NPR for PDAs

Nonperturbative renormalisation for low moments of light-meson distribution amplitudes

Jonathan Flynn (RBC/UKQCD Collaboration)
School of Physics & Astronomy
University of Southampton

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RBC/UKQCD

UKQCD
Rudy Arthur (Odense)
Peter Boyle (Edinburgh)
Luigi Del Debbio (Edinburgh)
Shane Drury (Southampton)
Jonathan Flynn (Southampton)
Julien Frison (Edinburgh)
Nicolas Garron (Dublin)
Jamie Hudspith (Toronto)
Tadeusz Janowski (Southampton)
Andreas Jüttner (Southampton)
Ava Kamseh (Edinburgh)
Richard Kenway (Edinburgh)
Andrew Lytle (TIFR)
Marina Marinkovic (Southampton)
Brian Pendleton (Edinburgh)
Antonin Portelli (Southampton)
Thomas Rae (Mainz)
Chris Sachrajda (Southampton)
Francesco Sanfilippo (Southampton)
Matthew Spraggs (Southampton)
Tobias Tsang (Southampton)

RBC
Ziyuan Bai (Columbia)
Thomas Blum (U Conn/RBRC)
Norman Christ (Columbia)
Xu Feng (Columbia)
Tomomi Ishikawa (RBRC)
Taku Izubuchi (RBRC/BNL)
Luchang Jin (Columbia)
Chulwoo Jung (BNL)
Taichi Kawanai (RBRC)
Chris Kelly (RBRC)
Hyung-Jin Kim (BNL)
Christoph Lehner (BNL)
Jasper Lin (Columbia)
Meifeng Lin (BNL)
Robert Mawhinney (Columbia)
Greg McGlynn (Columbia)
David Murphy (Columbia)
Shigemi Ohta (KEK)
Eigo Shintani (Mainz)
Amarjit Soni (BNL)
Sergey Syritsyn (RBRC)
Oliver Witzel (BU)
Hantao Yin (Columbia)
Jianglei Yu (Columbia)
Daiqian Zhang (Columbia)
RBC/UKQCD

R Arthur (Odense)
PA Boyle (Edinburgh)
D Brömmel (Jülich)
JM Flynn, A Jüttner, CTC Sachrajda (Southampton)
TD Rae (Mainz)
Introduction

Parton distribution amplitudes (PDAs) relevant for exclusive QCD processes at large momentum transfer, distances near the light cone

Give process-independent nonperturbative information on bound-state structure of hadrons

- Momentum-fraction distribution of partons in a particular Fock state of a hadron

Calculated in three main approaches

- Extract from experimental form factor data
- QCD sum rules
- Lattice QCD
Introduction

Low moments of PDAs can be computed from local matrix elements.

Example for pseudoscalar meson

$$\langle 0|S \bar{q}_a \gamma_\mu \gamma_5 \not{D}_\nu q_b |P(p)\rangle = \langle \xi^1 \rangle_P f_P S p_\mu p_\nu$$

$$\langle 0|S \bar{q}_a \gamma_\mu \gamma_5 \not{D}_\nu \not{D}_\rho q_b |P(p)\rangle = \langle \xi^2 \rangle_P f_P S p_\mu p_\nu p_\rho$$

$S$ means symmetrised and traceless in Lorentz indices.

Bare lattice operators need renormalisation and matching to a continuum scheme like $\overline{\text{MS}}$. 
Moments from non-forward matrix elements with momentum transferred at the operator insertion

For second moment get mixing of double-covariant derivative operator with double total-derivative operator

\[ S \bar{q}_a \gamma_\mu \gamma_5 \bar{D}_\nu \bar{D}_\sigma q_b \quad \text{and} \quad S \partial_\mu \partial_\nu (\bar{q}_a \gamma_\sigma \gamma_5 q_b) \]

Evaluate

\[ \langle \xi^1 \rangle^\text{MS} = \frac{Z_{D,D}}{Z_A} \langle \xi^1 \rangle^\text{bare} \]

\[ \langle \xi^2 \rangle^\text{MS} = \frac{Z_{DD,DD}}{Z_A} \langle \xi^2 \rangle^\text{bare} + \frac{Z_{DD,\partial\partial}}{Z_A} \]
Previous RBC/UKQCD calculation

Chiral extrapolations: $\alpha^{-1} = 1.73\text{ GeV blue, } 2.28\text{ GeV red}$

Kaon 1st moment
Small lattice-spacing dependence, NPR

Pion 2nd moment
More $\alpha$-dependence, but total-derivative operator perturbatively renormalised

prd83,074505 2011 and in preparation
\[
\langle \xi^1 \rangle^{\overline{\text{MS}}} = \frac{Z_{D,D}}{Z_A} \langle \xi^1 \rangle^{\text{bare}}
\]
\[
\langle \xi^2 \rangle^{\overline{\text{MS}}} = \frac{Z_{DD,DD}}{Z_A} \langle \xi^2 \rangle^{\text{bare}} + \frac{Z_{DD,\partial \partial}}{Z_A}
\]

Previously used RI’/MOM scheme

- No momentum transfer at insertion $\Rightarrow$ could not access mixing with total derivative operator for 2nd moment
- Where had both, PT and NPT renormalisation constants differed

\[\alpha^{-1} = 2.28 \text{ GeV lattice with } \overline{\text{MS}} \text{ scale } \mu = 2 \text{ GeV}\]

<table>
<thead>
<tr>
<th></th>
<th>$Z_{D,D}/Z_A$</th>
<th>$Z_{DD,DD}/Z_A$</th>
<th>$Z_{DD,\partial \partial}/Z_A$</th>
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<td>nonperturbative</td>
<td>1.50(2)</td>
<td>1.97(5)</td>
<td>—</td>
</tr>
<tr>
<td>mean-field imp PT</td>
<td>1.28(4)</td>
<td>1.51(6)</td>
<td>0.015(4)</td>
</tr>
</tbody>
</table>
Nonperturbative renormalisation (NPR)

- Use Rome–Southampton regularisation independent (RI) momentum-subtraction scheme (MOM)
- For operator $O$

$$\Lambda^O_R = \frac{1}{Z_q} Z_O \Lambda^O_B$$

- $\Lambda^O$ is amputated quark two-point function with insertion of $O$ (Lorentz indices implicit)
- $Z_O$ is operator renormalisation constant (can be matrix) and $Z_q$ is quark wavefunction renormalisation
- Impose renormalisation condition(s) at a particular momentum configuration, with associated scale $\mu$

$$\Lambda_{QCD} \ll \mu \ll 1/\alpha$$
RI MOM renormalisation

Evaluate amputated two-point quark Green functions at a specified Euclidean momentum configuration

- **RI’/MOM exceptional momentum**
  
  \[ q + p = 0, \quad q^2 = p^2 = \mu^2 \]

- **SMOM symmetric momentum**
  
  \[ q^2 = p^2 = (q + p)^2 = \mu^2, \quad q \cdot p = -\mu^2 / 2 \]
Use SMOM scheme
  ▶ nonexceptional momenta with momentum transfer at operator insertion
  ▶ suppresses contamination from IR effects
  ▶ allows mixing with total derivative operators (needed for our application)
  ▶ better-behaved (or at least not worse) perturbative series in conversion to $\overline{\text{MS}}$

Evaluate conversion functions from SMOM to $\overline{\text{MS}}$

Relevant continuum calculations done to 3 loops in $\overline{\text{MS}}$ and 2 loops in SMOM by J Gracey [epjc 71,1567 2011; jhep03,109 2011; prd84,016002 2011]
Operators

First moment

\[ X_2 = S \bar{\psi} \gamma_\mu \overset{\rightarrow}{D}_\nu \psi \]
\[ \partial X_2 = S \partial_\mu (\bar{\psi} \gamma_\nu \psi) \]

Second moment

\[ X_3 = S \bar{\psi} \gamma_\mu \overset{\rightarrow}{D}_\nu \overset{\rightarrow}{D}_\sigma \psi \]
\[ \partial \partial X_3 = S \partial_\mu \partial_\nu (\bar{\psi} \gamma_\sigma \psi) \]
\[ \partial X_3 = S \partial_\mu (\bar{\psi} \gamma_\nu \overset{\rightarrow}{D}_\sigma \psi) \]

- Work with C-eigenstate operators
  - \( X_2, \partial X_2, \partial X_3 \) all multiplicatively renormalised
  - \( X_3, \partial \partial X_3 \) mix

- Single total-derivative operators (\( \partial X_{2,3} \)) included for some checks
  - \( \partial X_2 \) and \( \partial \partial X_3 \) are total derivatives of vector current
  - \( \partial X_3 \) is total derivative of \( X_2 \)
Scalar coefficients

Expand Green functions as Lorentz tensors with scalar coefficients. Eg for first moment

$$\Lambda_{\mu\nu}(p, q)_{\text{sym}} = \sum_{i=1}^{10} P_{(i)}^{\mu\nu}(p, q) \Sigma_i(\mu^2)$$

$$\Sigma_i(\mu^2) = \frac{1}{\mu^2} \text{Tr} \left[ M_{ij} P_{(j)}^{\mu\nu}(p, q) \Lambda_{\mu\nu}(p, q)_{\text{sym}} \right]$$

where $P_{(i)}^{\mu\nu}(p, q)$ are 10 Lorentz tensor structures with

$$N_{ij} = \frac{1}{\mu^2} \text{Tr} \left[ P_{(i)}^{\mu\nu} P_{(j)\mu\nu} \right]_{\text{sym}} \quad M = N^{-1}$$

Similar decompositions for bilinear and second moment operators
Checking charge-conservation properties

- Gracey used a different basis of operators
- Changing to $\mathcal{C}$-conserving basis gives relations between Gracey’s 3-loop $\overline{\text{MS}}$ anomalous dimensions which are all satisfied (and determine one of them to one higher power in $g^2$)
- For amputated Green functions, charge-conjugation implies relations between scalar coefficients in our basis
  - Satisfied by the Gracey continuum calculations after the change of basis
  - Exactly satisfied by lattice data for a unit gauge field
  - Well-satisfied in our lattice data
SMOM renormalisation conditions

After tracing with some ‘projector’ $P$, the renormalised amputated Green function should give the tree-level result

$$\frac{1}{Z_q} \text{Tr}(Z_O \Lambda_{B,\text{sym}}^O P) = \text{Tr}(\Lambda_{\text{tree},\text{sym}}^O P)$$

- Choose $P$’s with charge-conjugation properties in mind
- Choose $P$’s for operators which are total derivatives of vector current to maintain Ward identity

Show examples...
**Vector current and derivatives**

- **SMOM renormalisation condition for vector current**
  \[
  \frac{1}{12\mu^2} \frac{Z_V}{Z_q} \text{Tr}(k_\mu \Lambda_{V,B}^\mu k) = 1 \quad \text{where} \quad k = q + p
  \]
  maintains WID \( k_\mu \Lambda_{V,R}^\mu = S_R^{-1}(-p) - S_R^{-1}(q) \)

- **Choose renormalisation conditions for total derivatives of the vector current**
  \[
  \frac{Z_{\partial \partial X_2}}{Z_q} \text{Tr}[(S k_\mu k_\nu) k \Lambda_{\partial \partial X_2,B}^{\mu\nu}] = 9i(\mu^2)^2
  \]
  \[
  \frac{Z_{\partial \partial X_3}}{Z_q} \text{Tr}[(S k_\mu k_\nu k_\rho) k \Lambda_{\partial \partial X_3,B}^{\mu\nu\rho}] = -6(\mu^2)^3
  \]

- **Conversion functions, SMOM to MS, for all three are then 1**
2nd moment operator

\[ X_3 = S\bar{\psi}\gamma_\mu \vec{D_\nu} \vec{D_\sigma}\psi \] mixes with \( \partial\partial X_3 = S\partial_\mu \partial_\nu(\bar{\psi}\gamma_\sigma\psi) \) and

\[ \Lambda_{\mu\nu\sigma}^{\vec{D_\sigma}}(p, q)_{\text{tree}} = -S(q_\mu - p_\mu)(q_\nu - p_\nu)\gamma_\sigma \]

\[ = \frac{\mu^2}{3}(P_{3,\mu\nu\sigma} + P_{1,\mu\nu\sigma} - P_{2,\mu\nu\sigma}) \]

Impose renormalisation condition

\[ \frac{1}{Z_q} \text{Tr}\left[ ((MP)_3 + (MP)_1 - (MP)_2)(Z_{DD,DD}^{\vec{D_\sigma}} + Z_{DD,\partial\partial}^{\vec{D_\sigma}}) \right] \]

\[ = \text{Tr}\left[ ((MP)_3 + (MP)_1 - (MP)_2)\Lambda_{\text{tree}}^{\vec{D_\sigma}} \right] = \mu^2 \]
2nd moment operator

Need another condition

\[
\frac{1}{Z_q} \text{Tr} \left[ \left( (MP)_3 + (MP)_1 + (MP)_2 \right) \left( Z_{DD,DD} \Lambda_{DD}^B + Z_{DD,\partial\partial} \Lambda_{\partial\partial}^B \right) \right]
\]

\[
= \text{Tr} \left[ \left( (MP)_3 + (MP)_1 + (MP)_2 \right) \Lambda_{\text{tree}}^{DD} \right] = \frac{\mu^2}{3}
\]

These fix \( Z_{DD,DD} \) and \( Z_{DD,\partial\partial} \) to get from bare lattice to SMOM
Conversion functions: SMOM to $\overline{\text{MS}}$

\[
O_{\overline{\text{MS}}} = C O_R \quad \Rightarrow \quad \Lambda_{\overline{\text{MS}}} = \frac{1}{C_q} C \Lambda_R \quad \text{where} \quad C_q \equiv \frac{Z_{q,\overline{\text{MS}}}}{Z_q}
\]

Expand Green function $\Lambda_a$ for operator $O_a$ in terms of tensors $P_i$ with scalar coefficients $\Sigma_{ai}$

\[
\Lambda_a = \sum_i \Sigma_{ai} P_i \quad \Sigma_{ai} = \text{Tr}[(MP)_i \Lambda_a]
\]

Renormalisation: tracing $\Lambda_{Ra}$ with some ‘projector’ $P_A$ gives tree-level (or other chosen) result, $T_A$

\[
\text{Tr} \left( \Lambda_{Ra} P_A \right) = T_{aA}
\]

May need several $P_A$ if operators mix
Conversion functions

Let $N_{iA}^P \equiv \text{Tr}(P_iP_A)$ and use $\Lambda_R = C_qC^{-1}\Lambda_{\overline{\text{MS}}}$

\[
C_qC^{-1}\sum_{bi}^{\overline{\text{MS}}} N_{iA}^P = T_{aA}
\]

- $\sum^{\overline{\text{MS}}}, N^P$ and $T$ (and $C_q$) known; impose enough conditions to solve for elements of $C$
- Combine $C$ with $Z$ from SMOM renormalisation conditions to get from lattice to $\overline{\text{MS}}$ at scale $\mu$
- Use $\overline{\text{MS}}$ anomalous dimensions to scale to a common value, say 2 GeV
- Can also compute the SMOM anomalous dimensions

\[
\gamma_{\text{SMOM}} = C^{-1}\gamma^{\overline{\text{MS}}}C - \mu \frac{dC^{-1}}{d\mu} C
\]
First moment

\[ C_{11} = 1 - (1.63903\alpha + 5.12484)\alpha \]
\[ - (3.8244\alpha^2 + 6.37866\alpha - 12.1458N_f + 106.359)\alpha^2 \]
\[ C_{22} = 1 \]

Second moment

\[ C_{11} = 1 - (2.18537\alpha + 8.24516)\alpha \]
\[ - (5.18357\alpha^2 + 2.38666\alpha - 19.8008N_f + 156.444)\alpha^2 \]
\[ C_{12} = (0.138749\alpha + 1.15755)\alpha \]
\[ + (0.419338\alpha^2 + 1.95065\alpha - 2.31945N_f + 20.0837)\alpha^2 \]
\[ C_{22} = 1 \]
\[ C_{33} = 1 - (1.63903\alpha + 5.12484)\alpha \]
\[ - (3.8244\alpha^2 + 6.37866\alpha - 12.1458N_f + 106.359)\alpha^2 \]
Aiming for 1st and 2nd moments of PDAs with fully nonperturbative renormalisation

Continuum calculations exist (Gracey) to allow the needed conversion from SMOM to MS once renormalistion conditions imposed

Will need care with renormalisation conditions for lattice Green functions when considering decomposition into hypercubic representations