NNLO Correction to Quark Quasi Distribution Functions

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Based on our recent works

- Next-to-next-to-leading order corrections to non-singlet quark Quasi distribution functions

  L.-B. Chen, W. Wang, R. Zhu, arXiv: 2006.14825

Long-Bin Chen Wei Wang Ruilin Zhu

- Master Integrals for two-loop QCD corrections to Quasi PDFs

  L.-B. Chen, W. Wang, R. Zhu, arXiv:2006.10917, to appear in JHEP

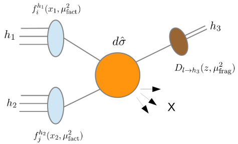
- Quasi parton distribution functions at NNLO: flavor non-diagonal quark contributions

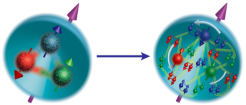
  L.-B. Chen, W. Wang, R. Zhu, arXiv:2005.13757, PRD102,011503(2020)

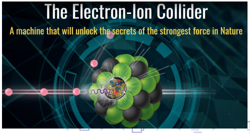
Also thank Feng Yuan for valuable discussions!

Parton Distribution Functions(PDFs)

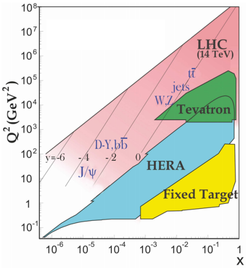
- Big fundamental
  - fundamental inputs to predict hadron-collider cross section
  - longitudinal momentum distributions
  - nonperturbative
- Big question
  - how are these partons arranged in a hadron?
- Big goal (1-D/3-D/5-D images) EIC, 1602.03922

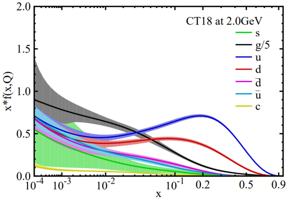


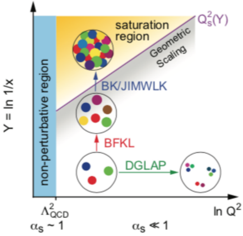




PDFs from the global fit



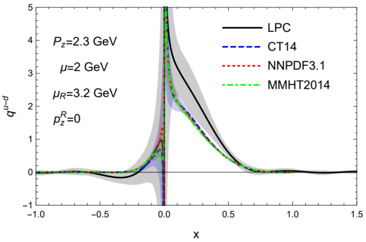




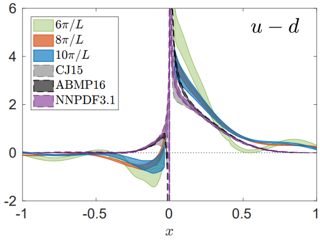
PDG2018 and 1912.10053

We know some (more on perturbative aspects) of the PDFs at many different facilities over 50 years effort, however, we understand less from first principle of QCD

Lattice QCD+LaMET



LPC,PRD101,034020(2020)



ETMC,PRL121,112001(2018)

a very attractive and active frontier to understand hadron structure from first principle of QCD

Outline

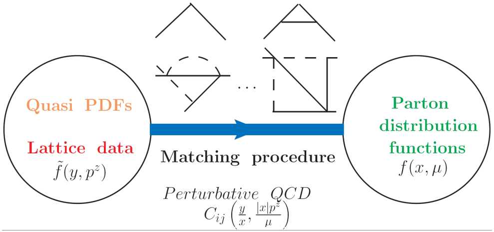
Large Momentum Effective Theory (LaMET)

- LaMET factorization formula

$$\tilde{f}_{i/H}(y, p^z) = \int_{-1}^{1} \frac{dx}{|x|} \left[C_{ij}\left(\frac{y}{x}, \frac{|x|p^z}{\mu}\right) f_{j/H}(x, \mu) \right] + \mathcal{O}\left(\frac{m_h^2}{p^{z2}}, \frac{\Lambda_{\rm QCD}^2}{p^{z2}} \right)$$

$x \in [-1, 1]$, $y \in [-\infty, \infty]$

X. Ji, PRL110,262002 (2013), ...



Perturbative calculation of $C_{ij}^{(0)}$, $C_{ij}^{(1)}$, $C_{ij}^{(2)}$, ...

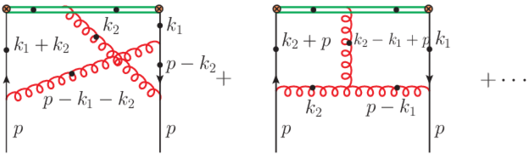
- $C_{ij} = C_{ij}^{(0)} + \frac{\alpha_s}{2\pi} C_{ij}^{(1)} + \left(\frac{\alpha_s}{2\pi}\right)^2 C_{ij}^{(2)} + \ldots$
- LO matching trivial: $C_{ij}^{(0)}(y) = \delta(1-y)$
- Higher-order matching and Quasi pdf are renormalization scheme dependent.
- NLO matching coefficient $C_{ij}^{(1)}(y, \frac{p^z}{\mu})$
  3 regions for y ($[-\infty, 0], [0,1], [1, +\infty]$)and 1 color factor C_F
  $\overline{MS}$: Izubuchi,Ji,Jin,Stewart,Zhao,1801.03917;
  $M\overline{MS}$: Alexandrou,Cichy,Constantinou,Jansen,Scapellato,Steffens, 1803.02685;
  RI/MOM: Stewart,Zhao,1709.04933; Wang,Zhang,Zhao,Zhu,1904.00978;
  Others: Ji,Xiong,Zhang,Zhao,1310.7471; Ma,Qiu,1404.6860, ...
- $C_{ij}^{(2)}$ is needed
  - higher-order corrections are important in QCD
  - If $\mu = 2GeV$, $\alpha_s(\mu = 2GeV) \sim 0.3$ and then α_s^2-correction is needed for a precision prediction
  - factorization proof at NNLO is nontrivial

Outline

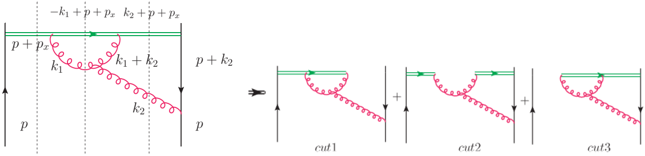
Two Loop Feynman Diagrams and Amplitudes

- Higher order corrections bring about a large number (79^+ at NNLO) of Feynman diagrams
- We treat Wilson line as an auxiliary quark field with linear propagator
- We use FeynRules and FeynArts to auto produce the Feynman diagrams and amplitudes

Christensen et al,1310.1921, T. Hahn,0012260



An example in Feynman gauge



- From auxiliary field back to Wilson line, we need to do the cuts. For cut1, we have $p_x = -p - k_2$

$$\mathcal{M}|_{cut1} = \mu^{4\epsilon} \int \int \frac{d^{4-2\epsilon} k_1}{(2\pi)^{4-2\epsilon}} \frac{d^{4-2\epsilon} k_2}{(2\pi)^{4-2\epsilon}} \text{ampcut1} \times \delta\left(k_2^z + p^z - yp^z\right),$$

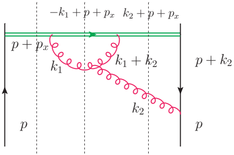
- For cut2, we have $p_x = -p + k_1$; both them give real contribution

$$\mathcal{M}|_{cut2} = \mu^{4\epsilon} \int \int \frac{d^{4-2\epsilon} k_1}{(2\pi)^{4-2\epsilon}} \frac{d^{4-2\epsilon} k_2}{(2\pi)^{4-2\epsilon}} \text{ampcut2} \times \delta\left(-k_1^z + p^z - yp^z\right),$$

- For cut3, we have $p_x = -p$ and it gives virtual contribution

$$\mathcal{M}|_{cut3} = \mu^{4\epsilon} \int \int \frac{d^{4-2\epsilon} k_1}{(2\pi)^{4-2\epsilon}} \frac{d^{4-2\epsilon} k_2}{(2\pi)^{4-2\epsilon}} \text{ampcut3} \times \delta\left(1 - y\right),$$

An example in Feynman gauge



- Use the identity

$$\frac{1}{k_1 \cdot n k_2 \cdot n} = \frac{1}{(k_1 \cdot n + k_2 \cdot n) k_2 \cdot n} + \frac{1}{k_1 \cdot n (k_1 \cdot n + k_2 \cdot n)},$$

- do the momentum transformation, then

$$\mathcal{M}|_{cut1+cut2+cut3} = \left[\mu^{4\epsilon} \int \int \frac{d^{3-2\epsilon} k_1}{(2\pi)^{4-2\epsilon}} \frac{d^{4-2\epsilon} k_2}{(2\pi)^{4-2\epsilon}} \mathrm{ampcut1}'|_{k_1^z = y p_z} \right]_+ + \left[\mu^{4\epsilon} \int \int \frac{d^{3-2\epsilon} k_1}{(2\pi)^{4-2\epsilon}} \frac{d^{4-2\epsilon} k_2}{(2\pi)^{4-2\epsilon}} \mathrm{ampcut2}'|_{k_1^z = y p_z} \right]_+$$

- It includes both the virtual and real contributions

Cutkosky rules

- Cutkosky rules, J.Math.Phys.1,429(1960)

$$\delta(k_z - xp_z) = \frac{1}{2\pi i}\left(\frac{1}{k_z - xp_z - i0} - \frac{1}{k_z - xp_z + i0}\right).$$

- All the real integrals become covariant integrals
- Solve real part of the real integrals $\Rightarrow$ Solve imaginary part of covariant integrals

$$\mu^{4\epsilon}\int\int \frac{d^{3-2\epsilon}k_1}{(2\pi)^{4-2\epsilon}}\frac{d^{4-2\epsilon}k_2}{(2\pi)^{4-2\epsilon}}\mathrm{ampcut1}|_{k_1^z=yp_z}$$

$$\Rightarrow\ Im\left[\mu^{4\epsilon}\int\int \frac{d^{4-2\epsilon}k_1}{(2\pi)^{4-2\epsilon}}\frac{d^{4-2\epsilon}k_2}{(2\pi)^{4-2\epsilon}}\frac{\mathrm{ampcut1}}{4\pi(k_1^z - yp_z)}\right]$$

Integration By Parts (IBP) reduction

- Simplify Feynman amplitudes to scalar integrals
- Reduce all the scalar integrals into Master Integrals (MIs)

$$\int \frac{d^d k_1}{(2\pi)^d} \cdots \frac{d^d k_L}{(2\pi)^d} \frac{\partial}{\partial k_i^\mu} \left[k_i^\mu I(k_i, L, p_i) \right] = 0$$

$$\int \frac{d^d k_1}{(2\pi)^d} \cdots \frac{d^d k_L}{(2\pi)^d} \frac{\partial}{\partial k_i^\mu} \left[p_i^\mu I(k_i, L, p_i) \right] = 0$$

Chetyrkin,Tkachov,NPB192,159(1981)

- In right box, $F(1)$ is called MIs. If the number of MIs is larger than 1, we may have a matrix relation.

$$F(a) = \int \frac{d^d k}{(k^2 - m^2)^a}.$$

↓

$$\int d^d k \frac{\partial}{\partial k} \cdot k \frac{1}{(k^2 - m^2)^a} = 0,$$

↓

$$(d - 2a) F(a) - 2am^2 F(a + 1) = 0.$$

↓

$$F(a) = \frac{d - 2a + 2}{2(a - 1)m^2} F(a - 1).$$

Outline

Master Integrals Calculation:Differential Equations

- To calculate MIs f_i, we can set up a differential equation with respect to Lorentz invariant kinematics z, for example $z = \frac{p^0}{p^z}$(or p^2)
- If the number of MIs is larger than 1, A is $n \times n$ coefficient matrix and depends on both z and ϵ

$$\frac{d}{dz}\begin{pmatrix} f_1(z,\epsilon) \\ \vdots \\ f_n(z,\epsilon) \end{pmatrix} = \begin{pmatrix} A_{11}(z,\epsilon) & \dots & A_{1n}(z,\epsilon) \\ \vdots & & \vdots \\ A_{n1}(z,\epsilon) & \dots & A_{nn}(z,\epsilon) \end{pmatrix} \begin{pmatrix} f_1(z,\epsilon) \\ \vdots \\ f_n(z,\epsilon) \end{pmatrix}$$

A.V.Kotikov, PLB254,158(1991); PLB267,123(1991)

- It is not easy to determine all the boundary condition for MIs f_i

A suitable choice of basis: Canonical basis

$$\frac{d}{dz}\begin{pmatrix} g_1(z;\epsilon) \\ \vdots \\ g_n(z;\epsilon) \end{pmatrix} = \epsilon \begin{pmatrix} B_{11}(z) & \dots & B_{1n}(z) \\ \vdots & & \vdots \\ B_{n1}(z) & \dots & B_{nn}(z) \end{pmatrix} \begin{pmatrix} g_1(z;\epsilon) \\ \vdots \\ g_n(z;\epsilon) \end{pmatrix}$$

where

$$\vec{f} = T\vec{g}$$
$$B = T^{-1}AT - T^{-1}\partial_z T$$

- New strategy in dimensional regularization with $D = 4 - 2\epsilon$
- A linear transformation of MIs to the canonical basis
- The coefficient matrix B only depends on z

J.M.Henn, PRL110, 251601 (2013)

Three families of MIs

- For quasi PDF, we have three families of MIs
- The first family of MIs is

$$I^1_{n_i} = \int\int \frac{\mathcal{D}^D k_1 \, \mathcal{D}^D k_2}{(k_1^2)^{n_1}(k_2^2)^{n_2}((k_2-p)^2)^{n_3}((k_1+k_2)^2)^{n_4}} \frac{1}{((k_1+k_2-p)^2)^{n_5}}$$
$$\times \left(\frac{1}{(P_1+i0)^{n_6}} - \frac{1}{(P_1-i0)^{n_6}}\right)\frac{1}{4\pi i}\left(\frac{1}{(Q_1+i0)^{n_7}} + \frac{1}{(Q_1-i0)^{n_7}}\right),$$

where the linear propagators are

$$P_1 = n\cdot k_1 + yn\cdot p, \qquad Q_1 = n\cdot k_2,$$

the integration measure is

$$\mathcal{D}^D k_i = \frac{1}{i\pi^{D/2} e^{-\frac{4-D}{2}\gamma_E}} \left(\frac{p_z^2}{\mu^2}\right)^{\left(\frac{4-D}{2}\right)} \mathrm{d}^D k_i\,,$$

Determine boundary condition

- Direct calculation(Feynman/Alpha parametrization, Residue theory, ...)
  For $g_3^1 = \epsilon(y-1)p_z I^1_{0,2,0,0,2,1,0}$,
  $$g_3^1 = \mathrm{Sgn}(y-1)\Big(-2+\epsilon\big[4\ln(4(y-1)^2)\big] - \frac{1}{3}\epsilon^2\big[12\ln((y-1)^2)\ln(16(y-1)^2)+5\pi^2+12\ln^2(4)\big]+\mathcal{O}(\epsilon^3)\Big).$$
- Regular condition
  $$\frac{\partial g_7^1}{\partial z} = \frac{\epsilon}{4}\left[\frac{8g_7^1}{z} - \frac{6g_7^1-g_8^1}{z-2y+1} - \frac{6g_7^1+g_8^1}{z+2y-1} + \frac{2g_3^1-6g_7^1+g_8^1}{z-1} - \frac{2g_3^1+6g_7^1+g_8^1}{z+1}\right].$$
  $z=\frac{p_0}{p_z}=0$ ($p^2=-p_z{}^2$) is regular point for all y regions, which implies $g_7^1|_{z=0}=0$. In addition $z=2y-1$ ($0<y<1$) and $z=1$ ($y>1$ or $y<0$) are also regular points.

Numerical check of MIs

- Check MIs using the numerical integration package FIESTA

  A.V.Smirnov,1511.03614

$$\boxed{\frac{p^2}{p_z^2} = -\frac{1}{2}, \quad y = \frac{1}{3}, \quad \frac{p_z}{\mu} = 1}$$

Analytic:

$$I^1_{1,1,0,0,2,1,0} = \frac{-2.492900960}{\epsilon} + 0.4498613241 + \epsilon(-21.287203876),$$

FIESTA:

$$I^1_{1,1,0,0,2,1,0} = \frac{-2.49290 \pm 0.0000652}{\epsilon} + 0.449836 \pm 0.000847 + \epsilon(-21.2872 \pm 0.004169).$$

Outline

Renormalization

- Renormalization procedure

$$\tilde{f}(y, \frac{p^z}{\mu}, \epsilon_{\mathrm{IR}}) = \int \frac{dy_1}{|y_1|} \left[Z_q \tilde{Z} \left(\frac{y}{y_1} \right) \right] \left[Z_q^{-1} \tilde{f} \left(y_1, \frac{p^z}{\mu}, \epsilon \right) \right].$$

Z_q is quark renormalization constant, $\tilde{Z}$ is quasi distribution renormalization factor

$$\tilde{Z}(\xi) = \delta(1-\xi) \left(1 + \frac{\alpha_s}{2\pi} \frac{\tilde{Z}^{(1)}}{\epsilon_{UV}} + \left(\frac{\alpha_s}{2\pi} \right)^2 \frac{\tilde{Z}^{(2)}}{\epsilon_{UV}^2} \right),$$

$$\tilde{Z}^{(1)} = -\frac{3 C_F S_\epsilon}{2}, \quad \tilde{Z}^{(2)} = S_\epsilon^2 \left(\frac{a + 9 C_F^2}{4} + \frac{b}{4} \epsilon \right)$$

X. Ji and J.H. Zhang, 1505.07699;

Braun, Chetyrkin and Kniehl, 2004.01043

IR behavior in Quasi PDF

- Soft divergences are cancelled
- Reducible collinear divergences

$$\tilde{f}^{(2)}_{q/q}(y, \frac{p^z}{\mu}, \epsilon_{\rm IR})|_{div.part.1} = C^{(1)}_{qq}\left(\frac{y}{x}, \frac{|x|p^z}{\mu}\right) \otimes \left[-\frac{(1+x^2)}{(1-x)}\right]_+ \frac{1}{\epsilon_{\rm IR}}.$$

- "Irreducible" collinear divergences

  the same as light cone PDFs, including both $\frac{1}{\epsilon_{\rm IR}}$ and $\left(\frac{1}{\epsilon_{\rm IR}}\right)^2$

$$\tilde{f}^{(2)}_{i/j}(y, \frac{p^z}{\mu}, \epsilon_{\rm IR})|_{div.part.2} = f^{(2)}_{i/j}(x, \epsilon_{\rm IR}).$$

$$f^{(2)}_{i/j}(x) = \frac{1}{2\epsilon^2_{\rm IR}}\left[\sum_k P^{(0)}_{ik}(z) \otimes P^{(0)}_{kj}(x) + \beta_0 P^{(0)}_{ij}(z)\right] - \frac{P^{(1)}_{ij}(x)}{\epsilon_{\rm IR}}$$

Factorization formula at NNLO

- Matching procedure between renormalized quasi and light-cone PDFs:

$$\begin{aligned}
\tilde{f}^{(0)}_{i/k}(y, \frac{p^z}{\mu}) =& C^{(0)}_{ij}\left(\frac{y}{x}, \frac{|x|p^z}{\mu}\right) \otimes f^{(0)}_{j/k}(x), \\
\tilde{f}^{(1)}_{i/k}(y, \frac{p^z}{\mu}, \epsilon_{\rm IR}) =& C^{(1)}_{ij}\left(\frac{y}{x}, \frac{|x|p^z}{\mu}\right) \otimes f^{(0)}_{j/k}(x) \\
&+ C^{(0)}_{ij}\left(\frac{y}{x}, \frac{|x|p^z}{\mu}\right) \otimes f^{(1)}_{j/k}(x, \epsilon_{\rm IR}), \\
\tilde{f}^{(2)}_{i/k}(y, \frac{p^z}{\mu}, \epsilon_{\rm IR}) =& C^{(2)}_{ij}\left(\frac{y}{x}, \frac{|x|p^z}{\mu}\right) \otimes f^{(0)}_{j/k}(x) \\
&+ C^{(1)}_{ij}\left(\frac{y}{x}, \frac{|x|p^z}{\mu}\right) \otimes f^{(1)}_{j/k}(x, \epsilon_{\rm IR}) \\
&+ C^{(0)}_{ij}\left(\frac{y}{x}, \frac{|x|p^z}{\mu}\right) \otimes f^{(2)}_{j/k}(x, \epsilon_{\rm IR}).
\end{aligned}$$

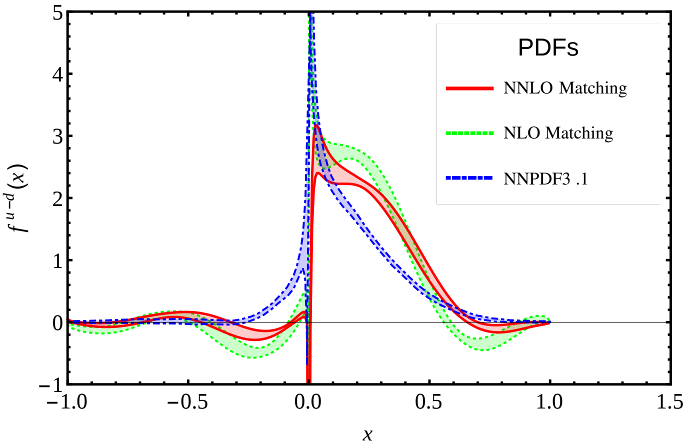
Outline

NNLO matching coefficients $C^{(2)}_{qq}$

- consistent results in $\overline{\mathrm{MS}}$ scheme by Li-Ma-Qiu (see Zheng-Yang's talk) Li,Ma,Qiu,2006.12370
- We also obtained $C^{(2)}_{qq}(y, \frac{p^z}{\mu})$ in both RI/MOM and $\mathrm{M}\overline{\mathrm{MS}}$ scheme
- 4 regions for y and 3 color structures $(C_F, C_A, nf\ T_F)C_F$
- the final asymptotic behavior: $C^{(2),\mathrm{M}\overline{\mathrm{MS}}}_{qq}|_{y\to\infty} \propto \frac{1}{y^2}$

$$
\begin{aligned}
&C^{(2),\mathrm{M}\overline{\mathrm{MS}}}_{qq}(y, \frac{p^z}{\mu}) \\
=&[C^{(2),\mathrm{M}\overline{\mathrm{MS}}}_{qq}(y, \frac{p^z}{\mu})|_{y>1}]_+ + [C^{(2),\mathrm{M}\overline{\mathrm{MS}}}_{qq}(y, \frac{p^z}{\mu})|_{0<y<1}]_+ \\
&+ [C^{(2),\mathrm{M}\overline{\mathrm{MS}}}_{qq}(y, \frac{p^z}{\mu})|_{-1<y<0}]_+ + [C^{(2),\mathrm{M}\overline{\mathrm{MS}}}_{qq}(y, \frac{p^z}{\mu})|_{y<-1}]_+
\end{aligned}
$$

PDFs from NNLO Matching



using LPC data with $z_{cut} = 10a$, $\mu = 2GeV$ and in modified $\overline{\mathrm{MS}}$ scheme; uncertainty is from lattice data

Summary

- NNLO correction is important
- NNLO matching coefficients of quark PDF are obtained
- Complete cancellation of IR divergence is confirmed, which nontrivially validates the LaMET factorization at NNLO
- NNLO matching improved the curves of quark PDF

- Outlook
  - *Gluon quasi distribution functions at NNLO*
  - *Pion quasi distribution amplitudes at NNLO*
  - *A new stage of lattice calculation of PDFs with NNLO matching*

Back up

Canonical bases for family-1 MIs, here only list 10 of them

$$
\begin{aligned}
g_1^1 &= \epsilon(y+1)p_z I^1_{0,0,2,2,0,1,0}\,,\\
g_2^1 &= \epsilon y p_z I^1_{0,2,0,2,0,1,0}\,,\\
g_3^1 &= \epsilon(y-1)p_z I^1_{0,2,0,0,2,1,0}\,,\\
g_4^1 &= \epsilon y p_z p_1^2 I^1_{2,2,1,0,0,1,0}\,,\\
g_5^1 &= \epsilon^2\sqrt{p_1^2+p_z^2}\, I^1_{0,1,1,0,2,1,0}\,,\\
g_6^1 &= \epsilon(p_1^2-4y(y-1)p_z^2) I^1_{0,1,1,0,2,2,0} + 8\epsilon^2(2y-1)p_z I^1_{0,1,1,0,2,1,0}\\
&\quad + \epsilon(y-1)p_z I^1_{0,2,0,0,2,1,0} + \epsilon y p_z I^1_{0,2,0,2,0,1,0}\,,\\
g_7^1 &= \epsilon^2\sqrt{p_1^2+p_z^2}\, I^1_{1,1,0,0,2,1,0}\,,\\
g_8^1 &= \epsilon(p_1^2-4y(y-1)p_z^2) I^1_{1,1,0,0,2,2,0} + 6\epsilon^2(2y-1)p_z I^1_{1,1,0,0,2,1,0}\,,\\
g_9^1 &= \epsilon^2\sqrt{p_1^2+p_z^2}\, I^1_{0,1,1,2,0,1,0}\,,\\
g_{10}^1 &= \epsilon(p_1^2-4y(y+1)p_z^2) I^1_{0,1,1,2,0,2,0} + 8\epsilon^2(2y+1)p_z I^1_{0,1,1,2,0,1,0}\\
&\quad + \epsilon(y+1)p_z I^1_{0,0,2,2,0,1,0} + \epsilon y p_z I^1_{0,2,0,2,0,1,0}\,,
\end{aligned}
$$

Back up

NLO matching coefficient in $\overline{MS}$ scheme

$$C_{qq}^{(1),\overline{MS}}\left(y,\frac{\mu}{p^z}\right)|_{\mathcal{O}(\epsilon^0)}$$
$$=C_F\begin{cases}\left[\frac{1+y^2}{1-y}\ln\frac{y}{y-1}+1+\frac{3}{2y}\right]_{+(1)}^{[1,\infty]}-\frac{3}{2y}, & y>1\\ \left[\frac{1+y^2}{1-y}\left(-\ln\frac{\mu^2}{4p^{z2}}+\ln(y(1-y))\right)-\frac{y(1+y)}{1-y}\right]_{+(1)}^{[0,1]}, & 0<y<1\\ \left[-\frac{1+y^2}{1-y}\ln\frac{y}{y-1}-1+\frac{3}{2(1-y)}\right]_{+(1)}^{[-\infty,0]}-\frac{3}{2(1-y)}, & y<0\end{cases}$$
$$+C_F\left[\delta(1-y)\left(\frac{3}{2}\ln\frac{\mu^2}{4p^{z2}}+\frac{5}{2}\right)\right]$$

NLO matching coefficient in $M\overline{MS}$ scheme

$$C\left(\xi,\frac{\xi\mu}{xP_3}\right)=\delta(1-\xi)+\frac{\alpha_s}{2\pi}C_F\begin{cases}\left[\frac{1+\xi^2}{1-\xi}\ln\frac{\xi}{\xi-1}+1+\frac{3}{2\xi}\right]_+ & \xi>1,\\ \left[\frac{1+\xi^2}{1-\xi}\ln\frac{x^2P_3^2}{\xi^2\mu^2}(4\xi(1-\xi))-\frac{\xi(1+\xi)}{1-\xi}+2\iota(1-\xi)\right]_+ & 0<\xi<1,\\ \left[-\frac{1+\xi^2}{1-\xi}\ln\frac{\xi}{\xi-1}-1+\frac{3}{2(1-\xi)}\right]_+ & \xi<0,\end{cases}$$

Back up

One loop quasi pdf in RI/MOM scheme

$$\tilde{f}^{(1)}_{q/q,\xi}(y,p^z)/C_F$$

$$= \begin{cases} -\dfrac{\left((\xi-2)\rho+2y^2-(\xi+1)\rho y+2\right)\log\left(\dfrac{\sqrt{1-\rho}+2y-1}{-\sqrt{1-\rho}+2y-1}\right)}{2(1-\rho)^{3/2}(y-1)} + \dfrac{\xi\rho\left(\rho(\rho+1)+16y^4-40y^3+4(\rho+8)y^2-2(3\rho+4)y\right)}{2(\rho-1)(y-1)\left(\rho+4y^2-4y\right)^2} \\ +\dfrac{y\left((4-5\rho)\rho-16y^4+8(\rho+3)y^3-24\rho y^2+4\left(\rho^2+3\rho-2\right)y\right)}{(\rho-1)(y-1)\left(\rho+4y^2-4y\right)^2} + \dfrac{3}{2-2y}, & y>1 \\ -\dfrac{\xi\rho+\xi-3\rho+4y^2-2(\xi+1)y+3}{2(\rho-1)(y-1)} - \dfrac{\log\left(\dfrac{\sqrt{1-\rho}+1}{1-\sqrt{1-\rho}}\right)\left((\xi-2)\rho+2y^2-(\xi+1)\rho y+2\right)}{2(1-\rho)^{3/2}(y-1)}, & 0<y<1, \\ \dfrac{\left((\xi-2)\rho+2y^2-(\xi+1)\rho y+2\right)\log\left(\dfrac{\sqrt{1-\rho}+2y-1}{-\sqrt{1-\rho}+2y-1}\right)}{2(1-\rho)^{3/2}(y-1)} - \dfrac{\xi\rho\left(\rho(\rho+1)+16y^4-40y^3+4(\rho+8)y^2-2(3\rho+4)y\right)}{2(\rho-1)(y-1)\left(\rho+4y^2-4y\right)^2} \\ -\dfrac{y\left((4-5\rho)\rho-16y^4+8(\rho+3)y^3-24\rho y^2+4\left(\rho^2+3\rho-2\right)y\right)}{(\rho-1)(y-1)\left(\rho+4y^2-4y\right)^2} - \dfrac{3}{2-2y}), & y<0. \end{cases}$$

$$\log\left(\frac{\sqrt{1-\rho}+1}{1-\sqrt{1-\rho}}\right) = -\log\left(\frac{1-\sqrt{z^2}}{\sqrt{2-z^2}+1}\right), \quad \log\left(\frac{\sqrt{1-\rho}+2y-1}{-\sqrt{1-\rho}+2y-1}\right) = -\log\left(\frac{2y-\sqrt{z^2}-1}{2y+\sqrt{2-z^2}-1}\right)$$

$$\int dy[h(y)]_+ g(y) = \int dy h(y)[g(y)-g(1)]$$

$$iD^{\mu\nu}_\xi(k) = -\frac{i}{k^2}\left[g^{\mu\nu}-(1-\xi)\frac{k^\mu k^\nu}{k^2}\right].$$

Back up

NNLO matching coefficients

$$
\begin{aligned}
C_{qq}^{(2),\mathrm{M\overline{MS}}}(y,\frac{p^z}{\mu})|_{y>1} =& C_{qq}^{(2),\overline{\mathrm{MS}}}(y,\frac{p^z}{\mu})|_{y>1} + \frac{C_F\left(11C_A+9C_F-2n_f\right)}{4y}\log\left(\frac{\mu^2}{p^{z2}}\right) - \frac{C_F n_f(5-4\log(2y))}{4y} \\
&- \frac{C_A C_F\left(132\log(2y)+4\pi^2-159\right)}{24y} + \frac{C_F^2\left(-108\log(2y)+16\pi^2+75\right)}{24y}, \\
C_{qq}^{(2),\mathrm{M\overline{MS}}}(y,\frac{p^z}{\mu})|_{0<y<1} =& C_{qq}^{(2),\overline{\mathrm{MS}}}(y,\frac{p^z}{\mu})|_{0<y<1}, \\
C_{qq}^{(2),\mathrm{M\overline{MS}}}(y,\frac{p^z}{\mu})|_{-1<y<0} =& C_{qq}^{(2),\overline{\mathrm{MS}}}(y,\frac{p^z}{\mu})|_{-1<y<0}, \\
C_{qq}^{(2),\mathrm{M\overline{MS}}}(y,\frac{p^z}{\mu})|_{y<-1} =& C_{qq}^{(2),\overline{\mathrm{MS}}}(y,\frac{p^z}{\mu})|_{y<-1} + \frac{C_F\left(11C_A+9C_F-2n_f\right)}{4(1-y)}\log\left(\frac{\mu^2}{p^{z2}}\right) - \frac{C_F n_f(5-4\log(-2y))}{4(1-y)} \\
&- \frac{C_A C_F\left(132\log(-2y)+4\pi^2-159\right)}{24(1-y)} + \frac{C_F^2\left(-108\log(-2y)+16\pi^2+75\right)}{24(1-y)}.
\end{aligned}
$$

$$
\begin{aligned}
C_{qq}^{(2),\overline{\mathrm{MS}}}(y,\frac{p^z}{\mu})|_{y>1} =& \left(C_F c_1^{C_F} + C_A c_1^{C_A} + 2T_F n_f c_1^{T_F}\right) C_F + \left(\Gamma_1(y)|_{y>1}\right)\log(\frac{\mu^2}{p_z^2}), \\
C_{qq}^{(2),\overline{\mathrm{MS}}}(y,\frac{p^z}{\mu})|_{0<y<1} =& C_F\left(C_F c_2^{C_F} + C_A c_2^{C_A} + 2T_F n_f c_2^{T_F}\right) + \left(\Gamma_2(y)\right)\log^2(\frac{\mu^2}{p_z^2}) \\
&+ \left(\left(\Gamma_1(y)|_{0<y<1}\right) - \left(P_{qq}^{(1),V}(y)|_{0<y<1}\right)\right)\log(\frac{\mu^2}{p_z^2}), \\
C_{qq}^{(2),\overline{\mathrm{MS}}}(y,\frac{p^z}{\mu})|_{-1<y<0} =& C_F\left(C_F c_3^{C_F} + C_A c_3^{C_A} + 2T_F n_f c_3^{T_F}\right) + \left(\left(\Gamma_1(y)|_{-1<y<0}\right) - P_{\bar{q}q}^{(1),V}(-y)\right)\log(\frac{\mu^2}{p_z^2}),
\end{aligned}
$$

$$
C_{qq}^{(2),\overline{\mathrm{MS}}}(y,\frac{p^z}{\mu})|_{y<-1} = -\, C_{qq}^{(2),\overline{\mathrm{MS}}}(y,\frac{p^z}{\mu})|_{y>1}\ .
$$

The complete analytic expression can be found in our paper.