Conjugate Directions in Landau and Coulomb Lattice Gauge Fixing

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Overview

- Matching lattice Green's functions to continuum perturbation theory, non perturbative renormalisation in Landau gauge.
- Coulomb gauge fixed wall source propagators, static potential measurements.
- Two schools

1. Los Alamos
   - Overrelaxation
2. Cornell
   - Fourier acceleration

Links, Fields and Notation

$$U_\mu\left(x + a\frac{\hat{\mu}}{2}\right) = e^{iag_0 A_\mu\left(x + a\frac{\hat{\mu}}{2}\right)}.$$

Generally for an SU(3) matrix $U = e^{iA} = f_0 + f_1 A + f_2 A^2$.

$$A \approx \frac{1}{2i}\left\{\left[U - U^\dagger\right] - \frac{1}{3}\mathrm{Tr}\left[U - U^\dagger\right]\cdot I_{3\times 3}\right\}.$$

$$A = \frac{f_2^* U - f_2 U^\dagger - 2i\Im(f_0 f_2^*)\cdot I_{3\times 3}}{2i\Im(f_1 f_2^*)}.$$

Links, Fields and Notation ... Cont'd

$$a\Delta_\mu A_\mu(x) = \sum_\mu \left(A_\mu \left(x + a\frac{\hat{\mu}}{2} \right) - A_\mu \left(x - a\frac{\hat{\mu}}{2} \right) \right).$$

$$\Theta^{(n)} = \frac{1}{N_c V} \sum_x \mathrm{Tr} \left[\left(a\Delta_\mu a g_0 A_\mu^{(n)}(x) \right)^2 \right].$$

Fixing on a lattice ... Steepest descent (SD)

Satisfying the Landau condition is the same as minimising a cost functional F(U).

$$F(U) = \frac{1}{N_d N_c V} \sum_{x,\mu} \mathrm{Tr}\left[\left(a g_0 A_\mu\left(x + a\frac{\hat{\mu}}{2}\right)\right)^2\right].$$

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$$x^{(n+1)} = x^{(n)} - \alpha f'(x^{(n)}).$$

$0 < \alpha < 1$ is a small tuning parameter.

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$$g(x) = e^{-i\alpha a \Delta_\mu a g_0 A_\mu^{(n)}(x)},$$

$$U_\mu^{(n+1)} \left(x + a\frac{\hat{\mu}}{2} \right) = g(x) U_\mu^{(n)} \left(x + a\frac{\hat{\mu}}{2} \right) g(x + a\hat{\mu})^\dagger.$$

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Fourier Acceleration (FA)

Davies et al. (1988) pointed out that the SD suffers from critical slowing down. Ameliorated by Fourier acceleration

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$$g(x) = e^{-i\alpha\,\tilde{F}\,\frac{p^2_{\mathrm{Max}}}{Vp^2}\,F\,a\Delta_\mu a g_0 A^{(n)}_\mu(x)}.$$

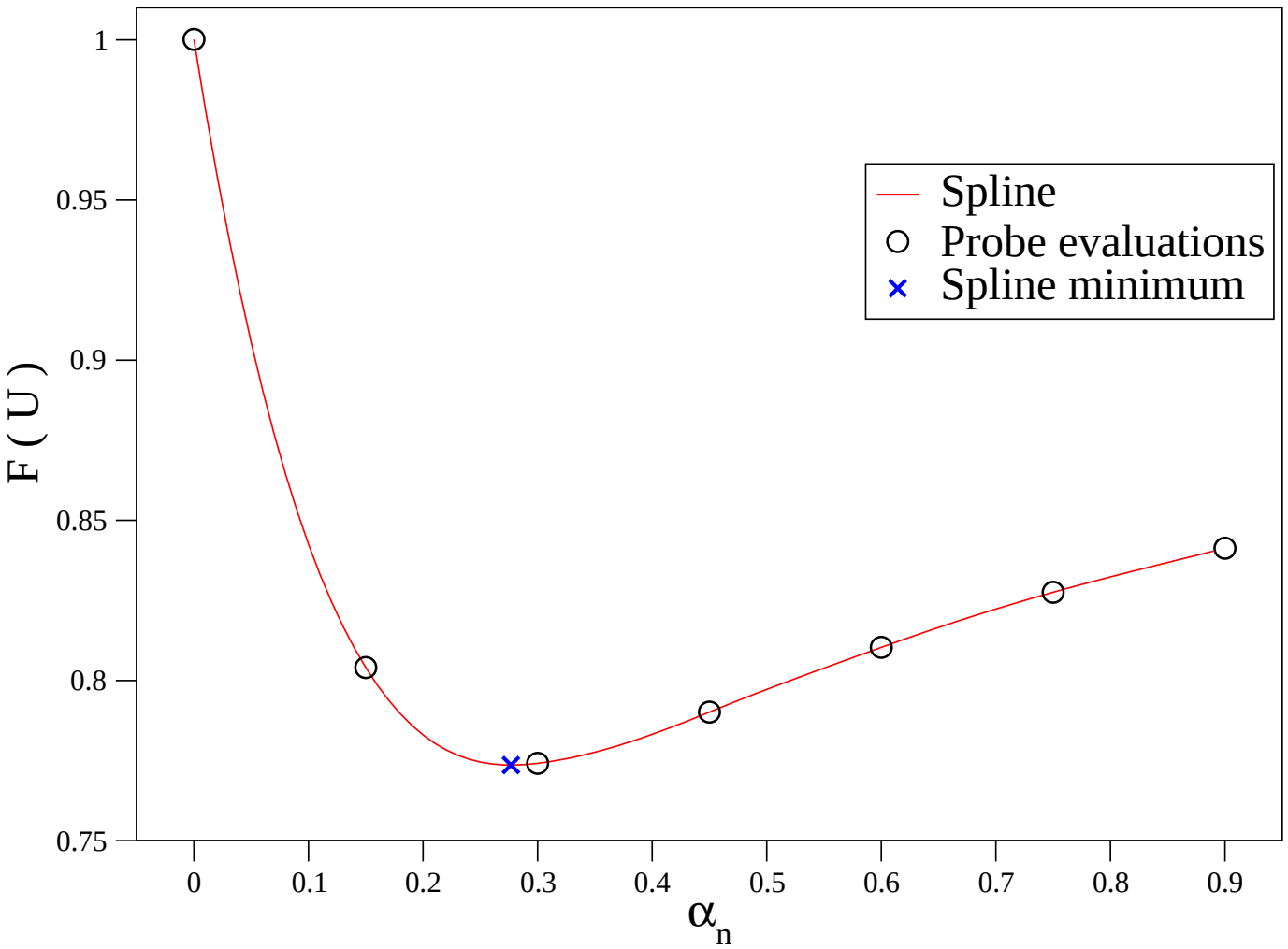
$$p^2 = 2\left(N_d - \sum_\mu \cos\left(\frac{2\pi n_\mu}{L_\mu}\right)\right).$$

We call this the FASD method.

Optimal SD is far from optimal

- What about $\alpha = \alpha_n$, tuned at each step?
- Evaluate "N" probe α_n''s and choose the one that minimises $F(U)$ best
- Use cubic spline interpolation and exact solve for minimum
- Cost is $\approx N\times$ that of fixed α.

Splines



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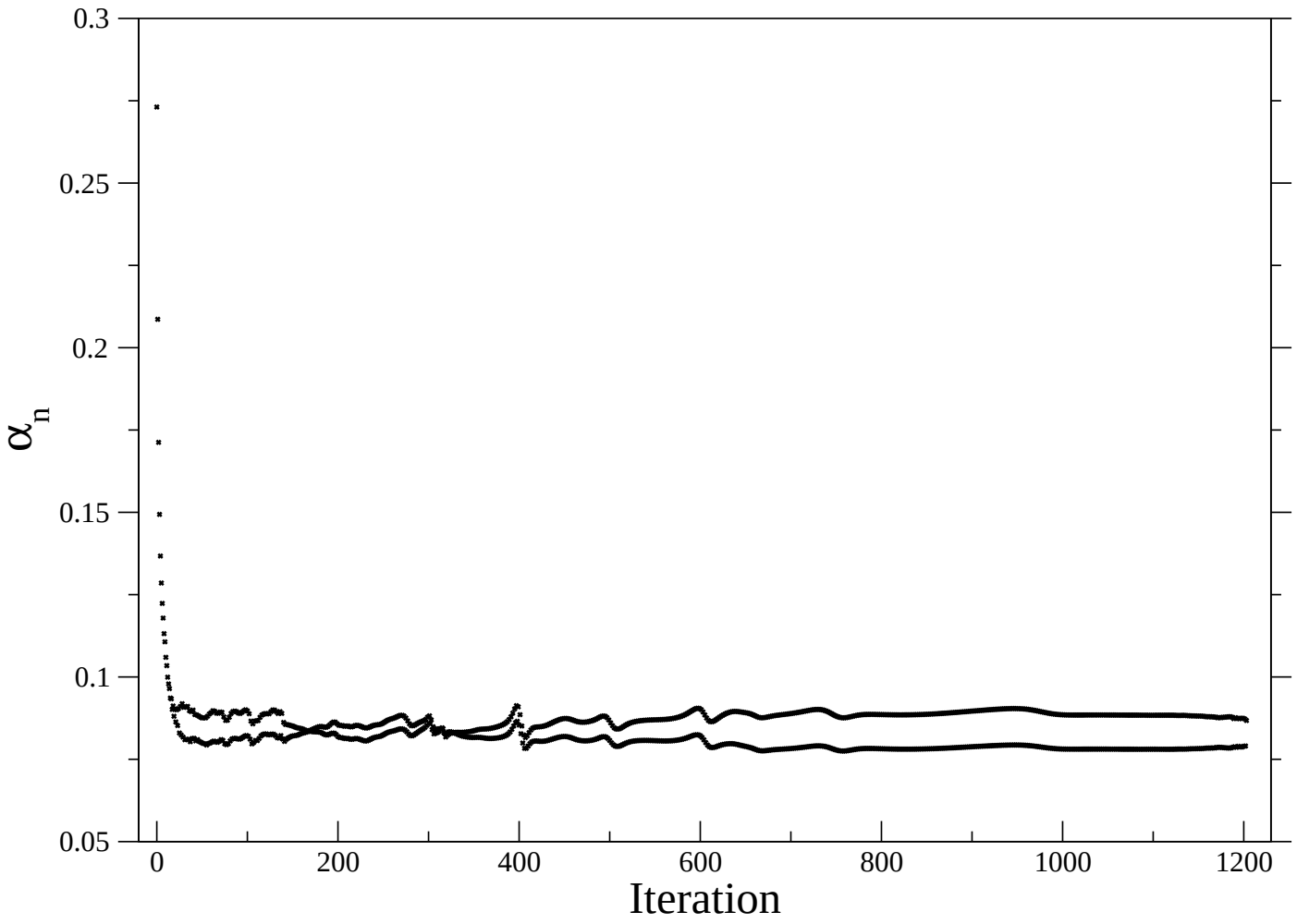

The general Non-linear CG procedure

Compute the gradient direction $f'(x_0)$
Perform a line search for α_0 s.t $\min(f(x_0 - \alpha_0 f'(x_0)))$
Perform the update $x_1 = x_0 - \alpha_0 f'(x_0)$
Set $s_0 = -f'(x_0)$
$n = 1$
while $|f'(x_n)|^2 >$ Tolerance **do**
    Compute the gradient $f'(x_n)$
    Compute $\beta_n = \max\left[0, \frac{f'(x_n)^T(f'(x_n)-f'(x_{n-1}))}{f'(x_{n-1})^T f'(x_{n-1})}\right]$
    Conjugate direction $s_n = -f'(x_n) + \beta_n s_{n-1}$
    Perform a line search for α_n s.t $\min(f(x_n + \alpha_n s_n))$
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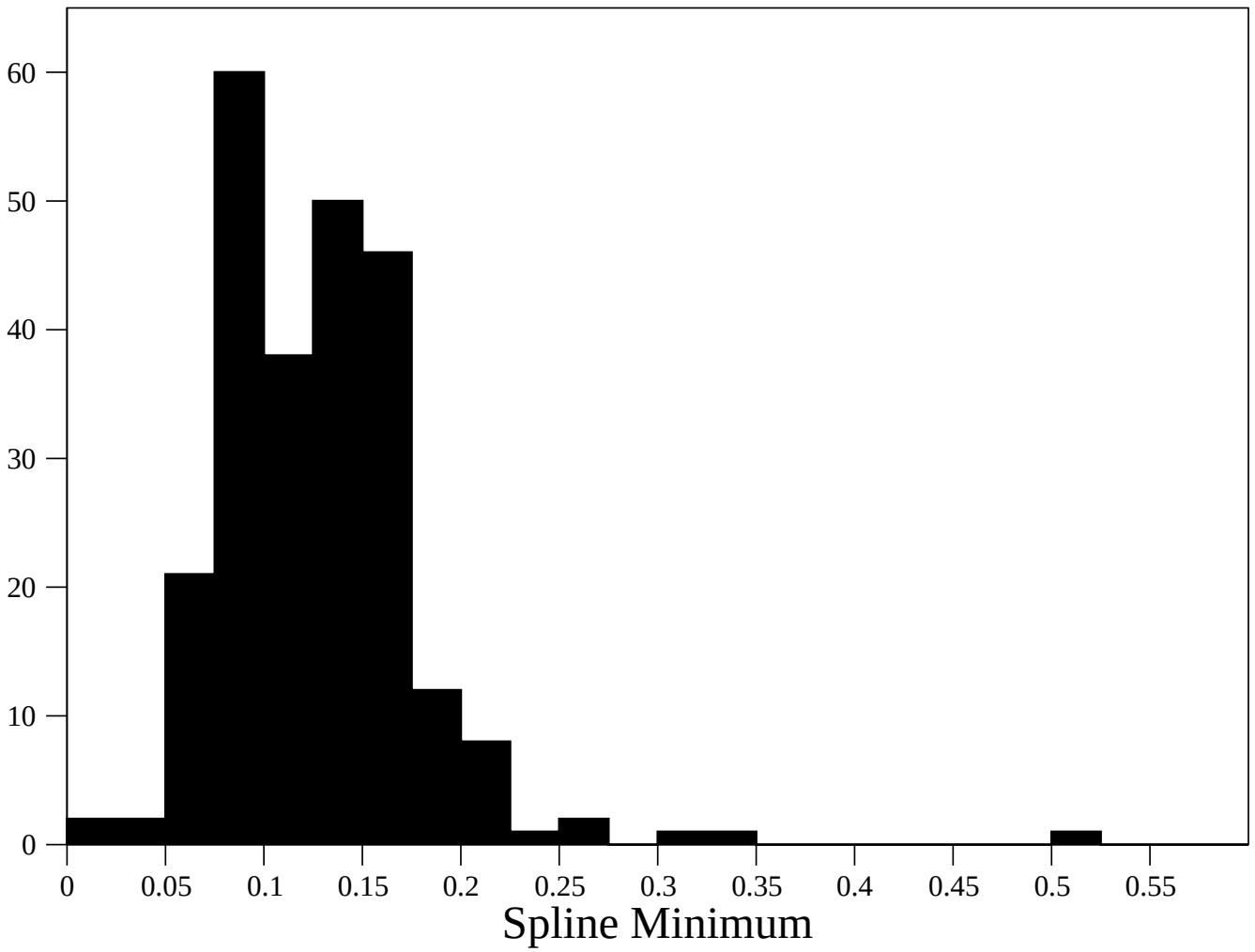
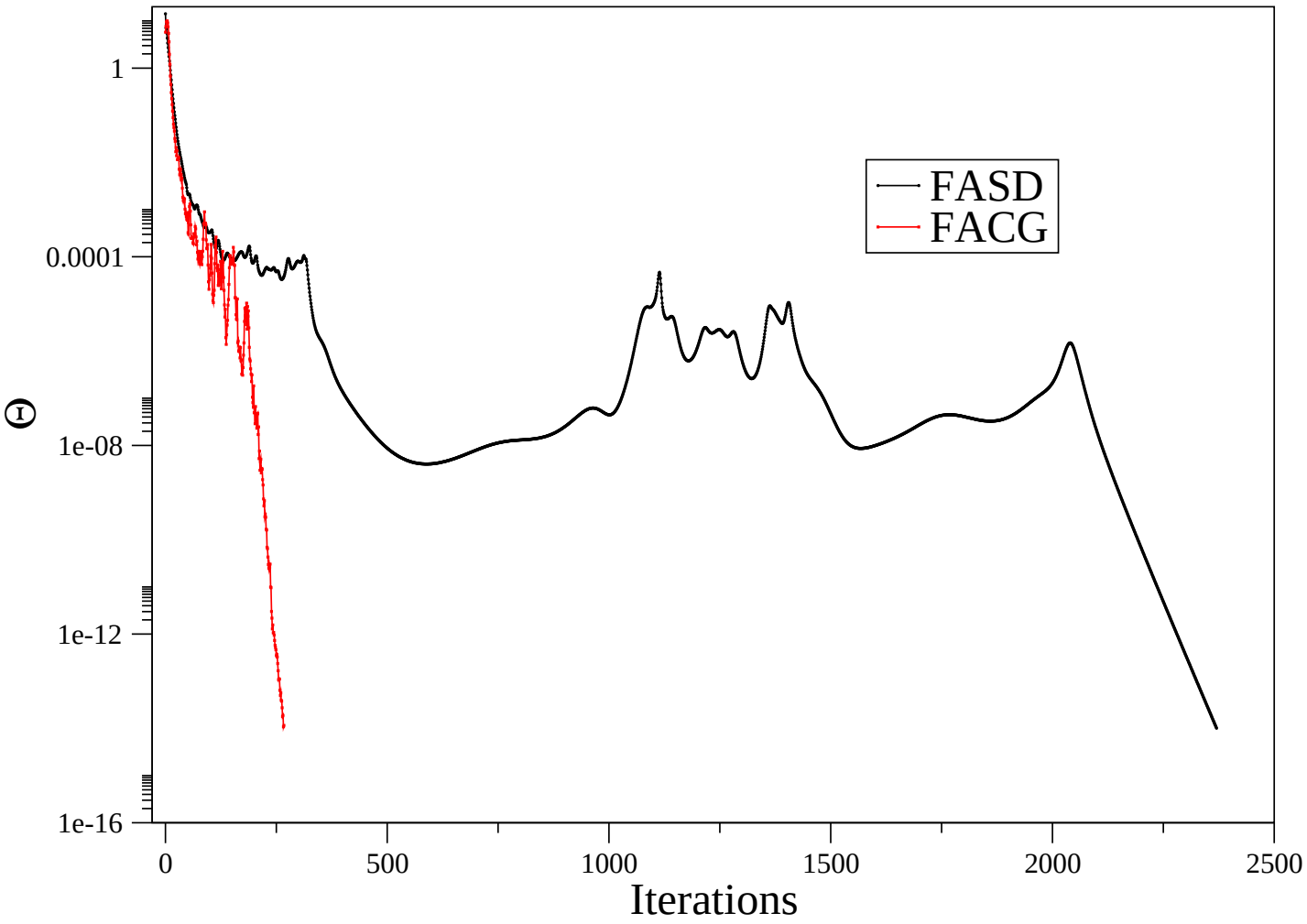
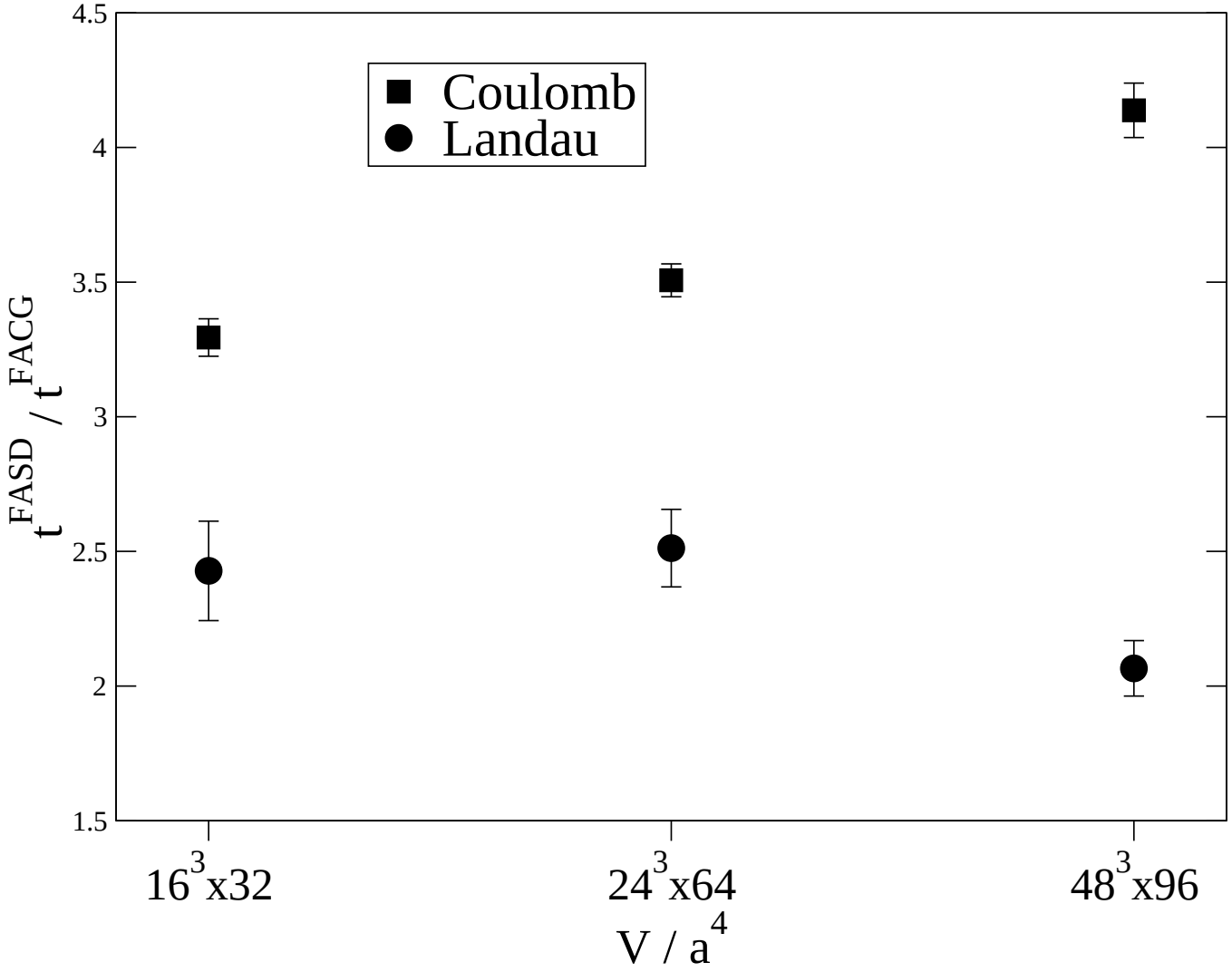
FACG spline history



FASD vs. FACG



Effective speed up



Timings

	Landau		Coulomb	
Volume	FASD (s)	FACG (s)	FASD (s)	FACG (s)
$16^3 \times 32$	25(1)	10.5(5)	27.4(6)	8.3(1)
$24^3 \times 64$	306(18)	122(7)	194(3)	55.4(4)
$48^3 \times 96$	12995(857)	6292(404)	10882(180)	2629(39)

Table: The time in seconds taken to achieve an accuracy of $\Theta^{(n)} < 10^{-14}$ for 25 randomly transformed copies of the same, well-thermalised configuration. Each configuration has the same lattice spacing. This measurement was performed using four 8-core AMD Opteron 6128 processors. All operations were performed in double precision.

Conclusions

- This method relies on fast functional evaluations and a line search
- At least a $2\times$ speed up from this procedure for all volumes considered
- Fewer FFTs compared to the FASD
- Method could be faster
- Check out www.github.com/RJHudspith/GLU/
- And my paper http://arxiv.org/abs/1405.5812