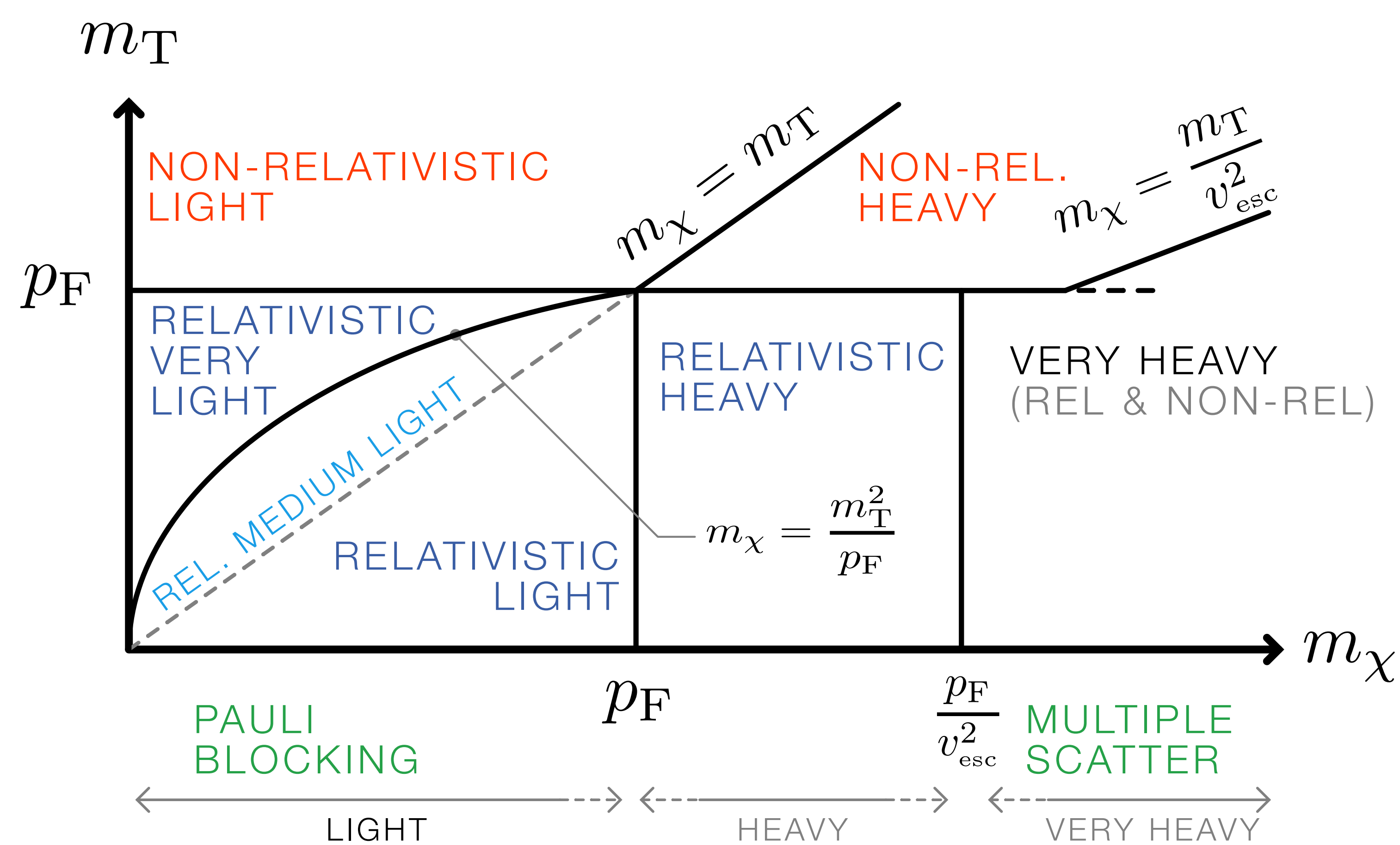
Larger range of accessible dark matter masses

No hierarchy between SI and SD scattering.

Outline







Julie Peasley, particlezoo.net

What else can we do?

Leptophilic?

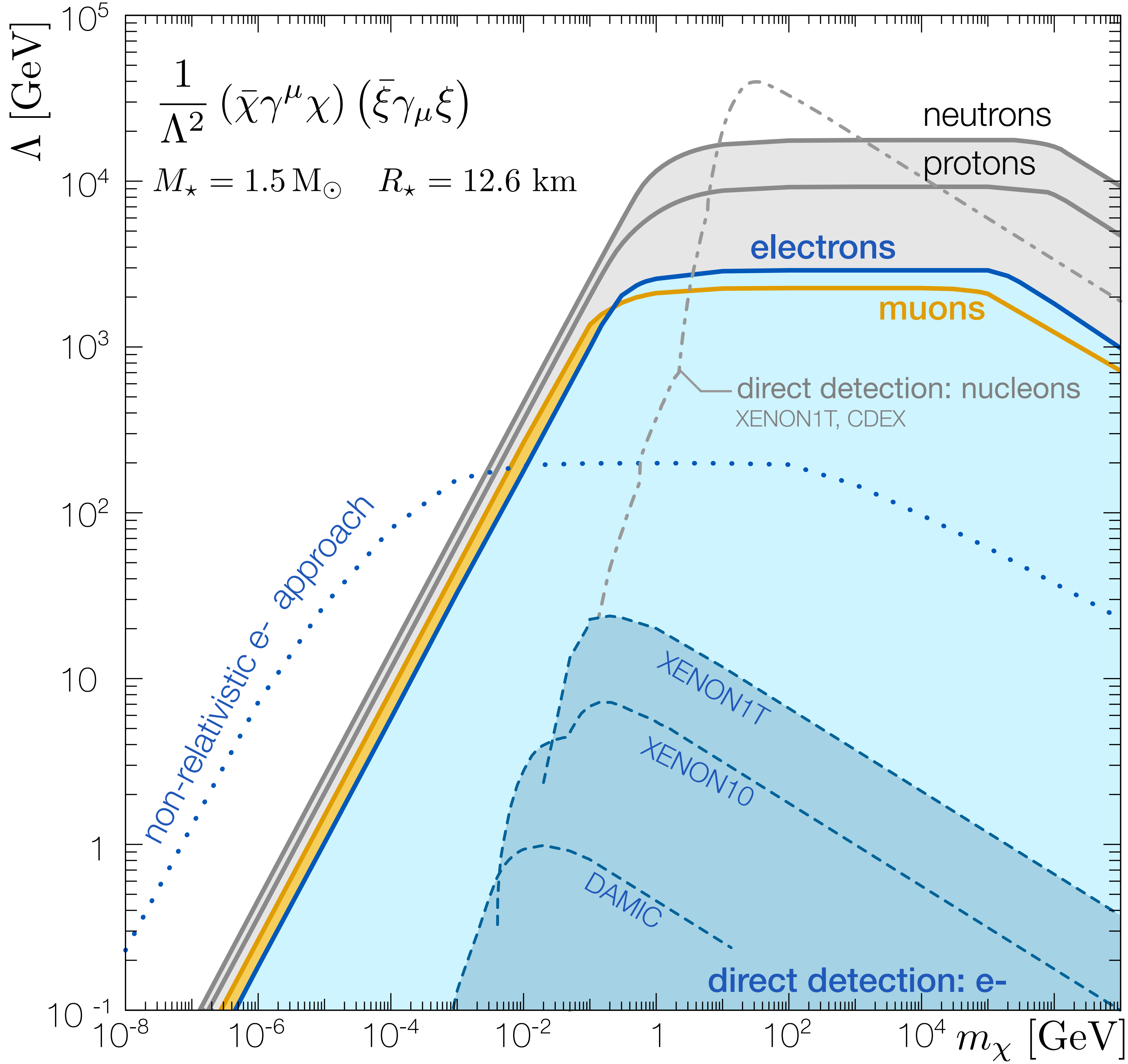
Julie Peasley, particlezoo.net

Result

can beat lepton direct detection

curiously similar reach to neutrons

dotted line?



Joglekar, Raj, FT, Yu, 1911.13293

Non-relativistic estimate

Capture of Leptophilic Dark Matter in Neutron Stars

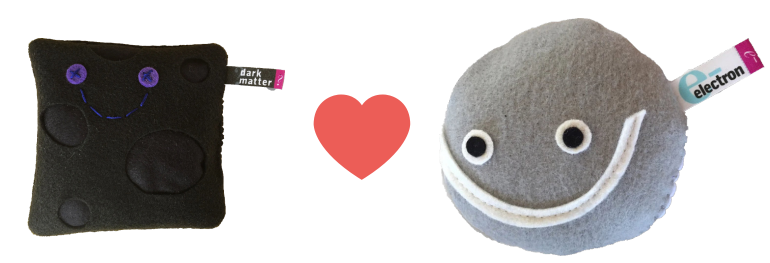
Nicole F. Bell,[a] Giorgio Busoni[b] and Sandra Robles[a]

Abstract. Dark matter particles will be captured in neutron stars if they undergo scattering interactions with nucleons or leptons. These collisions transfer the dark matter kinetic energy to the star, resulting in appreciable heating that is potentially observable by forthcoming infrared telescopes. While previous work considered scattering only on nucleons, neutron stars contain small abundances of other particle species, including electrons and muons. We perform a detailed analysis of the neutron star kinetic heating constraints on leptophilic dark matter. We also estimate the size of loop induced couplings to quarks, arising from the exchange of photons and Z bosons. Despite having relatively small lepton abundances, we find that an observation of an old, cold, neutron star would provide very strong limits on dark matter interactions with leptons, with the greatest reach arising from scattering off muons. The projected sensitivity is orders of magnitude more powerful than current dark matter-electron scattering bounds from terrestrial direct detection experiments.

treat electrons as little neutrons

KILLER APP

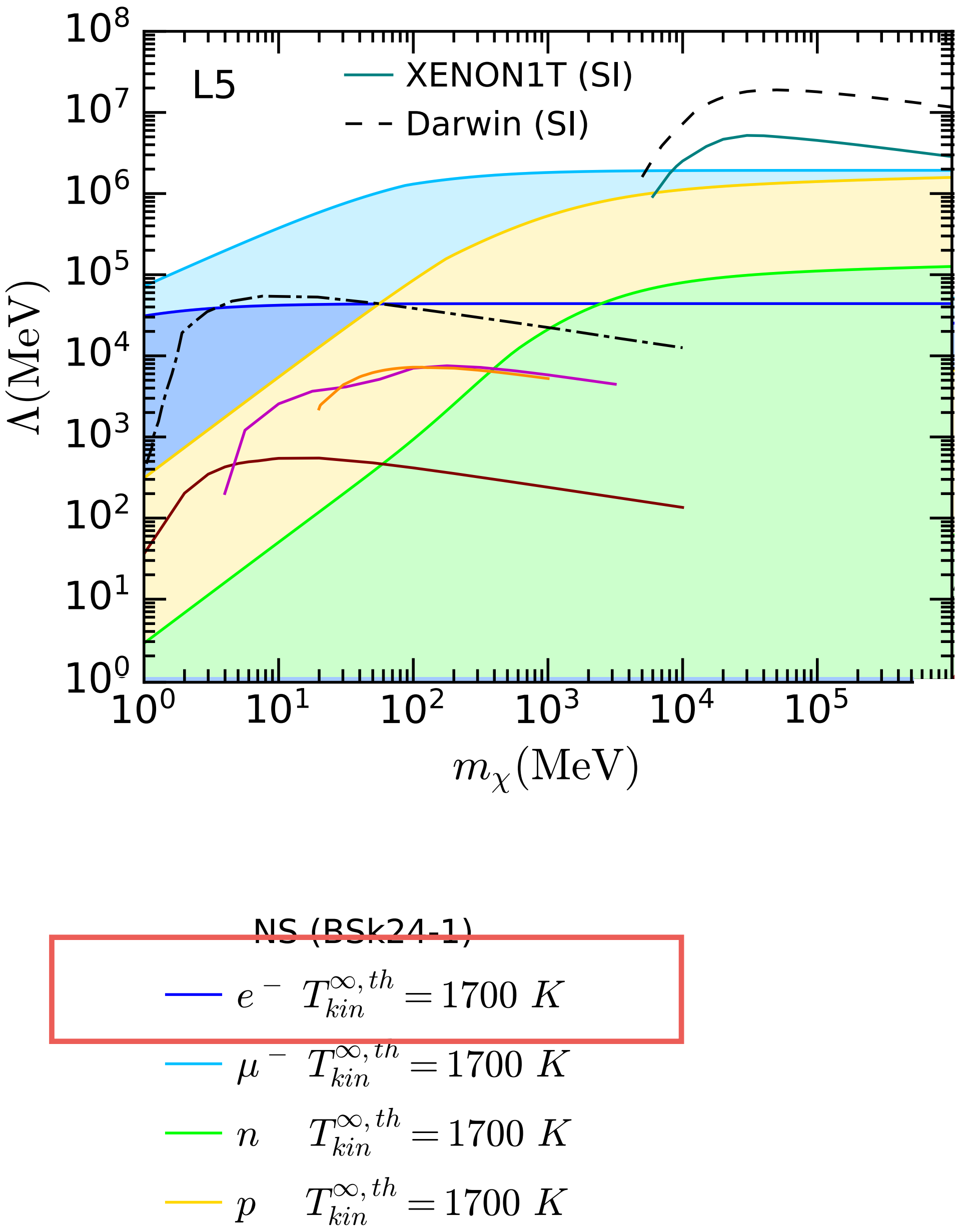
leptophilic dark matter

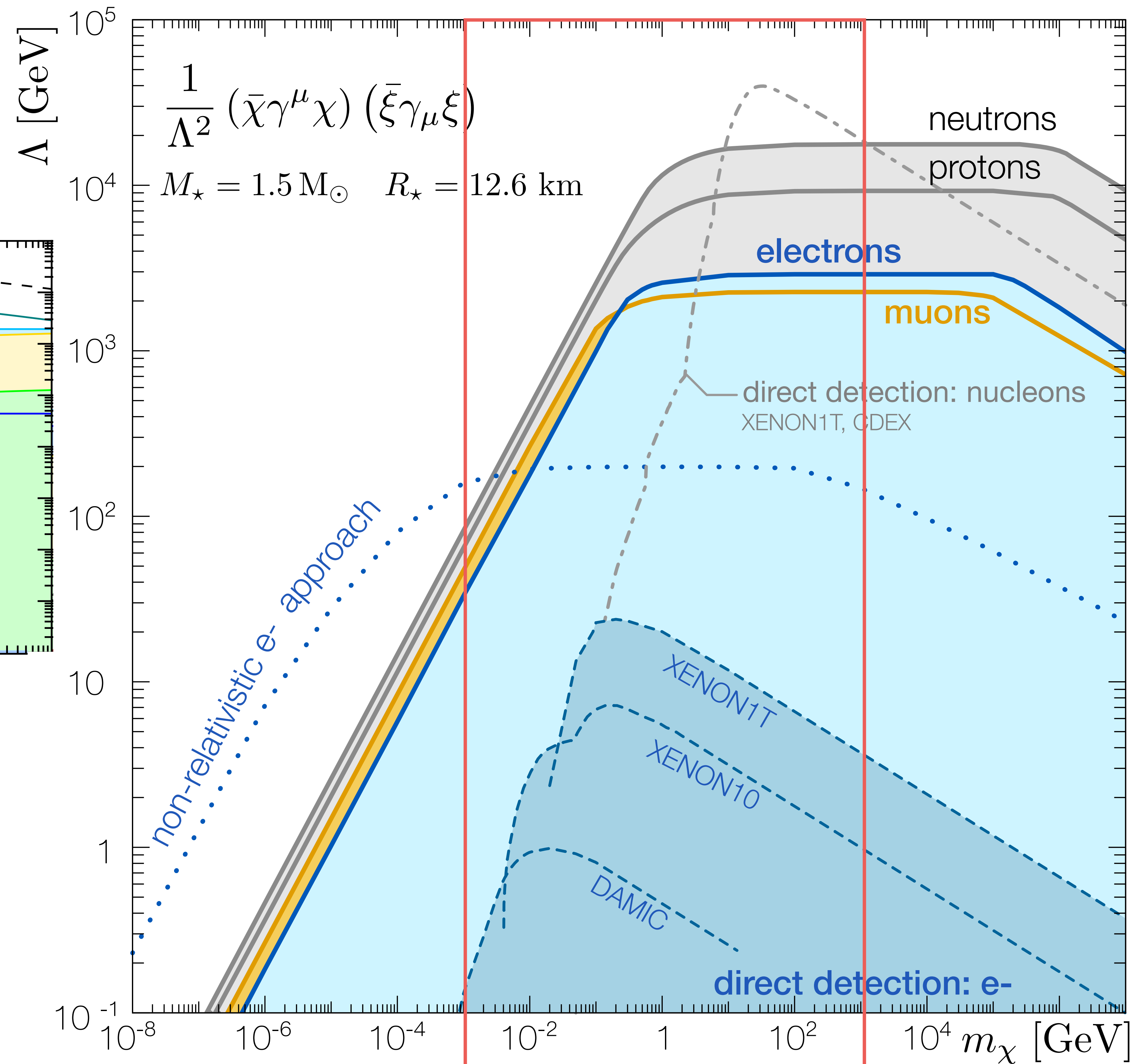


result: muons very promising, electrons are okay

1904.09803

comparison

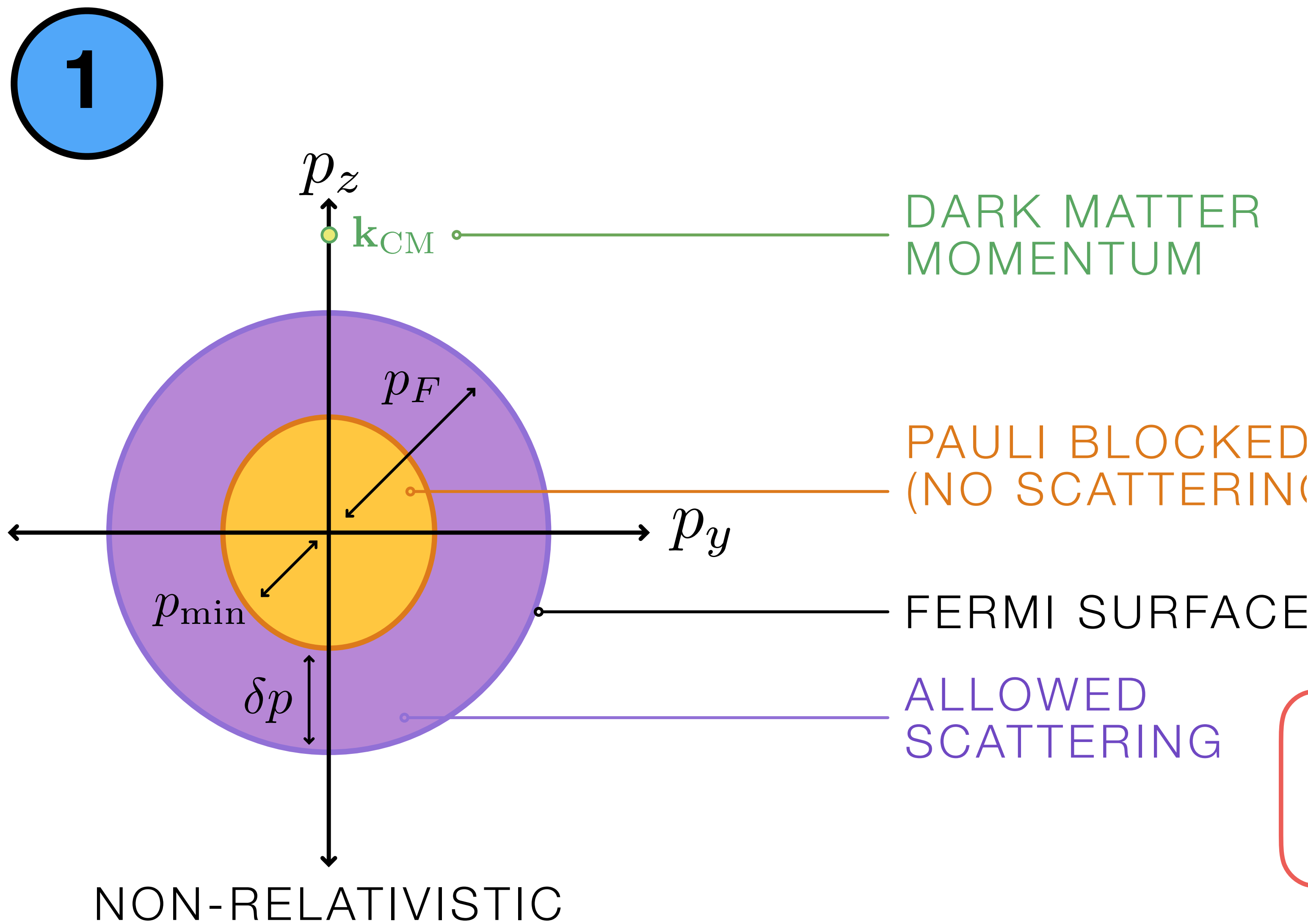


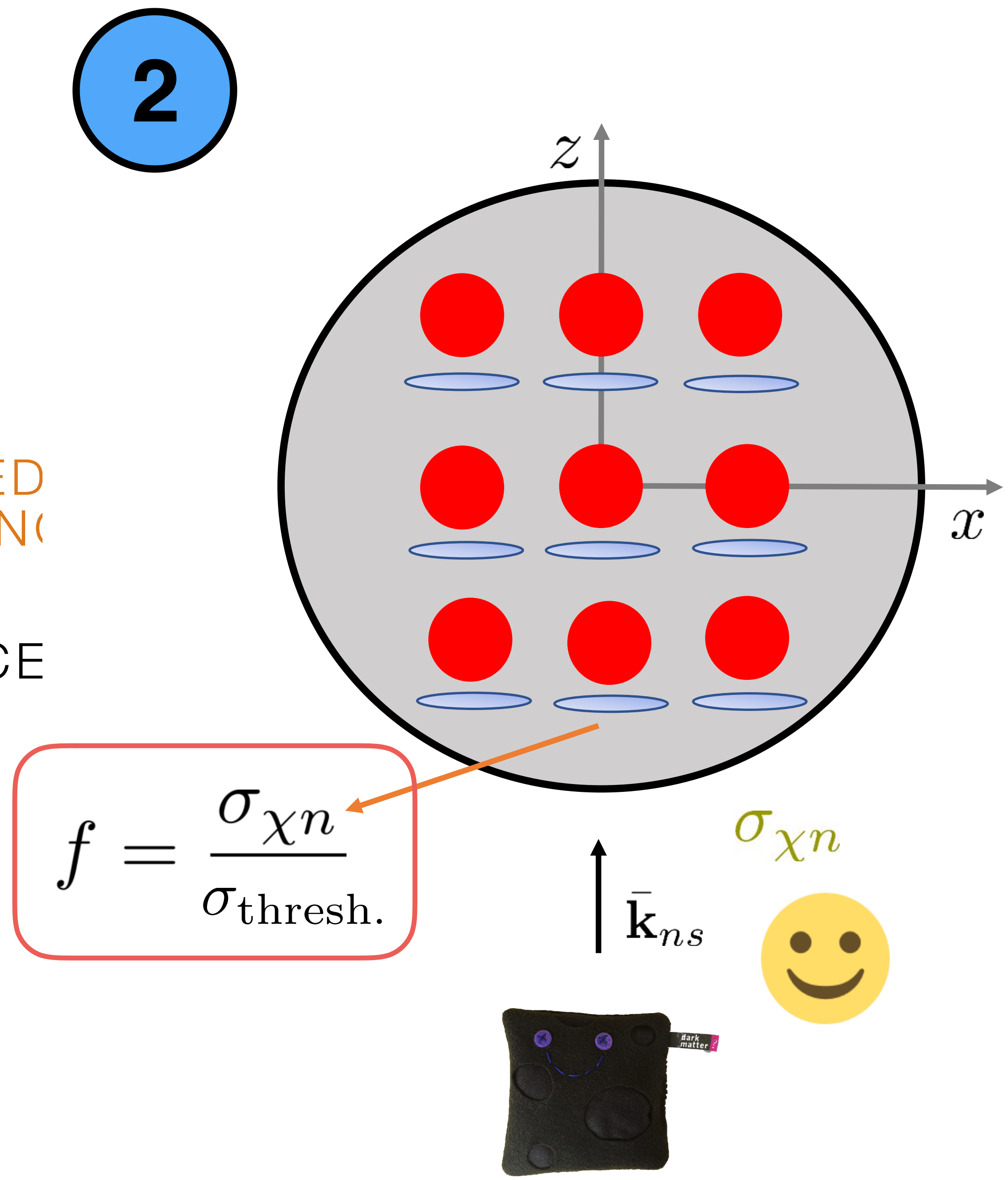


Bell et al. 1904.09803, Joglekar et al. 1911.13293 & coming soon

Simplifying Assumptions

non-relativistic limit

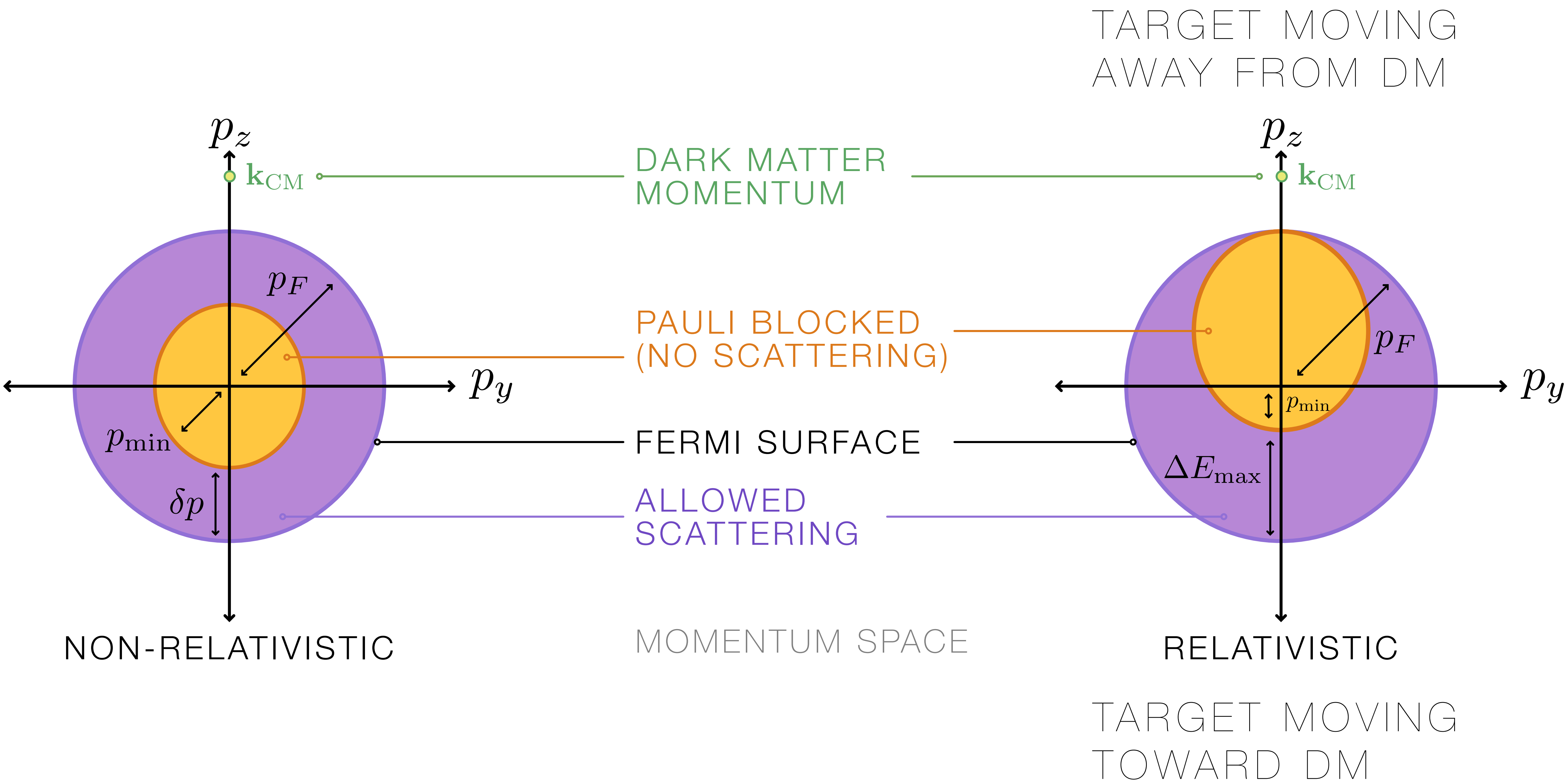




However: electrons are relativistic.

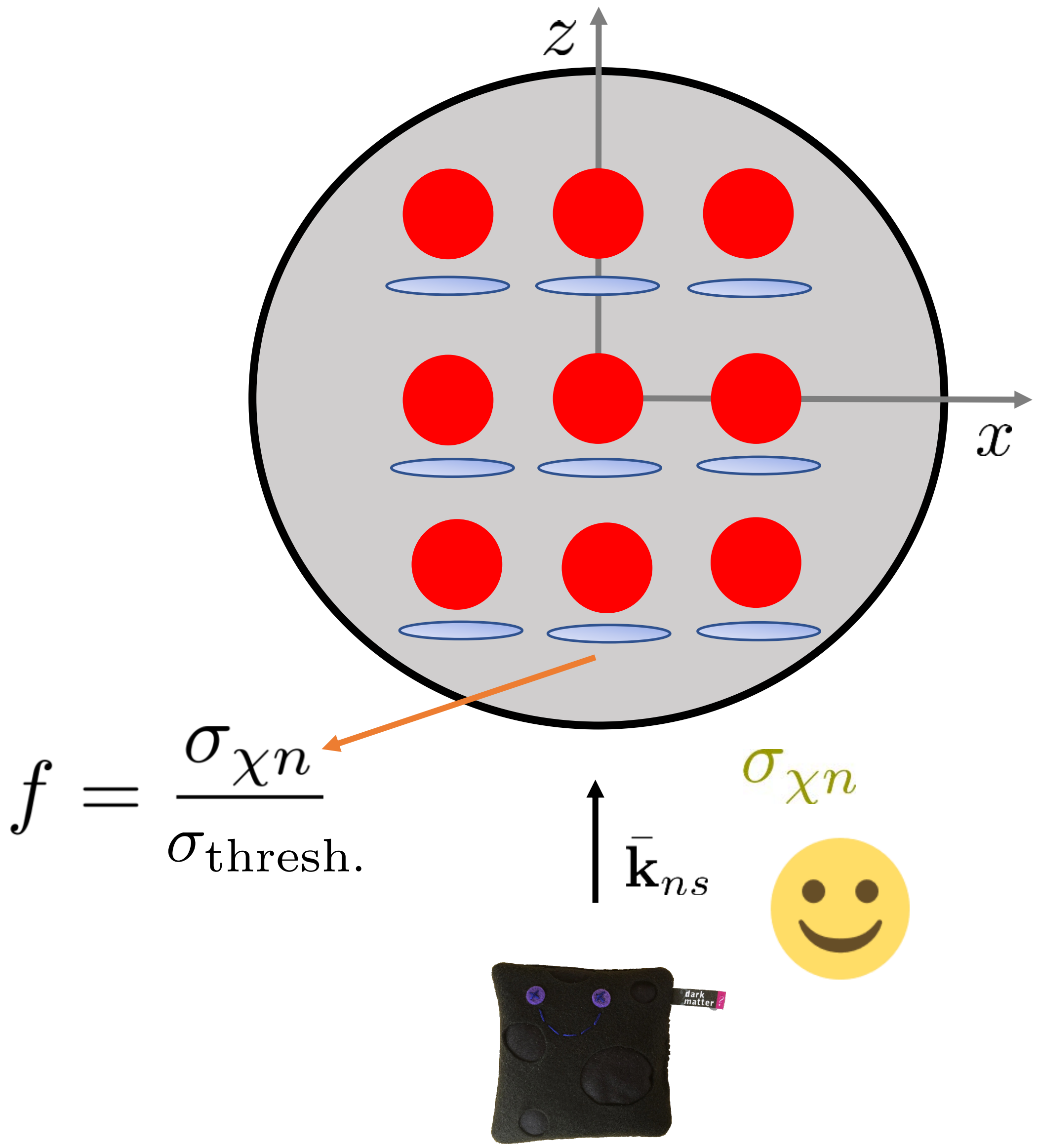
Right: Aniket Joglekar

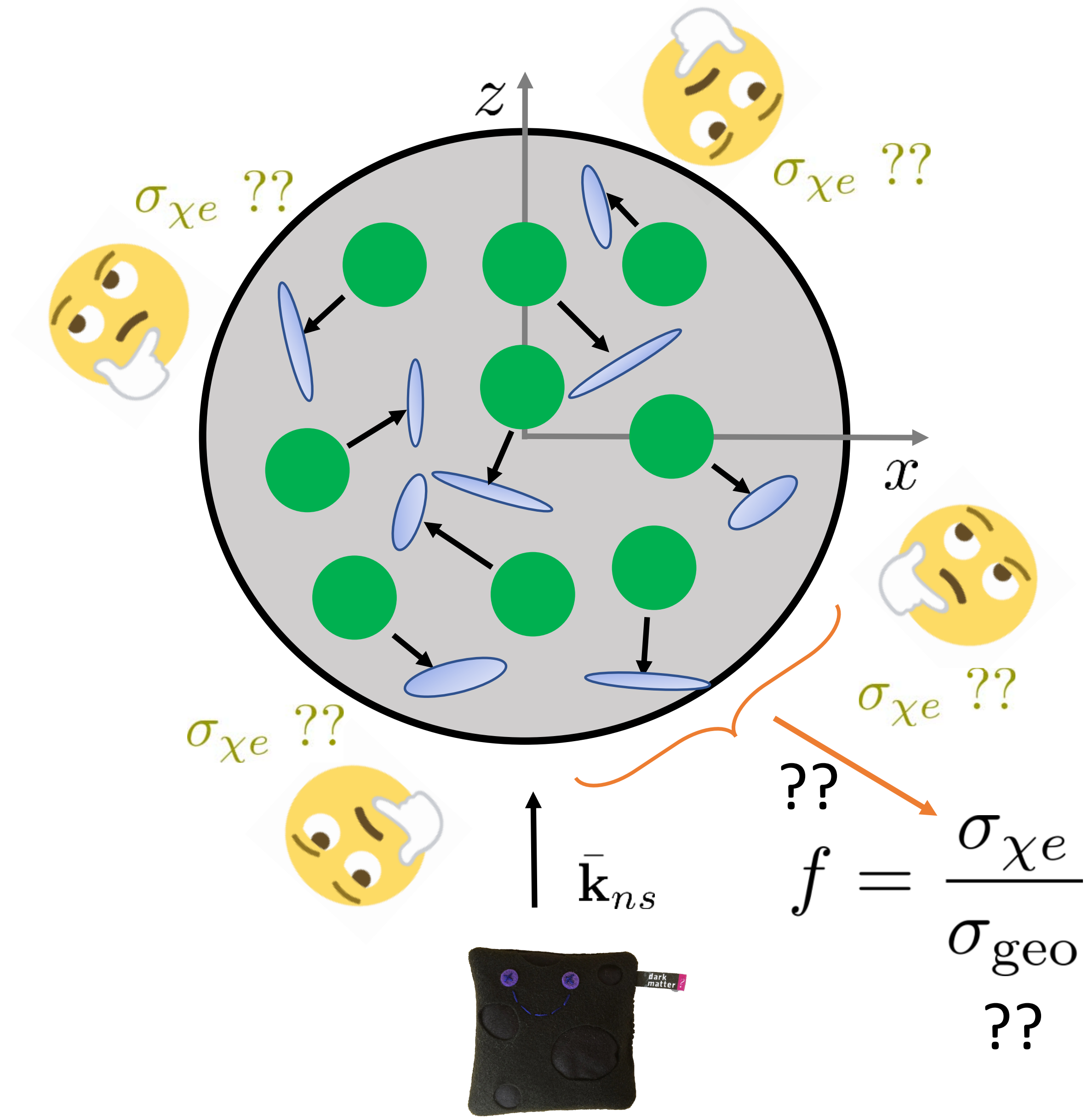
1. Relativistic Pauli Blocking



Some configurations favor capture.

2. … which cross section?

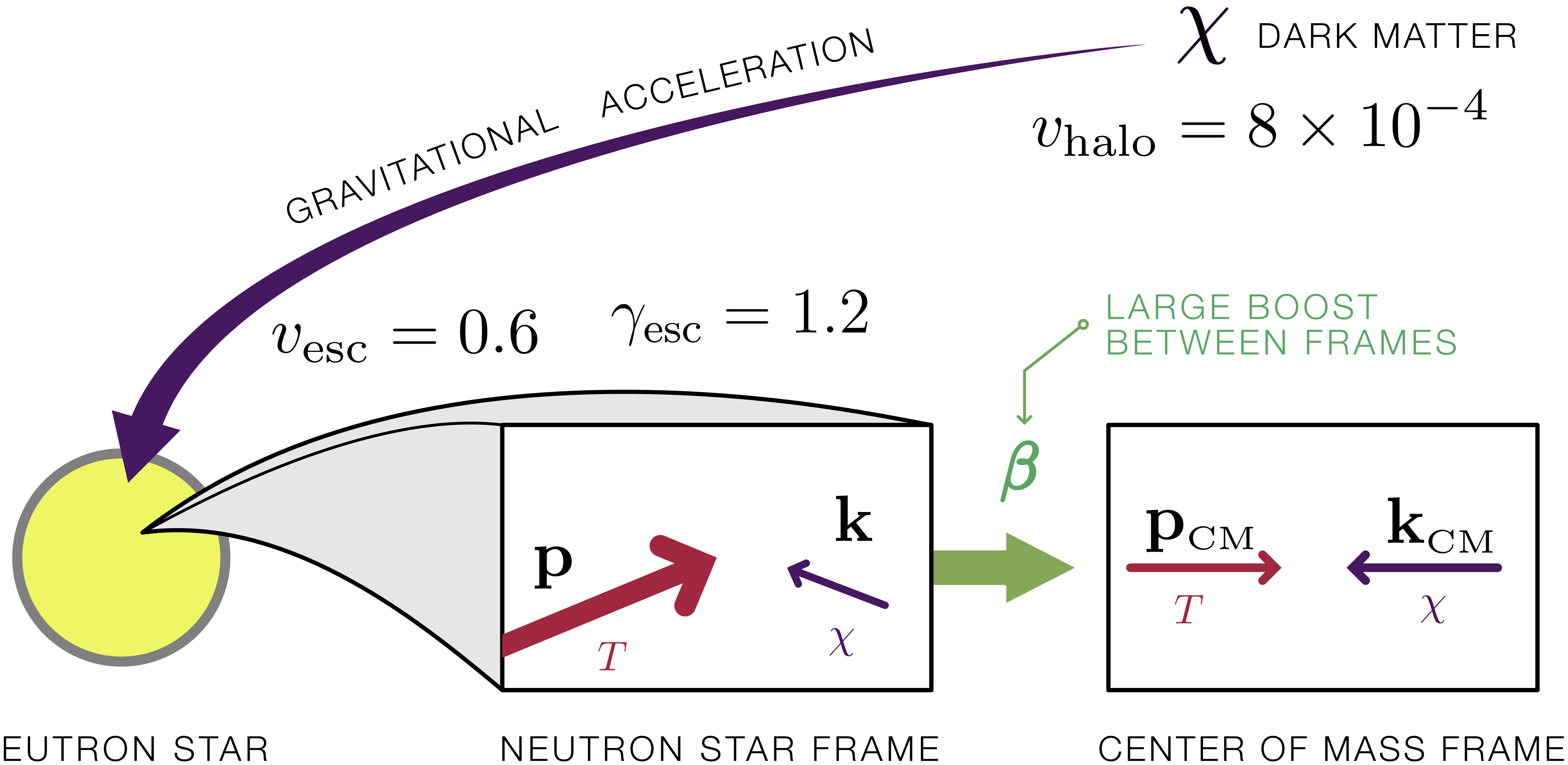




Cross section depends on kinematics

Images: Aniket Joglekar

A matter of frame



Relativistic Formalism

\# OF SCATTERS:

$$d\nu = d\sigma\, v_{\text{rel}}\, dn_{\text{T}}\, dn_\chi\, \Delta V\, \Delta t$$

$$\Delta t \approx 3.2\ R_\star$$

CAPTURE EFFICIENCY

$$df = \left.\frac{d\nu}{dN_\chi}\right|_{\textbf{capture}}$$

CAPTURE CONDITIONS

\# OF INCIDENT DM:

$$dN_\chi = dn_\chi \Delta V$$

Lorentz Invariant

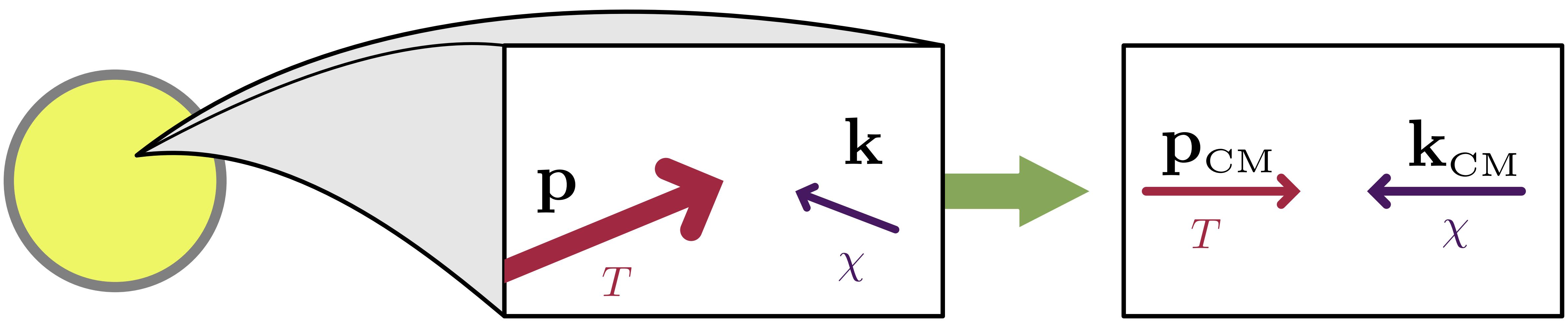
Frames and Lorentz Invariance

LORENTZ INVARIANT

$$df = d\sigma\, v_{\text{rel}}\; dn_{\text{T}}\, \Delta t\,\big|_{\text{capture}}$$

CENTER OF MASS FRAME?

NEUTRON STAR FRAME?



NS PROPERTIES

CROSS SECTION

Reminder: Möller Velocity

RELATIVISTIC
RELATIVE
VELOCITY

$$d\sigma \, v_{\text{rel}} = d\sigma_{\text{CM}} \, v_{\text{M\o l}}$$

FRAME | CM FRAME | FRAME

ANY FRAME

$$v_{\text{M\o l}} = \frac{\sqrt{(p\cdot k)^2 - m_{\text{T}}^2 m_\chi^2}}{E_p E_k}$$

see, e.g. Cannoni 1605.00569

Invariant capture efficiency

LORENTZ INVARIANT

$$df = d\sigma\, v_{\text{rel}}\; dn_{\text{T}}\, \Delta t\,\big|_{\text{capture}}$$

CENTER OF MASS FRAME

NEUTRON STAR FRAME

$$df = d\sigma_{\text{CM}}\; v_{\text{Møl}}\, dn_{\text{T}}\, \Delta t\,\big|_{\text{capture}}$$

CENTER OF MASS FRAME

NEUTRON STAR FRAME

Capture conditions

$$df = d\sigma_{\text{CM}}\, v_{\text{Møl}}\, dn_{\text{T}}\, \Delta t \,\big|_{\text{capture}}$$

$$df = \sum_{N_{\text{hit}}} d\sigma_{\text{CM}}\, v_{\text{Møl}}\, dn_{\text{T}} \frac{\Delta t}{N_{\text{hit}}}$$

$$\times\; \Theta\left(\Delta E - \frac{E_{\text{halo}}}{N_{\text{hit}}}\right) \Theta\left(\frac{\Delta E_{\text{min}}}{N_{\text{hit}}+1} - \Delta E\right)$$

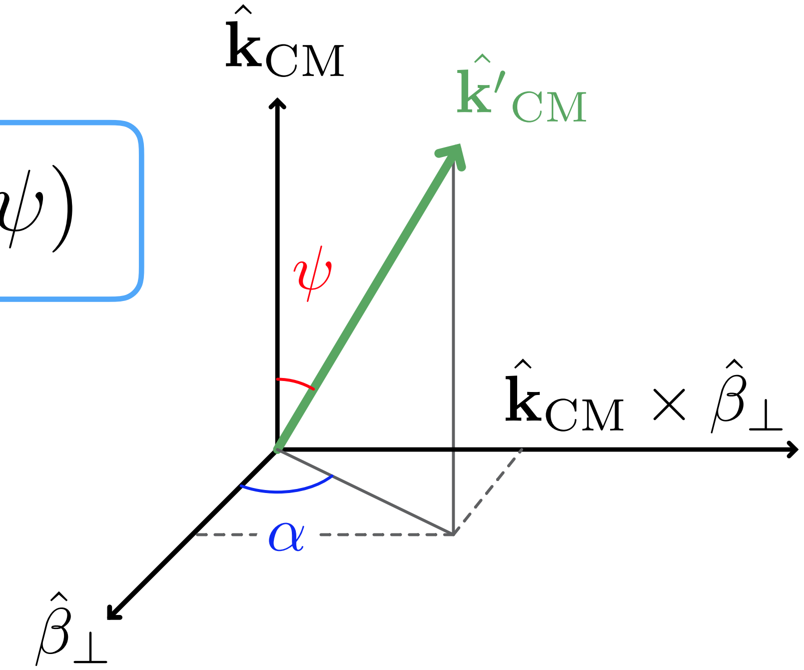
energy transfer leads to capture in N hits

$$\times\; \Theta\left(\Delta E + E_p - E_{\text{F}}\right)$$

Pauli blocking of final state

NIntegrate

$$d\Omega_{\text{CM}} = d\alpha\, d(\cos\psi)$$

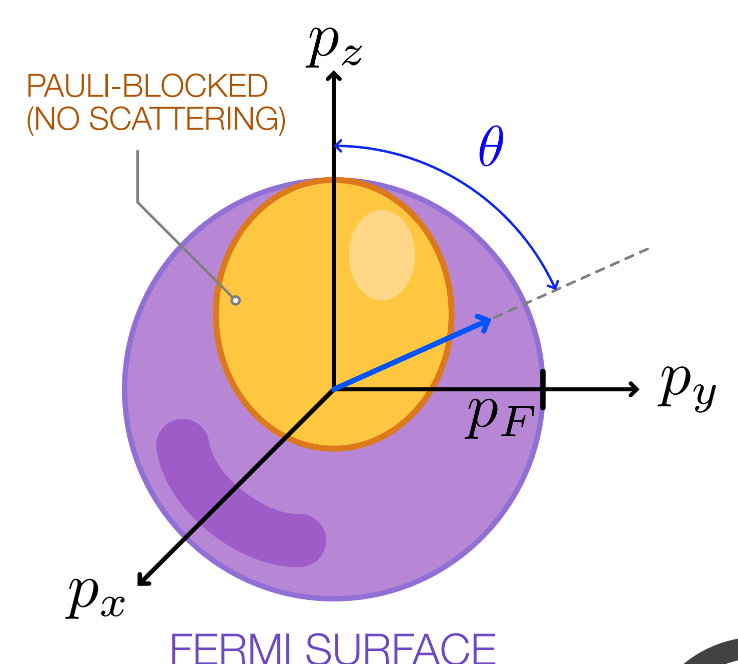


$$f = \sum_{N_{\text{hit}}} \frac{\langle n_{\text{T}} \rangle \Delta t}{N_{\text{hit}}} \int d\Omega_{\text{F}} \int_0^{p_{\text{F}}} \frac{p^2 dp}{V_{\text{F}}} \int d\Omega_{\text{CM}} \frac{d\sigma_{\text{CM}}}{d\Omega_{\text{CM}}} v_{\text{Møl}}\, \Theta^3(\Delta E)$$

$$dn_{\text{T}} = \langle n_{\text{T}} \rangle \frac{p^2 dp\, \Omega_{\text{F}}}{V_{\text{F}}}$$

$$d\Omega_{\text{F}} = d\varphi\, d(\cos\theta)$$

$$V_{\text{F}} = \frac{4}{3}\pi p_{\text{F}}^3$$



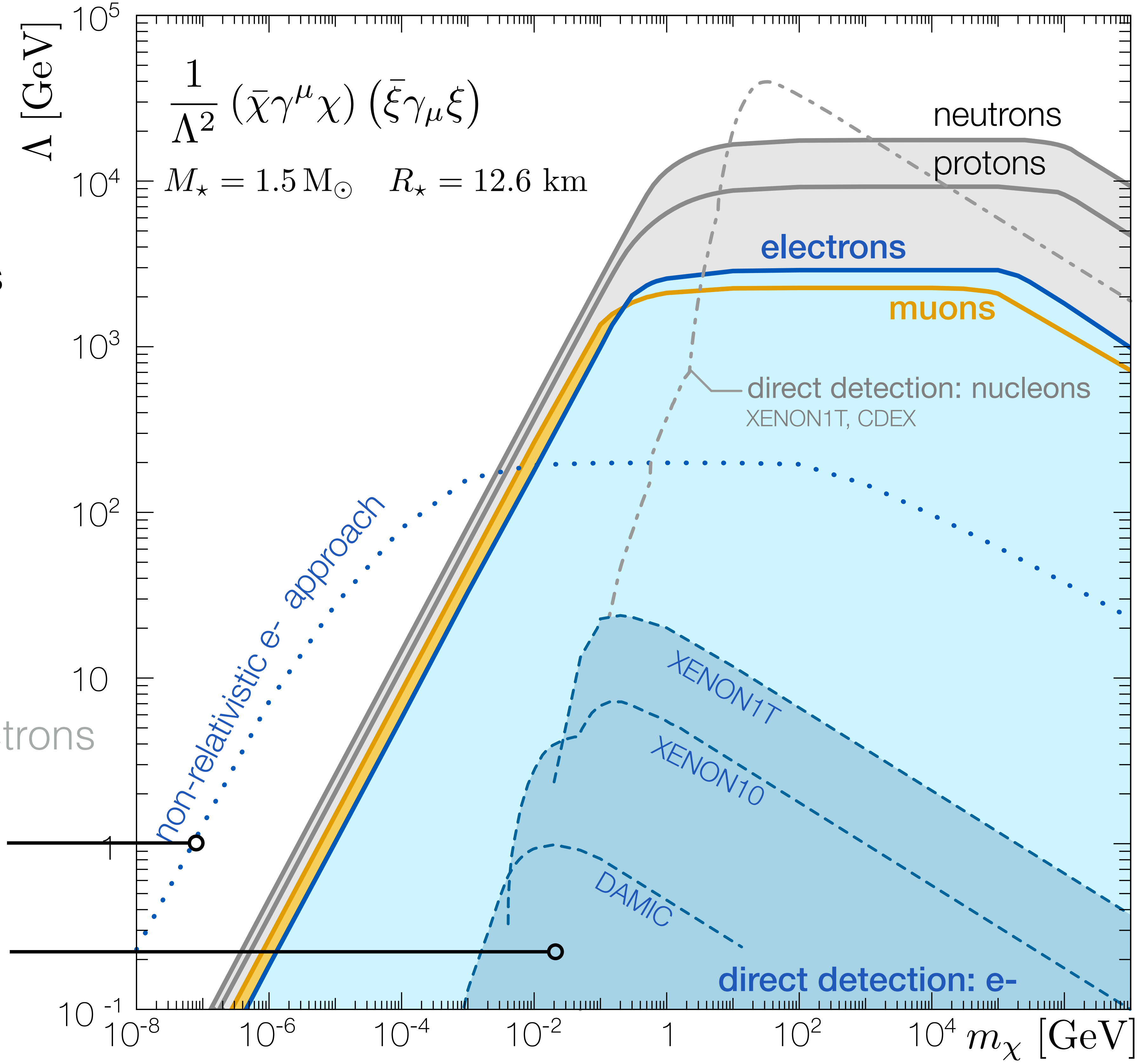
reach

ultra relativistic electrons
semi-relativistic muons

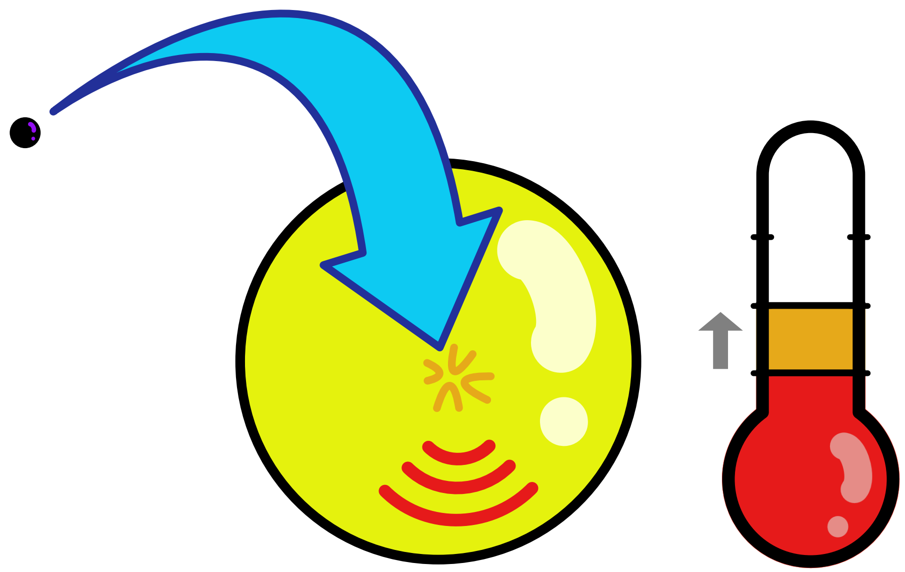
NR approx overestimates
muons (not shown) vs electrons

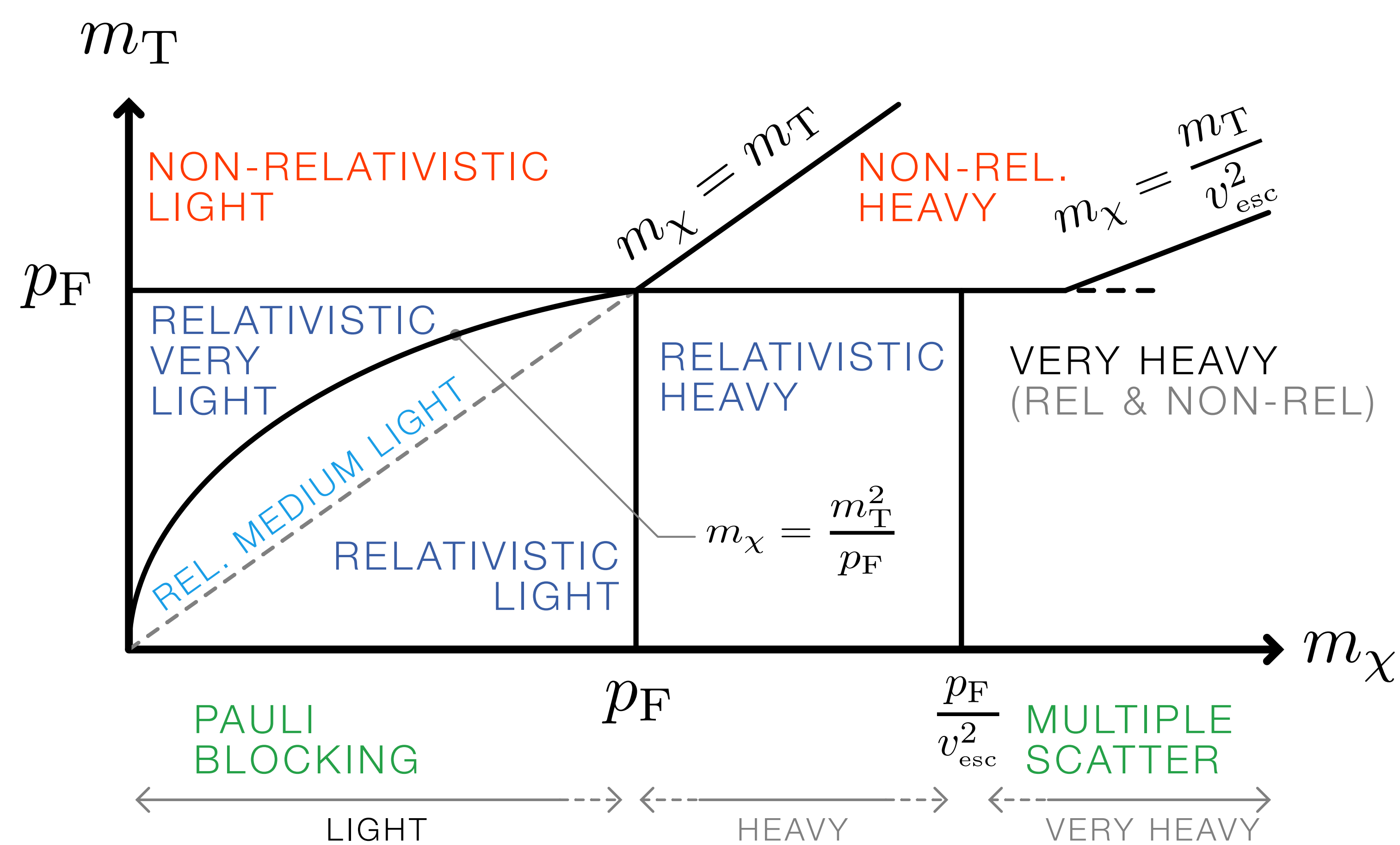
non-relativistic approx

terrestrial direct detection



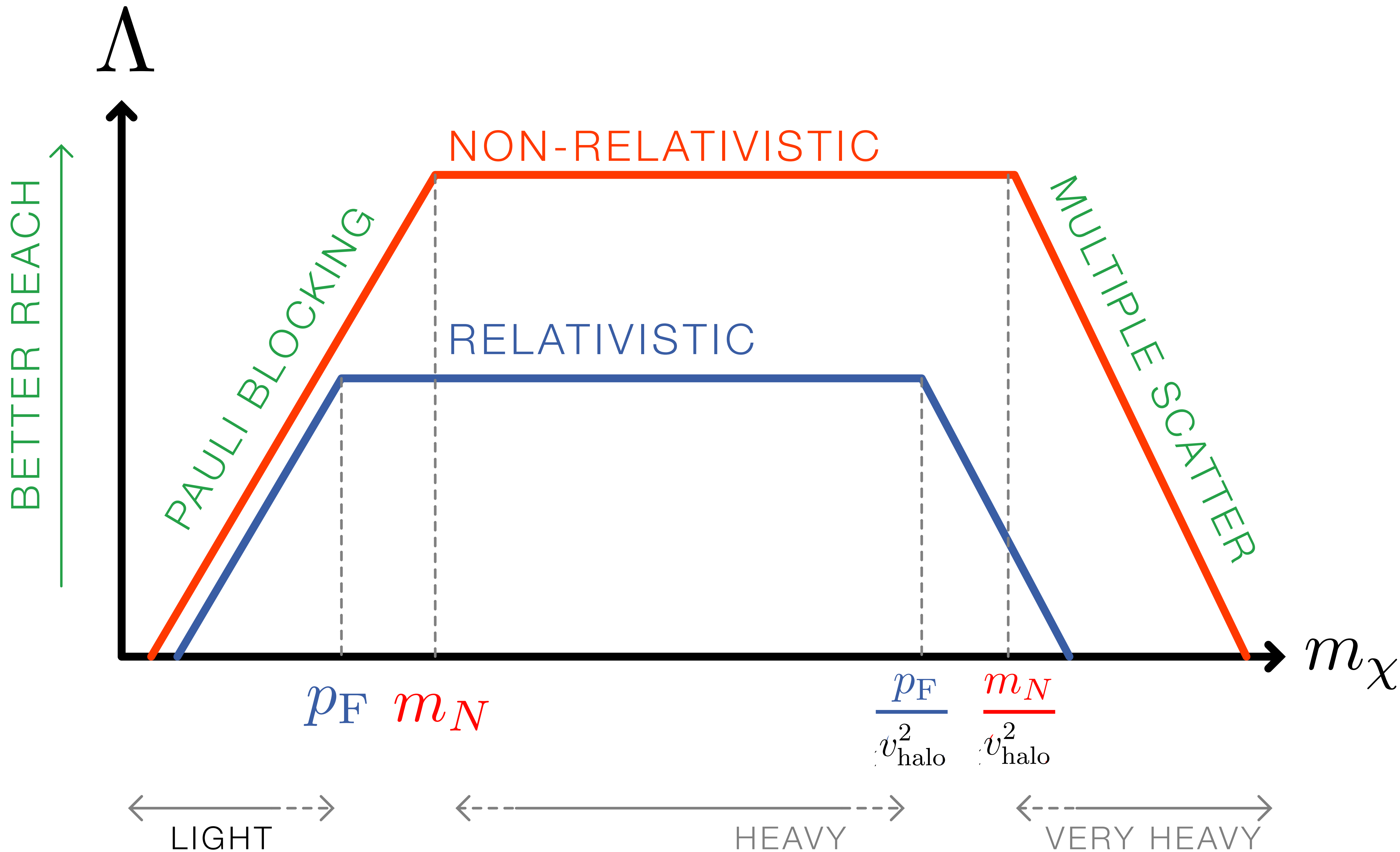
Outline





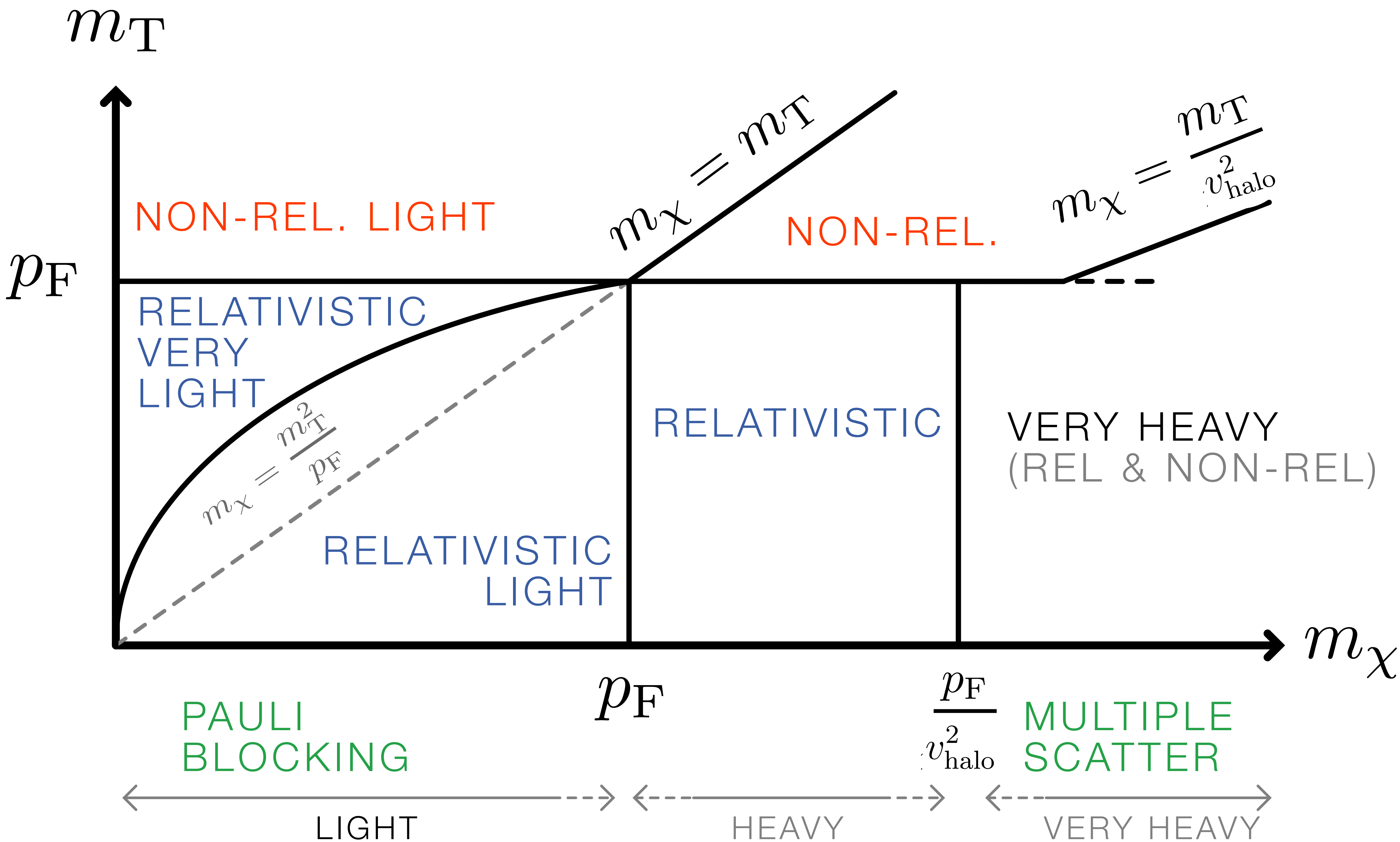
Julie Peasley, particlezoo.net

Why did we get this result?

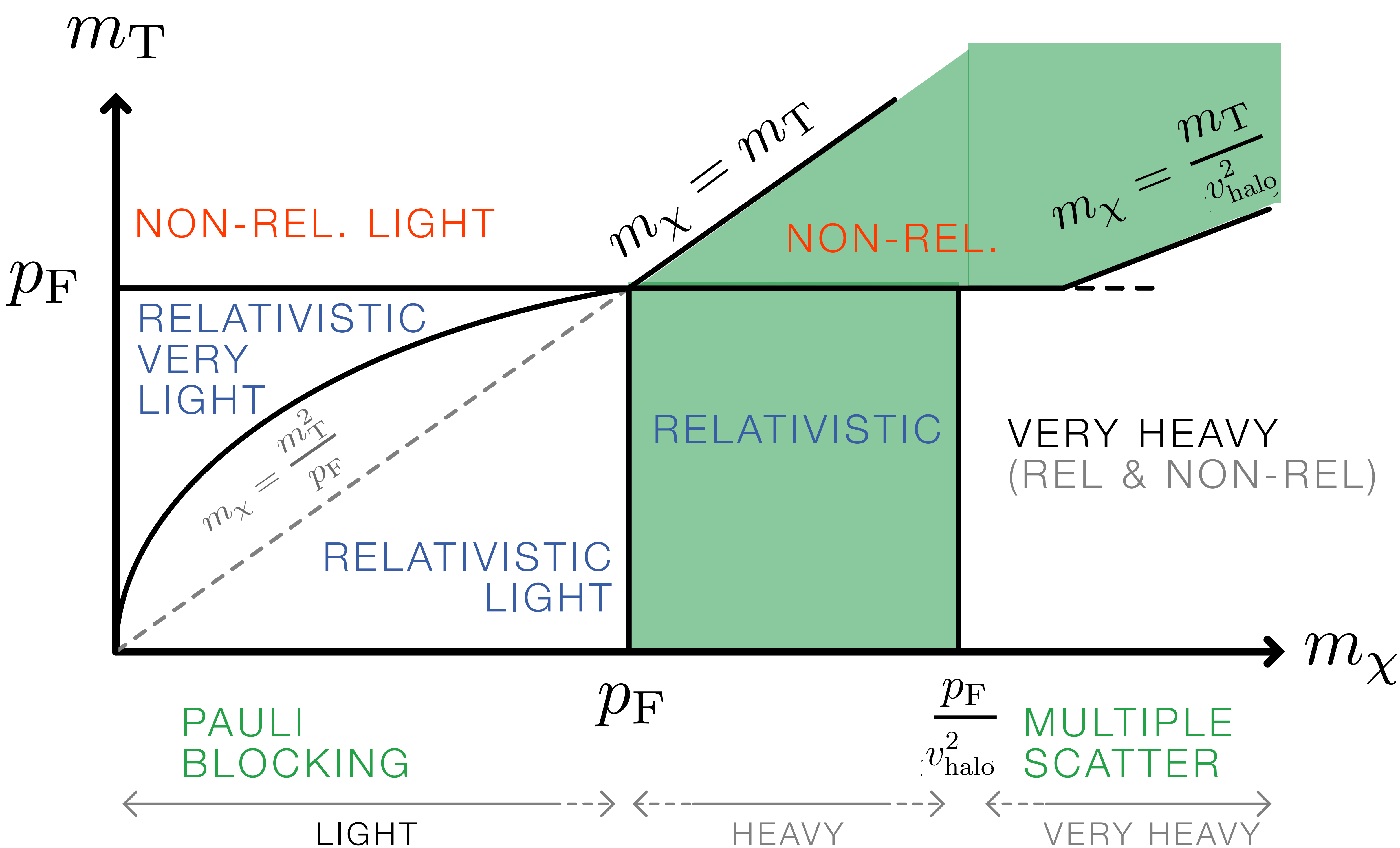


apples to oranges comparison

phase space of scattering

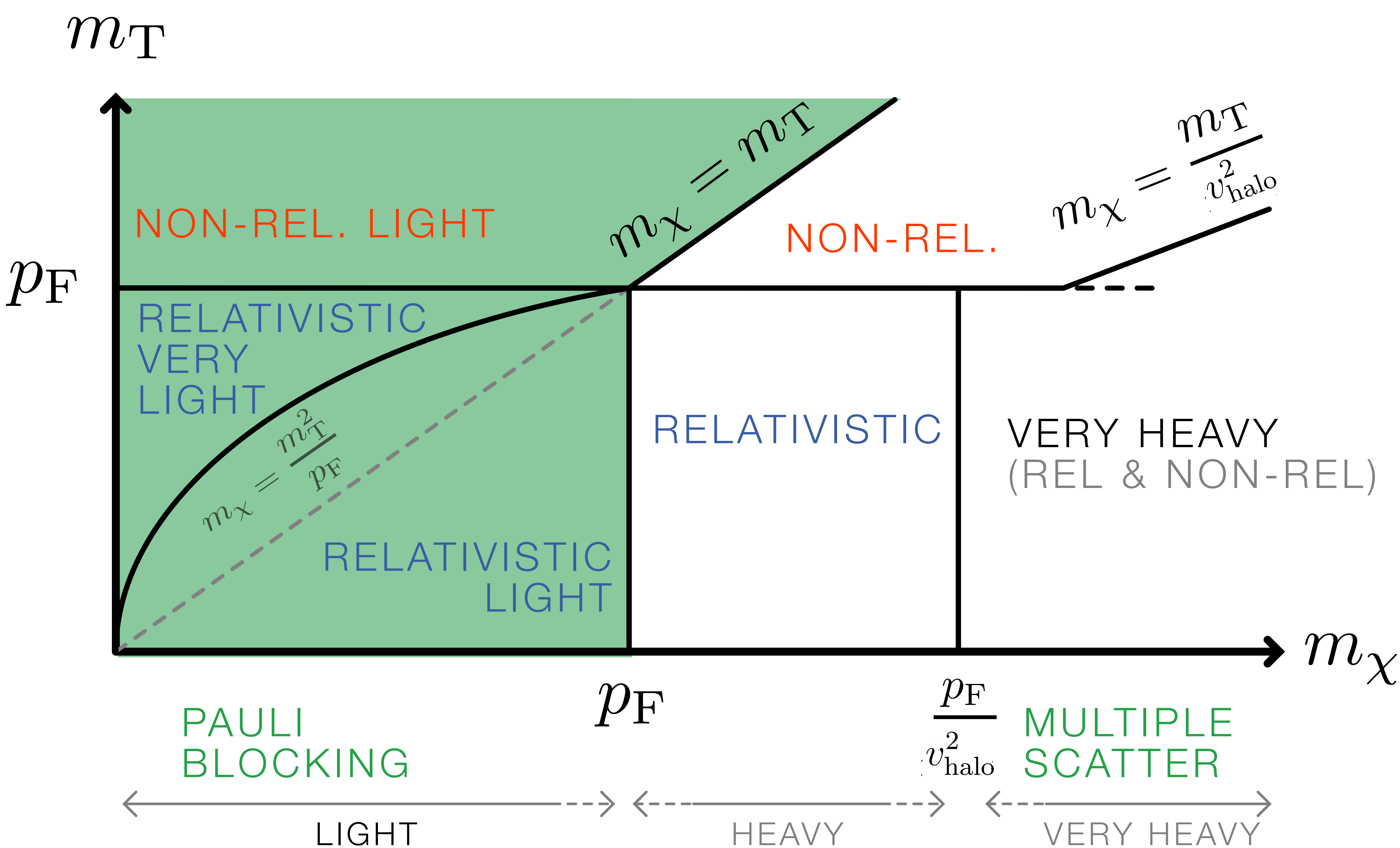


standard regime



light dark matter regime

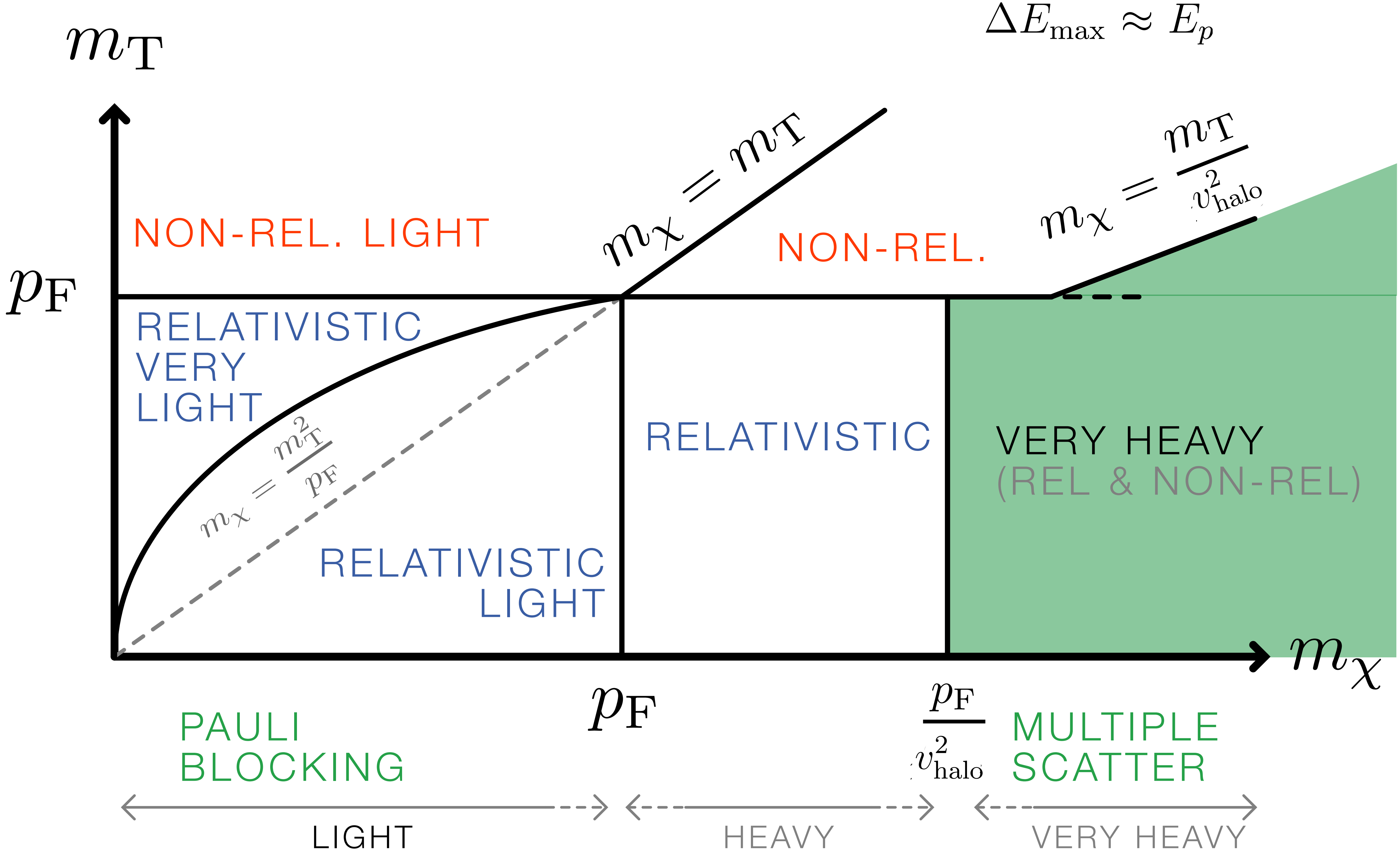
$\Delta E_{\max} \sim m_\chi$



Heavy regime

$$\Delta E_{\text{min}} = E_{\text{halo}} = \frac{1}{2} m_\chi v_{\text{halo}}^2$$

$$\Delta E_{\text{max}} \approx E_p$$



More careful analysis

1. How does the differential cross section scale with m_χ?
2. Is the phase space suppressed with m_χ?
3. Does capture require multiple scatters?

capture efficiency

$$f \sim \frac{1}{N_{\text{hit}}} \int_{\cos\psi_{\max}}^{1} d\cos\psi \int_{p_{\min}}^{p_{\text{F}}} \frac{p^2 dp}{p_{\text{F}}^3} \frac{|\mathcal{M}|^2}{s}$$

For each operator in each regime, check scaling with dark matter mass.

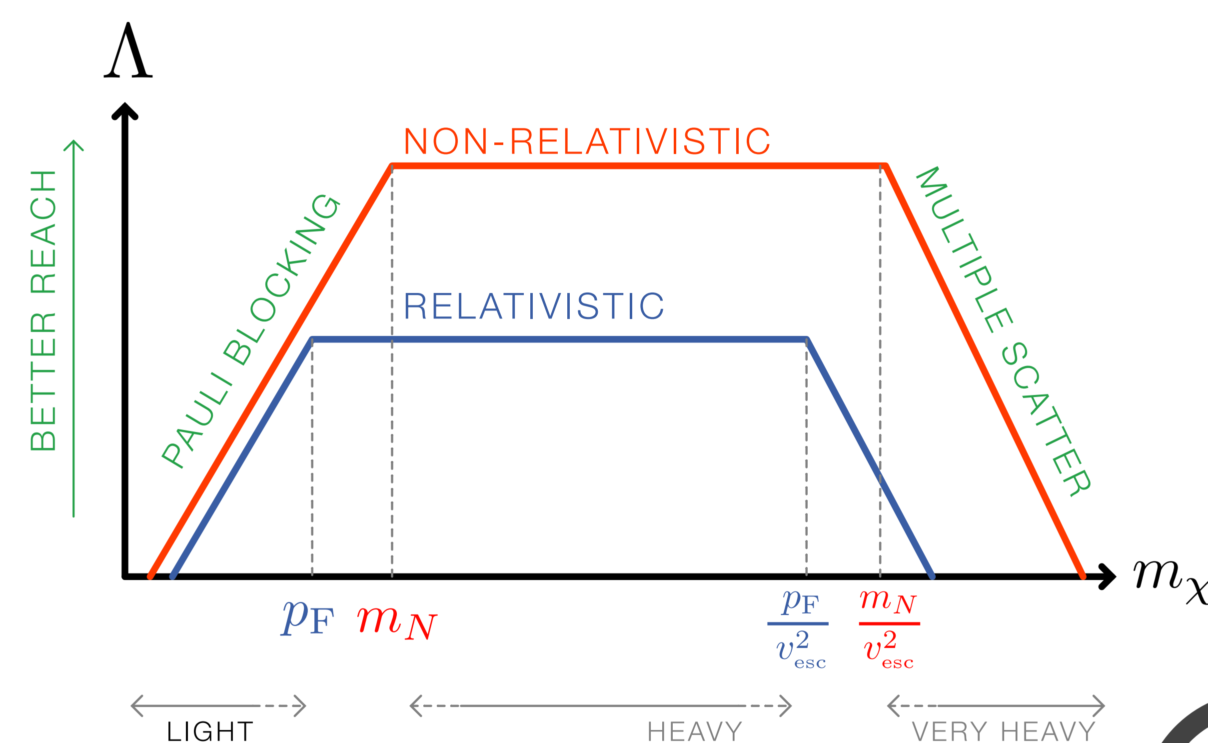
detailed version

$$\frac{|\mathcal{M}|^2}{s} \approx \frac{m_\chi^2 E_p^2}{s\Lambda^4} \approx \frac{m_\chi^2 m_\mathrm{T}^2}{s\Lambda^4}\left(1+\frac{E_\mathrm{F}^2}{m_\mathrm{T}^2}\right)$$

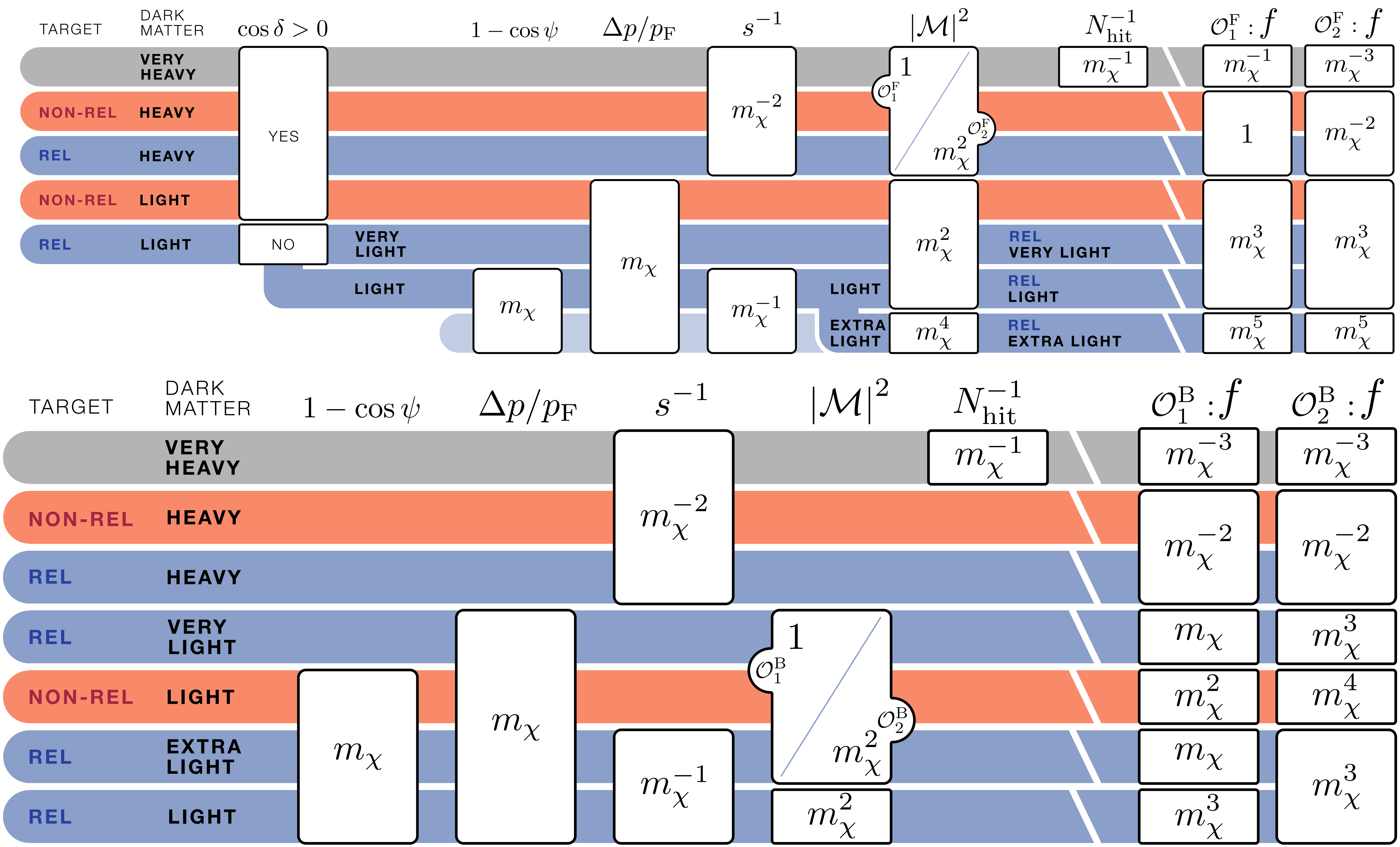
TARGET	DARK MATTER	$1-\cos\psi$	$\Delta p/p_\mathrm{F}$	s^{-1}	$\lvert\mathcal{M}\rvert^2$	N_hit^{-1}	f
	VERY HEAVY			m_χ^{-2}	m_χ^2	m_χ^{-1}	m_χ^{-1}
NON-REL	HEAVY			m_χ^{-2}	m_χ^2		1
REL	HEAVY			m_χ^{-2}	m_χ^2		1
NON-REL	LIGHT		m_χ		m_χ^2		m_χ^3
REL	VERY LIGHT		m_χ		m_χ^2		m_χ^3
REL	LIGHT	m_χ	m_χ	m_χ^{-1}	m_χ^2		m_χ^3

$$f \sim \frac{1}{N_\mathrm{hit}} \int_{\cos\psi_\mathrm{max}}^{1} d\cos\psi \int_{p_\mathrm{min}}^{p_\mathrm{F}} \frac{p^2\,dp}{p_\mathrm{F}^3}\,\frac{|\mathcal{M}|^2}{s}$$

$$s \approx \begin{cases} m_\mathrm{T}^2 & m_\chi \ll m_\mathrm{T}^2/p_\mathrm{F} \\ m_\chi E_p & m_\mathrm{T}^2/E_\mathrm{F} \ll m_\chi \ll p_\mathrm{F} \\ m_\chi^2 & p_\mathrm{F} \ll m_\chi \end{cases}$$



… some cases can be more complex



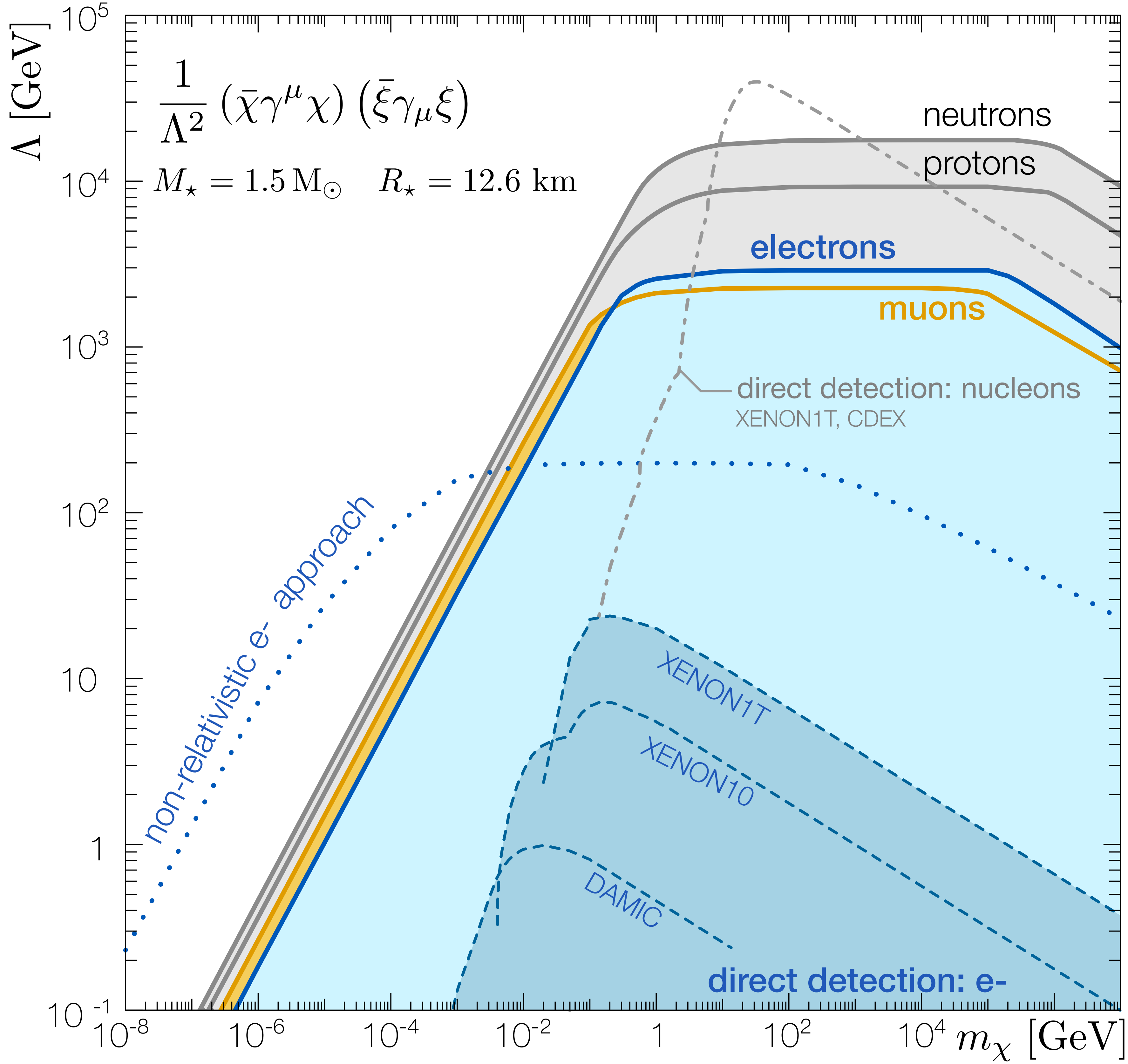
Result

electron scattering is a powerful probe of leptophilic DM!

esp compared to terrestrial experiments

relativistic scatter: some surprises; new formalism req.

Same benefits as neutron scatter…



Joglekar, Raj, FT, Yu, 1911.13293

How we win (vs direct detection)
how we complement existing program

Large volume, high density

Dark matter is accelerated
Better reach for momentum-suppressed interactions, inelastic scattering (up to 200 MeV)

No ceiling (strong int) **or floors** (neutrino BG)**.**

Larger range of accessible dark matter masses

No hierarchy between SI and SD scattering.

(repeated slide)

What's next: Light mediators

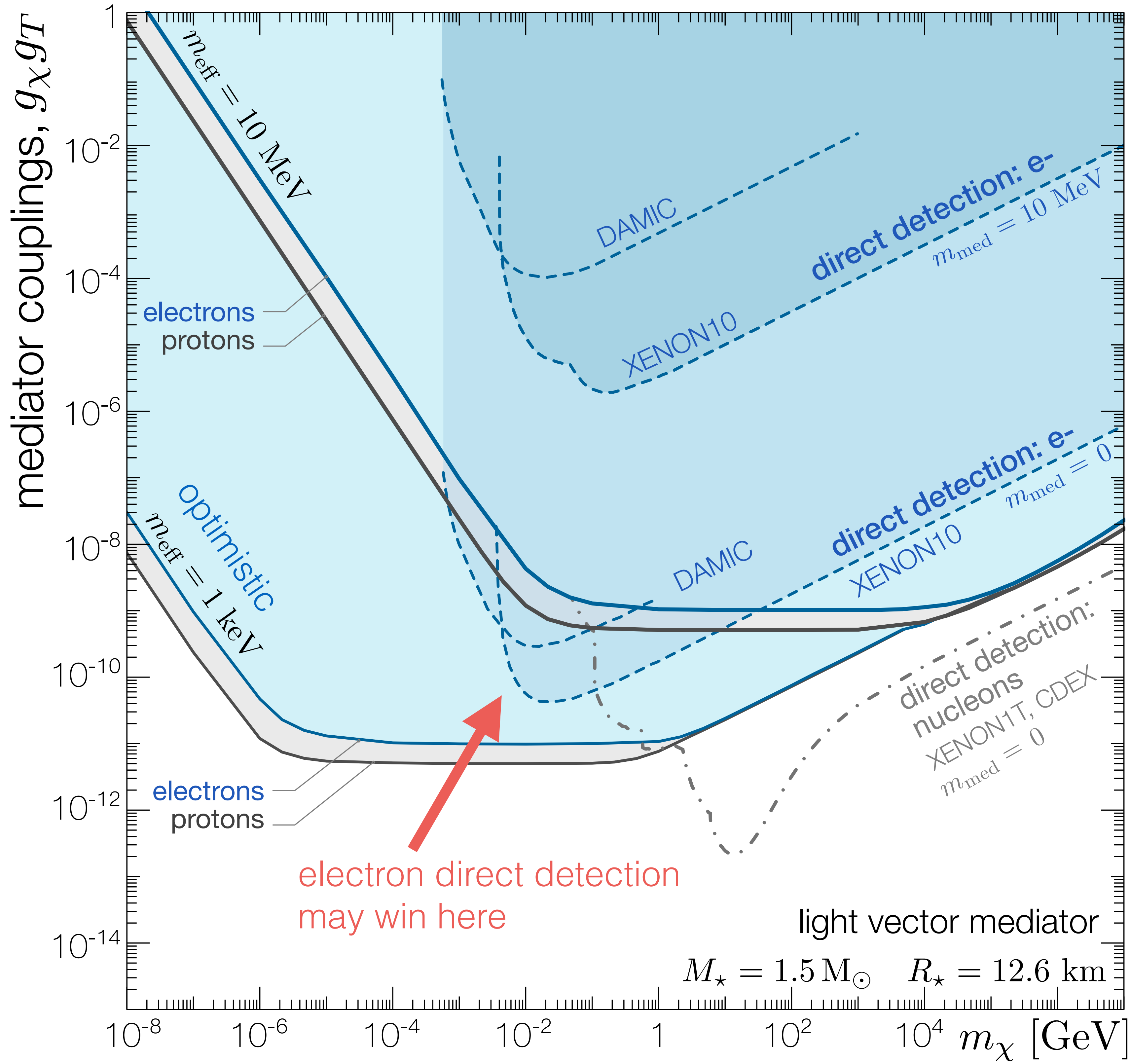
simplified model

effective mass from the medium (Debye)

$$\lambda_D^{-1} \sim e\sqrt{\frac{n_e}{T_{\text{eff}}}} \sim e\sqrt{\frac{n_e}{p_F}}$$

$$\approx 10 \text{ MeV}$$

massless mediator in DD has effective ass in NS



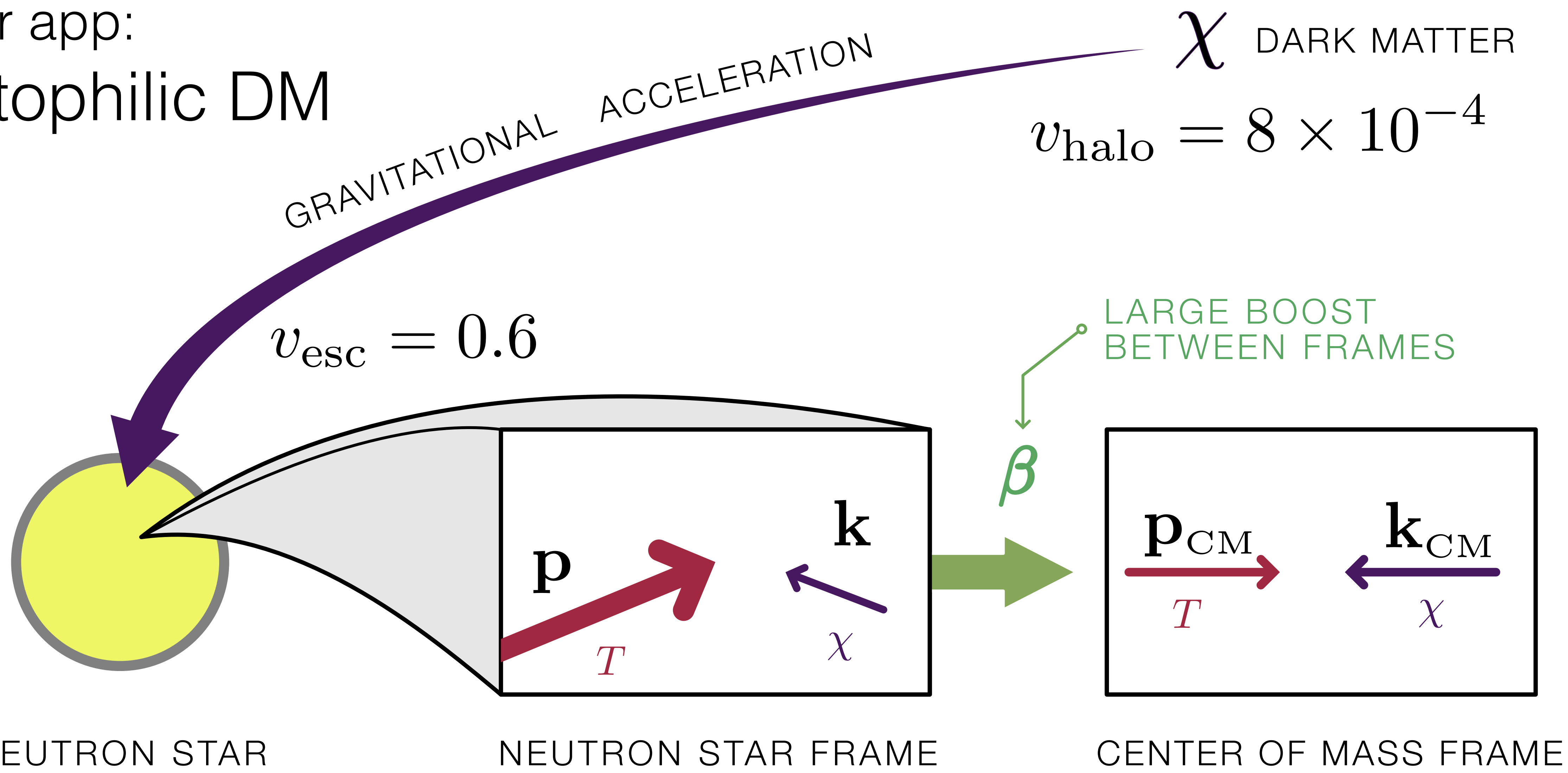
Summary

Opportunities for *direct detection* with neutron stars

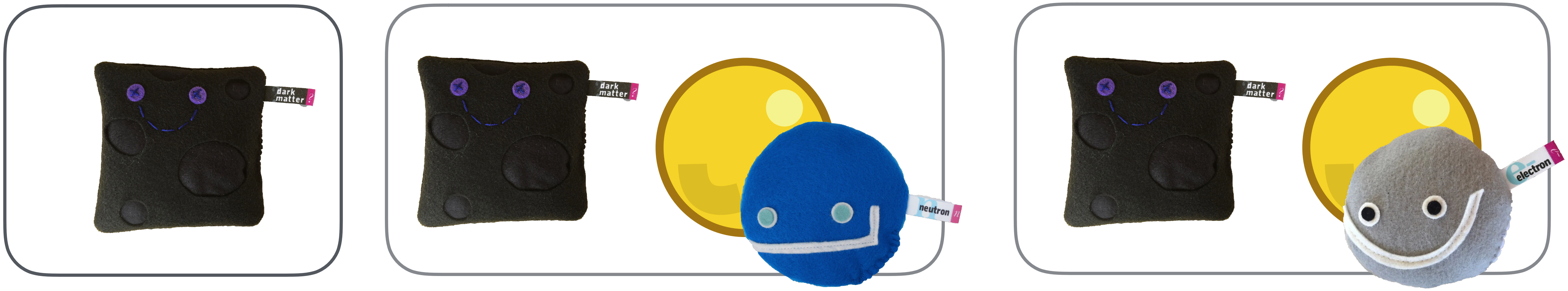
New formalism for relativistic, degenerate targets

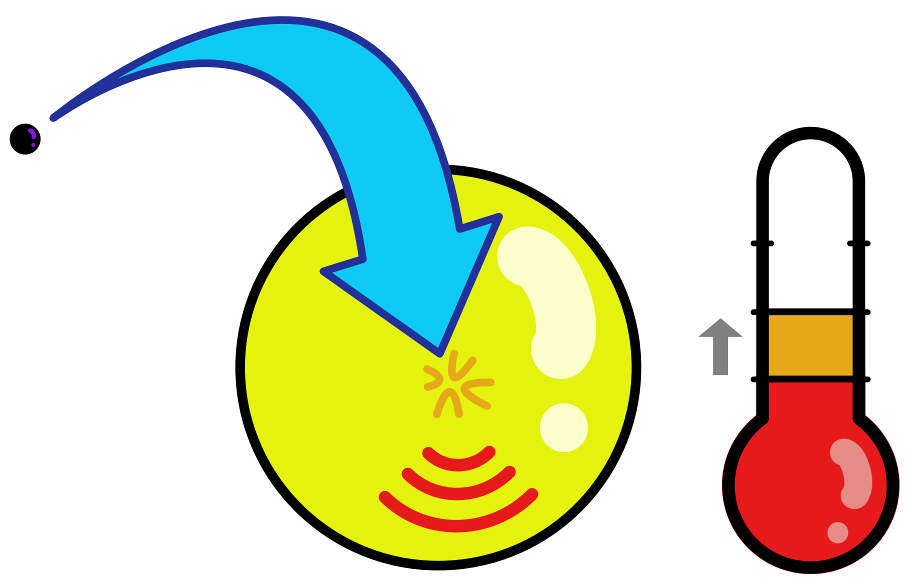
Killer app:

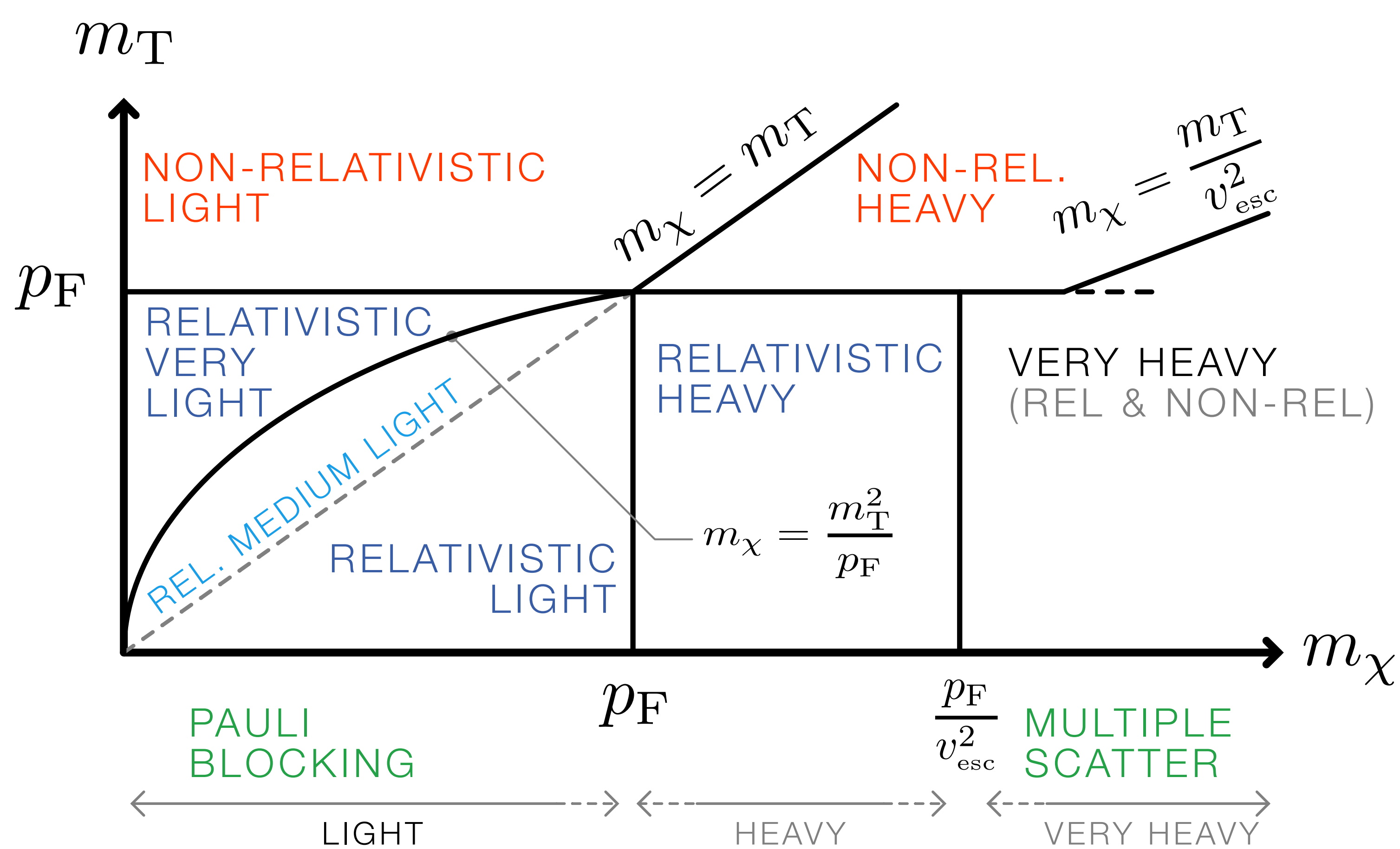
leptophilic DM



Thank you!







Julie Peasley, particlezoo.net

Additional Slides

Someone asked a clever question.

Möller Velocity Review

$$v_{\text{rel}} = \left.\frac{k}{E_k}\right|_{\text{T}} = \left.\frac{\sqrt{E_k^2 - m_\chi^2}}{E_k}\right|_{\text{T}} = \frac{\sqrt{(p\cdot k)^2 - m_{\text{T}}^2 m_\chi^2}}{p\cdot k} \qquad v_{\text{Møl}} = \frac{\sqrt{(p\cdot k)^2 - m_{\text{T}}^2 m_\chi^2}}{E_p E_k}$$

$$\mathcal{R} = \frac{d\nu}{\Delta V\,\Delta t} = (d\sigma\, v_{\text{rel}}\, dn_{\text{T}}\, dn_\chi)_{\text{T}} \quad\longrightarrow\quad A = \frac{p\cdot k}{E_T E_\chi}(d\sigma\, v_{\text{rel}})_{\text{T}}$$

$$\mathcal{R} = (A\, dn_{\text{T}} dn_\chi)_F = \left(A\frac{E_T E_\chi}{m_{\text{T}} m_\chi}\right)_F d\hat{n}_{\text{T}} d\hat{n}_\chi$$

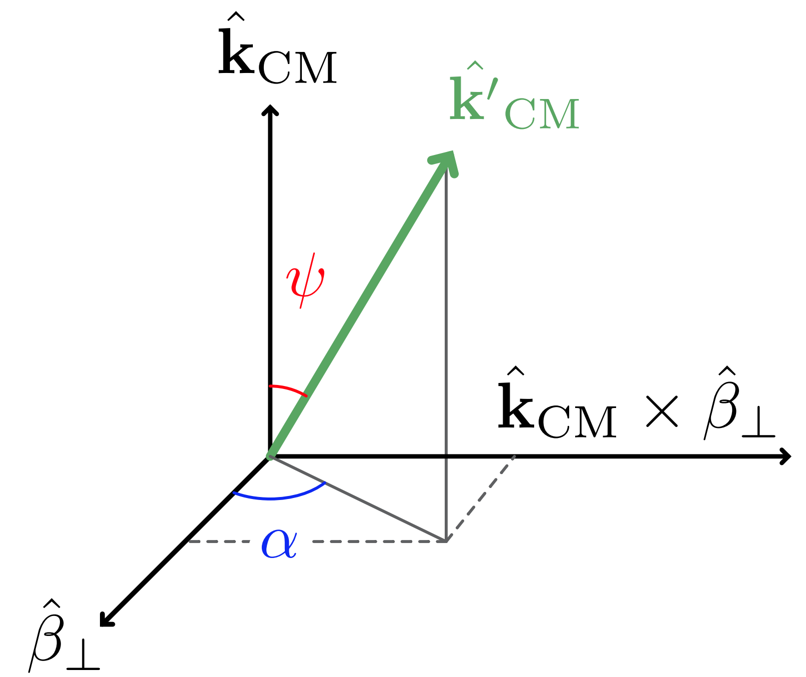
$$\mathcal{R} = d\sigma_{\text{CM}}\left(\frac{p\cdot k}{E_T E_\chi} v_{\text{rel}}\right) dn_{\text{T}} dn_\chi = d\sigma_{\text{CM}}\, v_{\text{Møl}}\, dn_{\text{T}} dn_\chi$$

see, e.g. Cannoni 1605.00569 for a review

Energy Transfer: non-relativistic

$$k^\mu_{\text{CM}} = \begin{pmatrix} \gamma & -\gamma\boldsymbol{\beta} \\ -\gamma\boldsymbol{\beta} & \gamma \end{pmatrix} \begin{pmatrix} E_k \\ \mathbf{k} \end{pmatrix} \qquad p^\mu_{\text{CM}} = \begin{pmatrix} \gamma & -\gamma\boldsymbol{\beta} \\ -\gamma\boldsymbol{\beta} & \gamma \end{pmatrix} \begin{pmatrix} m_{\text{T}} \\ \mathbf{0} \end{pmatrix}$$

$$\mathbf{p}_{\text{CM}} + \mathbf{k}_{\text{CM}} = 0 \qquad \boldsymbol{\beta} = \frac{\mathbf{k}}{E_{\text{esc}} + m_{\text{T}}}$$



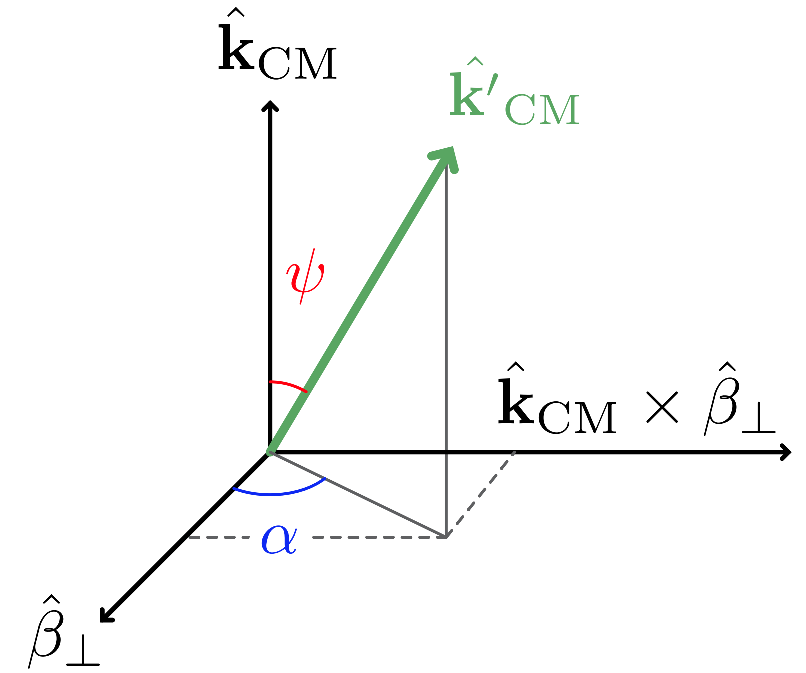
$$q^\mu_{\text{CM}} = k^\mu_{\text{CM}} - k'^\mu_{\text{CM}} = (0, \mathbf{q}_{\text{CM}})^T$$

$$\Delta E = q^0 = \gamma\boldsymbol{\beta}\cdot\mathbf{q}_{\text{CM}} = \frac{\gamma\mathbf{k}\cdot\mathbf{q}_{\text{CM}}}{E_k + m_{\text{T}}} = \frac{\gamma^2 m_{\text{T}} \mathbf{k}^2 (1-\cos\psi)}{(E_k + m_{\text{T}})^2}$$

$$\Delta E = \frac{m_{\text{T}} m_\chi^2}{m_\chi^2 + m_{\text{T}}^2 + 2\gamma_{\text{esc}} m_\chi m_{\text{T}}} \frac{v_{\text{esc}}^2}{1 - v_{\text{esc}}^2} (1 - \cos\psi) \ ,$$

Energy Transfer: Relativistic

$$\Delta E = E_k - E_{k'} = \gamma \left[(E_k)_{\text{CM}} + (E_k)_{\text{CM}} \right] + \gamma \boldsymbol{\beta} \cdot (\mathbf{k}_{\text{CM}} - \mathbf{k}'_{\text{CM}}) = \gamma \boldsymbol{\beta} \cdot \mathbf{q}_{\text{CM}}$$



$$\begin{aligned}\Delta E &= \gamma \boldsymbol{\beta} \cdot \left[\mathbf{k}_{\text{CM}} \, (1 - \cos\psi) - k_{\text{CM}} \sin\psi \, \cos\alpha \, \hat{\boldsymbol{\beta}}_\perp \right] \\ &= \gamma (\boldsymbol{\beta} \cdot \mathbf{k}_{\text{CM}}) \, (1 - \cos\psi) - \gamma \sqrt{\boldsymbol{\beta}^2 \, \mathbf{k}^2_{\text{CM}} - (\boldsymbol{\beta} \cdot \mathbf{k}_{\text{CM}})^2} \, \sin\psi \, \cos\alpha \, .\end{aligned}$$

$$\boldsymbol{\beta} \cdot \mathbf{k}_{\text{CM}} \equiv \beta k_{\text{CM}} \cos\delta = \frac{E_p k^2 - E_k p^2 + (E_p - E_k) \mathbf{p} \cdot \mathbf{k}}{E E_{\text{CM}}}$$

Maximum Energy Transfer

$$\frac{\Delta E}{\gamma\beta k_{\text{CM}}} = \cos\delta\ (1-\cos\psi) - |\sin\delta|\cos\alpha\ \sin\psi$$

We may succinctly write the conditions for the maximum energy transfer as

$$\cos\alpha = -1 \qquad \cos\psi = -\cos\delta \qquad \sin\psi = |\sin\delta| = \sqrt{1-\cos^2\delta}$$

$$\frac{\Delta E_{\text{max}}}{\gamma\beta k_{\text{CM}}} = \cos\delta(1+\cos\delta) + \sin^2\delta = \cos\delta + 1$$

$$\cos\delta = \frac{E_p k^2 - E_k p^2 + (E_p - E_k)\mathbf{p}\cdot\mathbf{k}}{E\beta E_{\text{CM}} k_{\text{CM}}}$$

One may then evaluate this in various limits.

Heuristics for Phase Space Scaling

Rule of Thumb 1 (Independent Integration Assumption). *We assume that the phase space integrals are independent of one another. For simplicity, we ignore the dependence on phase space integrals in the differential cross section,* $d\sigma/d\Omega_{\text{CM}}$*. This is sufficient to understand the scaling behavior with respect to the dark matter mass.*

Rule of Thumb 2 (Weak Condition). *First* $\Delta E > 0$*. This is a sufficient, but not necessary condition.*

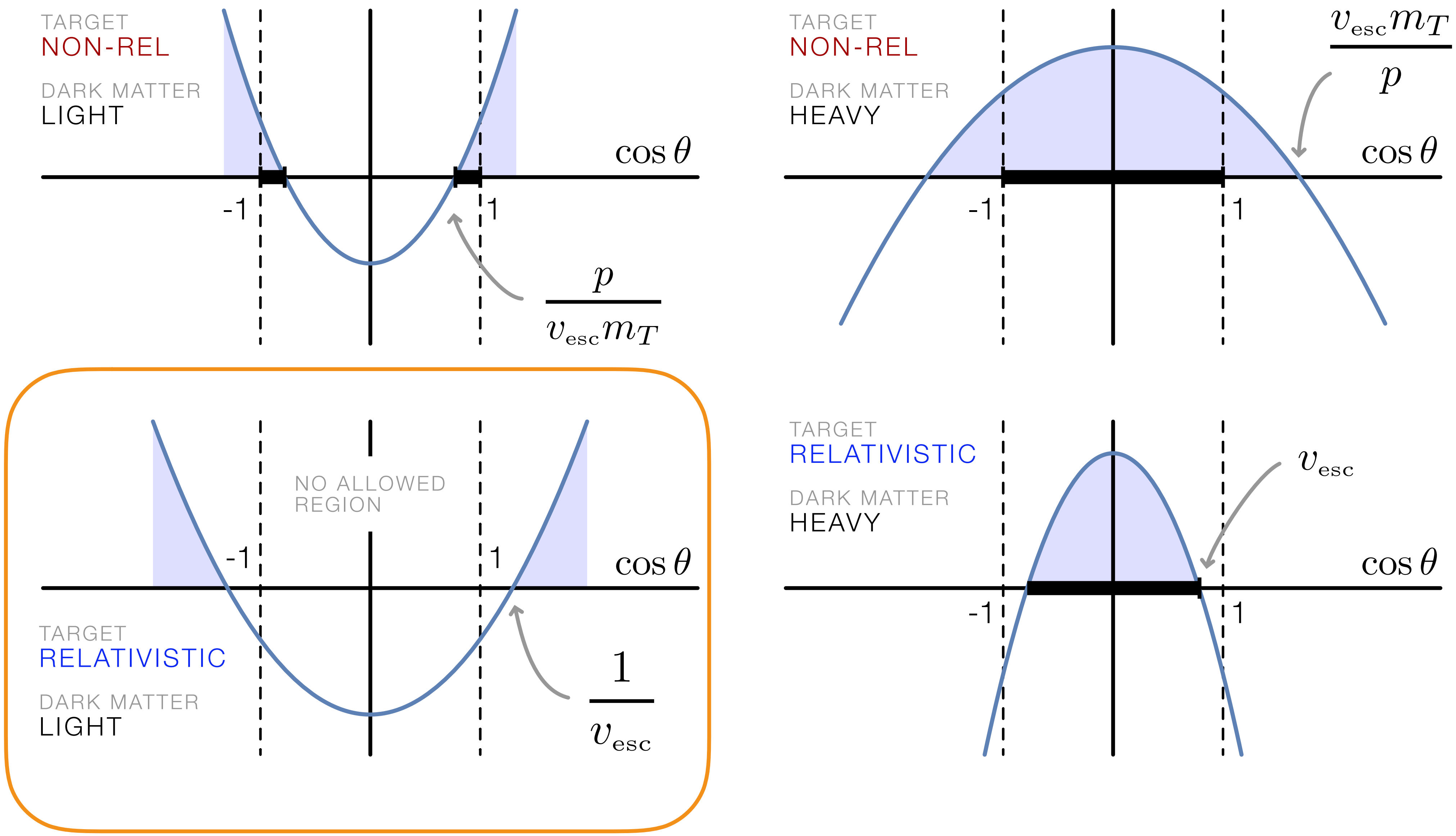
Corollary of Thumb 1 (Weaker condition). $\cos\delta > 0$ *is a sufficient condition that* $\Delta E > 0$ *for a unsuppressed part of phase space. This is a sufficient, but not necessary condition.*

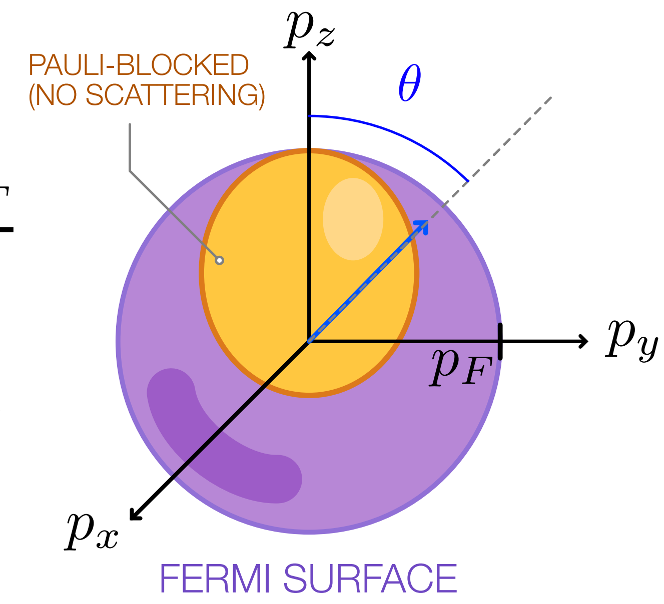
Proof. This comes from positivity of the right-hand side and the range $0 \leq \psi \leq \pi$, since ψ is a polar angle. □

Rule of Thumb 3 (Strong Condition). *The phase space for the initial target momentum must be large enough that the outgoing target after scattering has momentum larger than the Fermi momentum. For this diagnostic, we check relative to the maximum kinematically allowed energy transfer,* ΔE_{max}*:*

$$p + \Delta E_{max} > p_F \ . \tag{F.4}$$

Initial State Suppression





Corollary of Thumb 1 (Weaker condition). $\cos\delta > 0$ *is a sufficient condition that* $\Delta E > 0$ *for a unsuppressed part of phase space. This is a sufficient, but not necessary condition.*

$$\left(m_{\text{T}}^2 + p^2\right)\left(\gamma_{\text{esc}}^2 v_{\text{esc}}^2 m_\chi^2 + p\gamma_{\text{esc}} v_{\text{esc}} m_\chi \cos\theta\right)^2 > \gamma_{\text{esc}}^2 m_\chi^2 \left(p^2 + p\gamma_{\text{esc}} v_{\text{esc}} m_\chi \cos\theta\right)$$