Connecting quasi and pseudo distributions in nongauge theories

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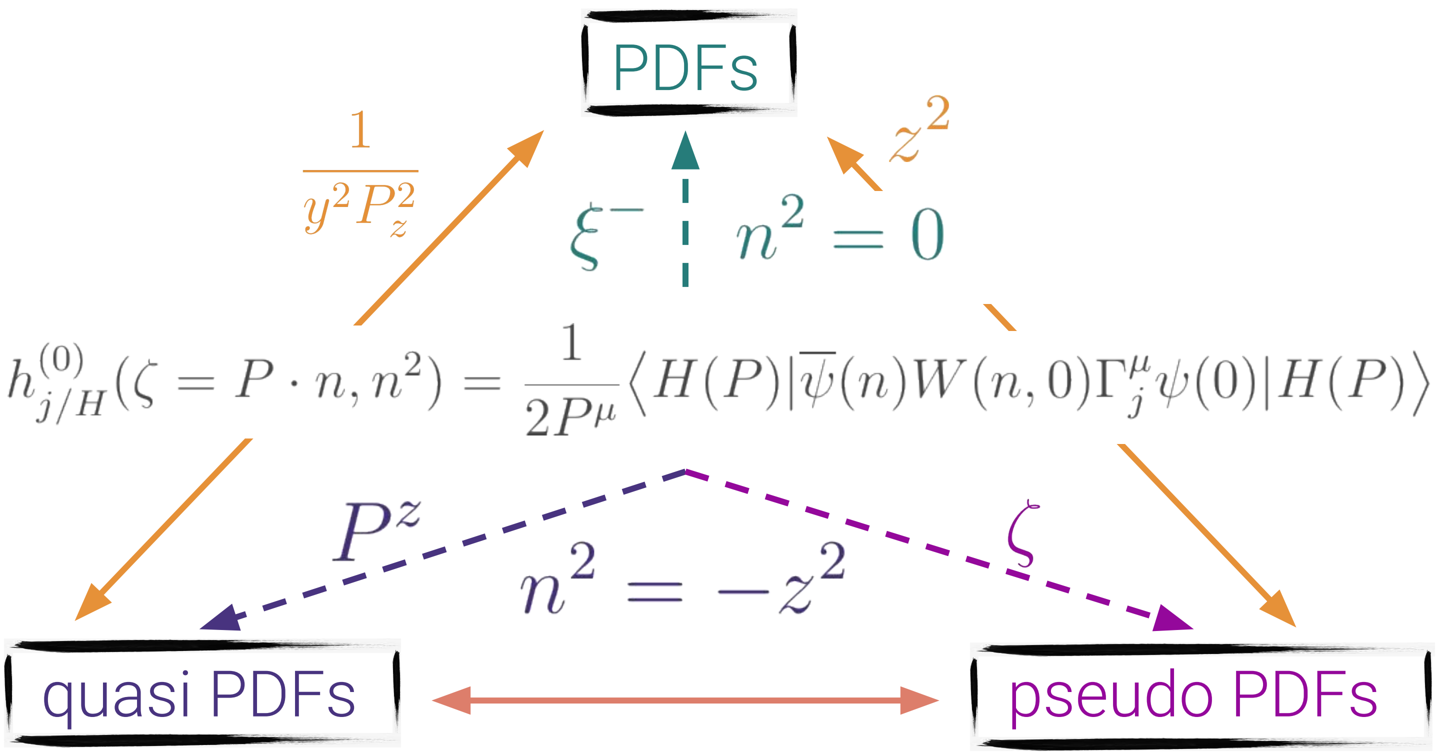
arXiv:2007.02131 [to appear in JHEP]
with Tommaso Giani and Luigi Del Debbio

The distribution landscape

$$f^{(0)}_{j/H}(\xi) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega^-}{4\pi} e^{-i\xi P^+\omega^-} \langle H(P)\,|\overline{\psi}(0,\omega^-,\mathbf{0}_\mathrm{T}) W(\omega^-,0)\Gamma_j \psi(0)|\,H(P)\rangle$$



$$\widetilde{f}^{(0)}_{j/H}(\xi, P^z) = \int_{-\infty}^{\infty} \frac{\mathrm{d}z}{4\pi} e^{i\xi P_z z} \langle H(P)\,|\overline{\psi}(0,z,\mathbf{0}_\mathrm{T}) W(z,0)\Gamma_j \psi(0)|\,H(P)\rangle$$

Ji, PRL 110 (2013) 262002

$$\widetilde{p}^{(0)}_{j/H}(\xi, z^2) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\zeta}{4\pi} e^{i\xi\zeta} \langle H(P)\,|\overline{\psi}(0,z,\mathbf{0}_\mathrm{T}) W(z,0)\Gamma_j \psi(0)|\,H(P)\rangle$$

Radyushkin, PRD 96 (2017) 034025

Giani, Del Debbio & Monahan, arXiv:2007.02131

Motivation

1. Simplify quasi-PDF and pseudo-PDF landscape by working in scalar field theory

2. Map out key relationships explicitly at one loop

3. Generalise Collin's study of Ioffe-time matrix elements to z^2 nonzero

Collins, PRD 21 (1980) 2962

4. Address criticisms from Rossi and Testa…

Rossi & Testa, PRD 96 (2017) 014507
Rossi & Testa, PRD 98 (2018) 054028

… which have also been previously addressed

Ji, Zhang & Zhao, NPB 924 (2017) 366
Karpie, Orginos & Zafeiropoulos, JHEP 11 (2018) 178
Radyushkin, PLB 788 (2019) 380

but…

Work in six-dimensions, with Lagrangian (density)

$$\mathcal{L} = \frac{1}{2}\left(\partial\phi\right)^2 - \frac{m^2}{2}\phi^2 - \frac{g}{3!}\phi^3$$

Consider Ioffe-time matrix elements

$$\mathcal{M} = \langle P|\phi\left(z\right)\phi\left(0\right)|P\rangle \ = \ \mathcal{M}\left(\nu, z^2\right) \qquad \nu \ = \ P \cdot z$$

but in perturbation theory, we work with the partonic matrix element

$$\widehat{\mathcal{M}}\left(\nu, z^2\right) = \langle p|\phi\left(z\right)\phi\left(0\right)|p\rangle \qquad \nu \ = \ p \cdot z$$

For z^2 lightlike, we define the parton distribution functions (PDFs) through

$$f\left(x\right) = xP^+ \int \frac{dz^-}{2\pi}\, e^{-ixP^+z^-} \langle P|\phi\left(z\right)\phi\left(0\right)|P\rangle \qquad z^2 \ = \ 0$$

For z^2 spacelike, we can calculate the matrix element on the lattice

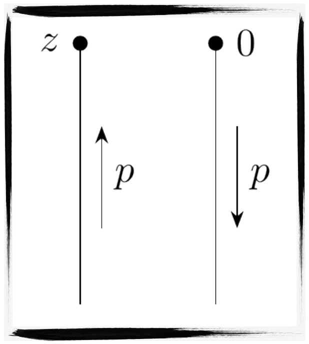
Extract Ioffe-time matrix element through LSZ reduction

$$\widehat{\mathcal{M}}\left(\nu, z^2\right) = \langle p | T\left[\phi\left(z\right)\phi\left(0\right)\right]|p\rangle$$
$$= \lim_{p^2\to m^2_{\mathrm{pole}}} \left(p^2 - m^2_{\mathrm{pole}} + i\epsilon\right)^2 \int dz_1\, dz_2\, e^{-ip\cdot z_1} e^{ip\cdot z_2} \langle 0|T\left[\phi\left(z\right)\phi\left(0\right)\phi\left(z_1\right)\phi\left(z_2\right)\right]|0\rangle$$

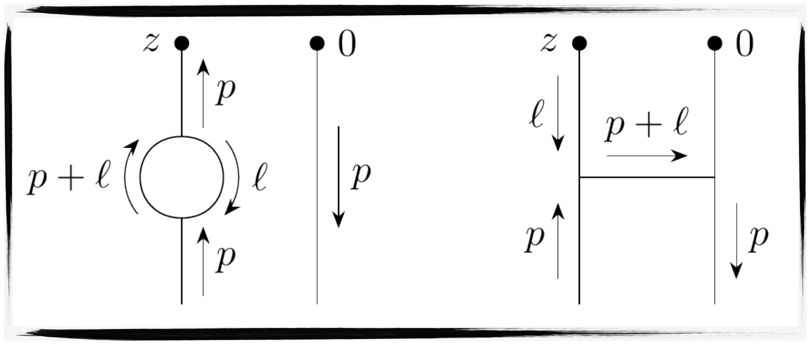
For spacelike separations, we know the result will be equivalent to the matrix element extracted from the long-time behaviour of Euclidean correlation functions

Briceno, Hansen & CJM, PRD 96 (2017) 014502

To one-loop:

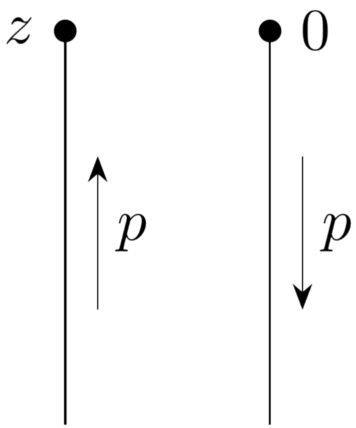


Tree-level



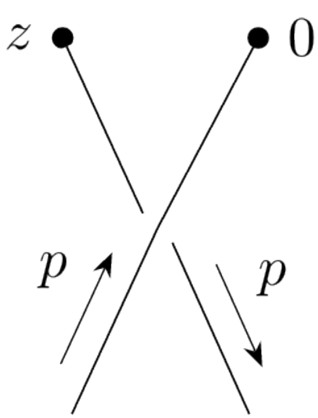
One-loop

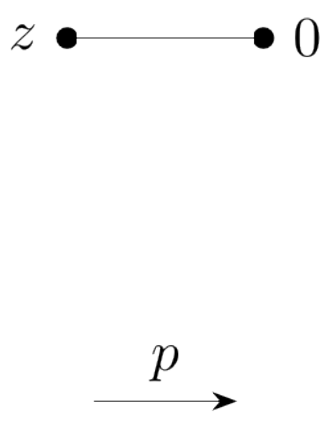
Leading order contribution



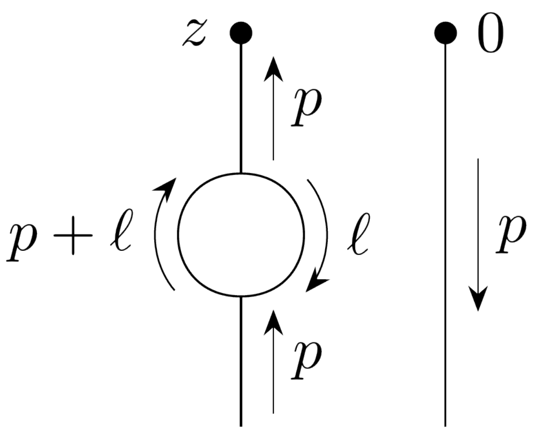
$$\widehat{\mathcal{M}}^{(0)}\left(\nu, z^2\right) = -e^{-i\nu} \equiv \widehat{\mathcal{M}}^{(0)}(\nu, 0)$$

Ignore topologies of the form





First topology: wavefunction-type diagram



$$\widehat{\mathcal{M}}_{\rm self}\left(\nu, z^2\right) = R\,\widehat{\mathcal{M}}^{(0)}\left(\nu, 0\right)$$

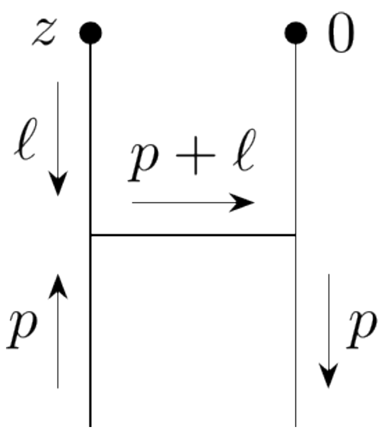
$$R = \frac{d\Pi\left(l^2\right)}{dl^2}\bigg|_{l^2=p^2_{\rm pole}} = \alpha\left[\frac{1}{12}\log\frac{m^2}{\mu^2} + \frac{1}{12}\frac{1}{\epsilon} + \frac{b}{2}\right]$$

Leads to

$$\widehat{\mathcal{M}}_{\rm self}\left(\nu, z^2\right) = \left[1 + \alpha\left(\frac{1}{6}\log\frac{m^2}{\mu^2} + \frac{1}{6}\frac{1}{\epsilon} + b\right)\right]\widehat{\mathcal{M}}^{(0)}\left(\nu, 0\right) + \mathcal{O}\left(\alpha^2\right)$$

which is, of course, independent of the separation z^2

Second topology: vertex-type diagram



$$\widehat{\mathcal{M}}^{(1)}\left(\nu, z^2\right) = -i\, g^2 \int_k \frac{e^{-ik\cdot z}}{\left(k^2 - m^2 + i\epsilon\right)^2} \frac{1}{\left(p-k\right)^2 - m^2 + i\epsilon}$$
$$= g^2 \int_0^1 d\xi\ \left(1-\xi\right) K\left(z^2, M^2\right)\ \widehat{\mathcal{M}}^{(0)}\left(\xi\nu, 0\right)\ ,$$

with

$$K\left(z^2, M^2\right) = 2i \int_q \frac{e^{-iq\cdot z}}{\left(q^2 - M^2 + i\epsilon\right)^3} \qquad M^2 = m^2\left(1 - \xi + \xi^2\right)$$

Leads to

$$\widehat{\mathcal{M}}\left(\nu, 0\right) = \left[1 + \alpha\left(\frac{1}{6}\log\frac{m^2}{\mu^2} + \frac{1}{6}\frac{1}{\epsilon} + b\right)\right] \widehat{\mathcal{M}}^{(0)}\left(\nu, 0\right)$$
$$+ \alpha \int_0^1 d\xi\ \left(1-\xi\right)\left(\frac{1}{\epsilon} + \log\frac{\mu^2}{m^2\left(1-\xi+\xi^2\right)}\right) \widehat{\mathcal{M}}^{(0)}\left(\xi\nu, 0\right)$$

to one loop

Renormalised matrix element given by

$$\widehat{\mathcal{M}}^{(1)}\left(\nu, z^2\right) = -i\, g^2 \int_k \frac{e^{-ik\cdot z}}{\left(k^2 - m^2 + i\epsilon\right)^2} \frac{1}{\left(p-k\right)^2 - m^2 + i\epsilon}$$

$$= g^2 \int_0^1 d\xi\ (1-\xi)\, K\left(z^2, M^2\right)\ \widehat{\mathcal{M}}^{(0)}\left(\xi\nu, 0\right)\ ,$$

and satisfies

$$\mu^2 \frac{d}{d\mu^2} \widehat{\mathcal{M}}_R\left(\nu, 0, \mu^2\right) = \alpha \int_0^1 d\xi\, P\left(\xi\right)\ \widehat{\mathcal{M}}_R\left(\xi\nu, 0, \mu^2\right) + \mathcal{O}\left(\alpha^2\right)$$

where

$$P\left(\xi\right) = \left(1-\xi\right) - \frac{1}{6}\delta\left(1-\xi\right) = \left(1-\xi\right)_+ + \frac{1}{3}\delta\left(1-\xi\right)$$

$$\mu^2 \frac{d}{d\mu^2} \widehat{f}\left(x, \mu^2\right) = \alpha \int_x^1 \frac{d\xi}{\xi} P\left(\xi\right)\ \widehat{f}\left(\frac{x}{\xi}, \mu^2\right)$$

$$\widehat{\mathcal{M}}_R\left(\nu, 0, \mu^2\right) = \int_0^1 dx\, e^{ix\nu} \widehat{f}\left(x, \mu^2\right)$$

Writing out the one-loop matrix element explicitly

$$\widehat{\mathcal{M}}\left(\nu,-z_3^2\right)=\left[1+\alpha\left(\frac{1}{6}\log\frac{m^2}{\mu^2}+\frac{1}{6}\frac{1}{\epsilon}+b\right)\right]\widehat{\mathcal{M}}^{(0)}\left(\nu,0\right)$$
$$+\alpha\int_0^1 d\xi\,\left(1-\xi\right)2K_0\left(Mz_3\right)\widehat{\mathcal{M}}^{(0)}\left(x\nu,0\right)$$

where the kernel is Bessel function

$$K\left(-z_3^2,M^2\right)=\frac{1}{64\pi^3}\int_0^\infty\frac{dT}{T}e^{-T}e^{-\frac{(Mz_3)^2}{4T}}=\frac{1}{64\pi^3}\,2K_0\left(Mz_3\right)$$

Renormalised matrix element given by

$$\widehat{\mathcal{M}}_R\left(\nu,-z_3^2;\,\mu^2\right)=\int_0^1 dy\,\tilde{\mathcal{K}}\left(y\right)\widehat{\mathcal{M}}\left(y\nu,-z_3^2\right)$$

where now the kernel is

$$\tilde{\mathcal{K}}\left(y\right)=\delta\left(1-y\right)\left[1-\alpha\frac{1}{6\,\epsilon}\right]$$

We can now factorise our renormalised matrix elements

$$\widehat{\mathcal{M}}_R\left(\nu, -z_3^2;\, \mu^2\right) = \widehat{\mathcal{M}}_R\left(\nu, 0, \mu^2\right) + \alpha \int_0^1 d\xi\, \left(1-\xi\right) \left(2K_0\left(Mz_3\right) - \log \frac{\mu^2}{M^2}\right) \widehat{\mathcal{M}}_R\left(\xi\nu, 0, \mu^2\right)$$

writing this in terms of the PDF, we have

$$\widehat{\mathcal{M}}_R\left(\nu, -z_3^2; \mu^2\right) = \int_0^1 dx\, \tilde{C}\left(x\nu, mz_3, \frac{\mu^2}{m^2}\right) \widehat{f}\left(x, \mu^2\right)$$

where the factorisation coefficient is

$$\tilde{C}\left(x\nu, mz_3, \frac{\mu^2}{m^2}\right) = e^{ix\nu} - \alpha \int_0^1 d\xi\, \left(1-\xi\right) \left(2K_0\left(Mz_3\right) - \log \frac{\mu^2}{M^2}\right) e^{i\xi x\nu}$$

In the short-distance limit, the coefficient

$$\tilde{C}\left(x\nu, mz_3, \frac{\mu^2}{m^2}\right) = e^{ix\nu} - \alpha \int_0^1 d\xi\ (1-\xi)\left(2K_0\left(Mz_3\right) - \log\frac{\mu^2}{M^2}\right) e^{i\xi x\nu}$$

we expand

$$2K_0\left(Mz_3\right) = -\log\left(M^2 z_3^2\right) + 2\log\left(2e^{-\gamma_E}\right) + \mathcal{O}\left(M^2 z_3^2\right)$$

So our coefficient becomes

$$\tilde{C}\left(x\nu, \mu^2 z_3^2\right) = e^{ix\nu} - \alpha \int_0^1 d\xi\ (1-\xi)\log\left(\mu^2 z_3^2 \frac{e^{2\gamma_E}}{4}\right) e^{i\xi x\nu} + \mathcal{O}\left(m^2 z_3^2\right)$$

$$\mathcal{M}_R\left(\nu, -z_3^2;\ \mu^2\right) = \int_0^1 dx\, \tilde{C}\left(x\nu, \mu^2 z_3^2\right) f\left(x, \mu^2\right)$$

Natural to introduce RGI ratio $\mathfrak{M}\left(\nu, -z_3^2\right) = \dfrac{\mathcal{M}_R\left(\nu, -z_3^2; \mu^2\right)}{\mathcal{M}_R\left(0, -z_3^2; \mu^2\right)}$

then the coefficient becomes

$$\tilde{C}_+\left(x\nu, \mu^2 z_3^2\right) = e^{ix\nu} - \alpha\ \log\left(\mu^2 z_3^2 \frac{e^{2\gamma_E}}{4}\right)\int_0^1 d\xi\ (1-\xi)_+\, e^{i\xi x\nu} + \mathcal{O}\left(m^2 z_3^2\right)$$

Relation at arbitrary momenta

$$q\left(y,\mu^2,P_3^2\right)=\int_0^1\frac{dx}{x}\,f\left(x,\mu^2\right)C\left(\frac{y}{x},\frac{m^2}{x^2P_3^2},\frac{\mu^2}{m^2}\right)$$

$$C\left(\eta,\frac{m^2}{x^2P_3^2},\frac{\mu^2}{m^2}\right)=\int_0^1 d\xi\,\left(1-\xi\right)\left[\frac{1}{\sqrt{\left(\eta-\xi\right)^2+\frac{M^2}{x^2P_3^2}}}-\delta\left(\xi-\eta\right)\log\frac{\mu^2}{M^2}\right]$$

and at large momenta

$$q\left(y,\mu^2,P_3^2\right)=\int_0^1\frac{dx}{x}\,f\left(x,\mu^2\right)C\left(\frac{y}{x},\frac{\mu^2}{x^2P_3^2}\right)+\mathcal{O}\left(\frac{m^2}{P_3^2}\right)$$

$$C\left(\eta,\frac{\mu^2}{x^2P_3^2}\right)=\delta\left(1-\eta\right)+\alpha\begin{cases}\left(1-\eta\right)\log\frac{\eta}{\eta-1}+1 & \eta>1\\ \left(1-\eta\right)\log 4\eta\left(1-\eta\right)\frac{x^2P_3^2}{\mu^2}+2\eta-1 & 0<\eta<1\\ -\left(1-\eta\right)\log\frac{\eta}{\eta-1}-1 & \eta<0\end{cases}$$

Confirmed equivalence by explicit Fourier transform

Demonstrated factorisation at one loop for smeared distribution

$$\widehat{\mathcal{M}}_t\left(\nu, -z_3^2; \mu^2\right) = \int_0^1 dx\, \overline{C}\left(x\nu, \mu^2 z_3^2\right) \widehat{f}\left(x, \mu^2\right) \qquad \widehat{\mathcal{M}}_t\left(\nu, \overline{z}^2\right) = \langle p | \rho\left(t; z\right) \rho\left(t; 0\right) | p \rangle$$

where at one loop we have

$$\begin{aligned}\widehat{\mathcal{M}}_t^{(1)}\left(\nu, -\overline{z}_3^2\right) &= g^2 \int_{k_E} e^{-2k_E^2 t} \frac{e^{-ik_{\mathrm{E}} z_3}}{\left(k_{\mathrm{E}}^2 + m^2\right)^2} \frac{1}{\left(p_{\mathrm{E}} - k_{\mathrm{E}}\right)^2 + m^2} \\ &= g^2 \int_0^1 d\xi\, \left(1 - \xi\right) K_t\left(-\overline{z}_3^2, \overline{M}^2\right) \widehat{\mathcal{M}}^{(0)}\left(\xi\nu, 0\right)\end{aligned}$$

$$K_t\left(-\overline{z}_3^2, \overline{M}^2\right) = \frac{\mu^{6-d}}{(4\pi)^{d/2}} e^{-2m^2 t\xi} \int_0^\infty \mathrm{d}T\, \frac{T^2}{(T + 2t)^{d/2}}\, e^{-TM^2} e^{(4\xi t p_{\mathrm{E}} - i z_{\mathrm{E}})^2/(4(T+2t))}$$

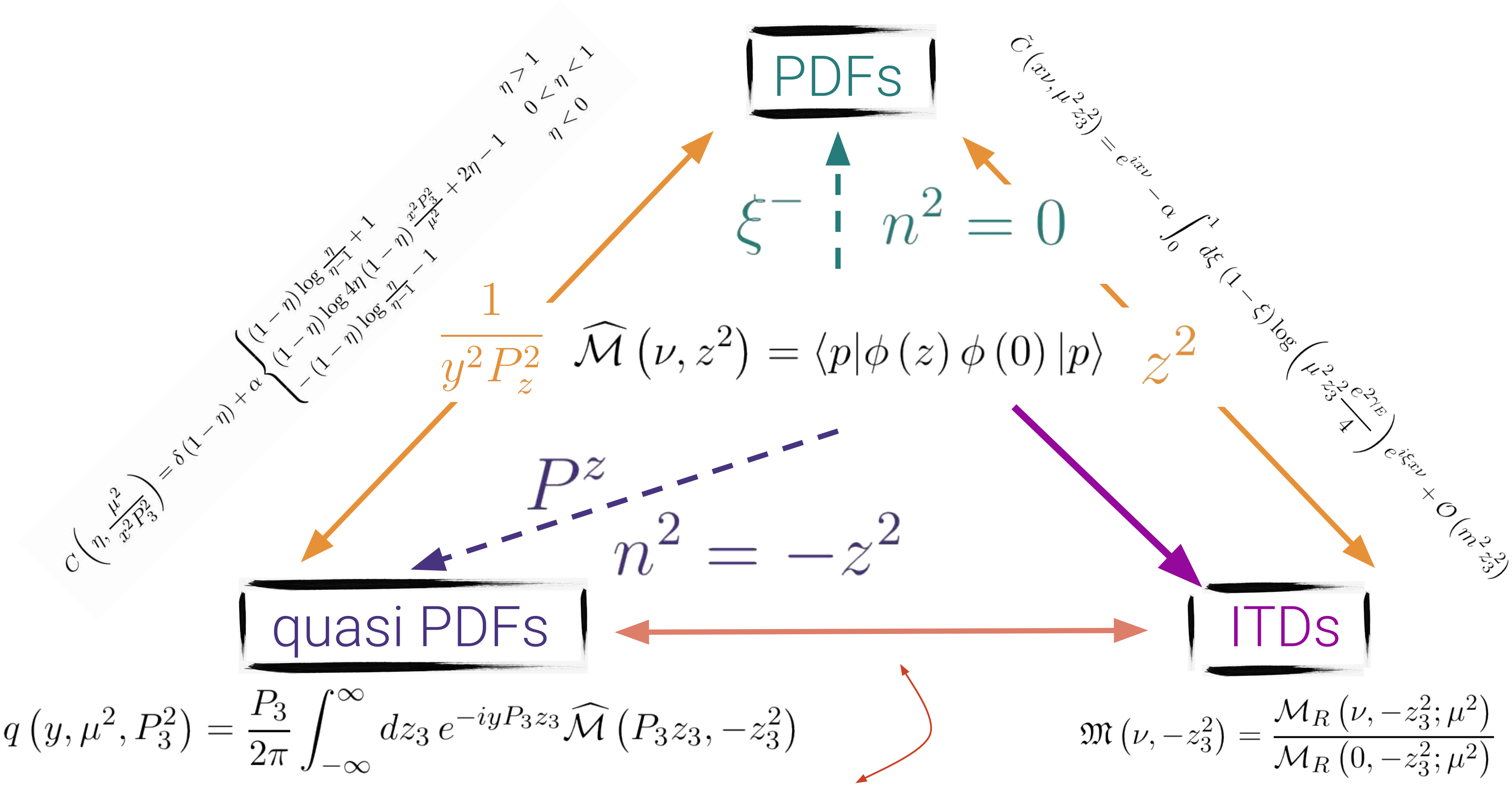
Perturbative factorisation coefficient given by

$$\begin{aligned}\overline{C}\left(x\nu, \mu^2 z_3^2\right) &= e^{ix\nu} - \alpha \int_0^1 d\xi\, \left(1 - \xi\right) \left[\log\left(\mu^2 z_3^2 \frac{e^{2\gamma_E}}{4}\right) - \mathcal{R}(M z_3)\right] e^{i\xi x\nu} \\ &\quad + \mathcal{O}\left(m^2 z_3^2, \frac{t^2 m^2}{z_E^2}, 1/\overline{z}^2\right) .\end{aligned}$$

Giani, Del Debbio & Monahan, arXiv:2007.02131

The distribution landscape

$$\widehat{\mathcal{M}}(\nu,0) = \left[1+\alpha\left(\frac{1}{6}\log\frac{m^2}{\mu^2}+\frac{1}{6}\frac{1}{\epsilon}+b\right)\right]\widehat{\mathcal{M}}^{(0)}(\nu,0) + \alpha\int_0^1 d\xi\,(1-\xi)\left(\frac{1}{\epsilon}+\log\frac{\mu^2}{m^2(1-\xi+\xi^2)}\right)\widehat{\mathcal{M}}^{(0)}(\xi\nu,0)$$



$$q(y,\mu^2,P_3^2)=\frac{P_3}{2\pi}\int_{-\infty}^{\infty}dz_3\,e^{-iyP_3z_3}\widehat{\mathcal{M}}(P_3z_3,-z_3^2)$$

$$\mathfrak{M}(\nu,-z_3^2)=\frac{\mathcal{M}_R(\nu,-z_3^2;\mu^2)}{\mathcal{M}_R(0,-z_3^2;\mu^2)}$$

$$\frac{1}{x}C\left(\eta,\frac{\mu^2}{x^2P_3^2}\right)=\frac{P_3}{2\pi}\int_{-\infty}^{\infty}dz_3\,e^{-iyP_3z_3}\,\tilde{C}(x\nu,\mu^2z_3^2)$$

Conclusions

Scalar field theory provides a natural theatre in which to study partonic physics

- we generalized Collins' study to off-light-cone matrix elements

In leading twist approximation and at one loop, explicitly

- calculate renormalised matrix elements on and off the light-cone
- demonstrate factorisation at small separations, large momenta, small flow times
- calculate perturbative relation between factorisation coefficients

Everything really works as expected

- at one loop and leading twist, pseudo and quasi PDFs equivalent

Neglects systematic uncertainties that affect practical implementations

Most natural

- work directly with Ioffe-time distributions
- factorise in real space
- treat ITD as "data" and apply global fitting techniques

conceptual equivalence does not mean practical interchangeability

Thank you

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