A study of the anomalous magnetic moment of the muon computed from the Adler function

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1. The Adler function

2. Results for the Adler function

3. Combined fits to the Adler function

4. Alternative method to determine $\alpha_\mu$

5. Summary and Outlook
The Adler function is defined as

\[ \frac{D(q^2)}{q^2} = \frac{3\pi}{\alpha} \frac{d}{dq^2} \Delta \alpha_{\text{had}}(q^2), \]

and it can be measured in \( e^+e^- \rightarrow \text{hadrons} \). The Adler function is related to the vacuum polarization by

\[ D(q^2) = 12\pi^2 q^2 \frac{d\Pi(q^2)}{dq^2}, \]

and if we use \( D(q^2) \) to determine \( \hat{\Pi}(q^2) = \Pi(q^2) - \Pi(0) \) we avoid the extrapolation to \( \Pi(q^2 = 0) \).

Steps of the analysis:

1. determine \( \Pi(q^2) \) on the lattice
2. take the numerical derivative of \( \Pi(q^2) \rightarrow D(q^2) \)
3. in order to describe the \( q^2 \)-dependence apply a fit to \( D(q^2) \)
4. determine \( a_{\mu}^{HLO} \) from this fit

\[ ^1 \text{Adler, Phys. Rev. D 10, 3714, 1974} \]
In our study we use $O(a)$–improved Wilson-fermions with $N_f = 2$ with partially twisted boundary conditions. The strange and charm quarks are partially quenched.

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$^*$ [arXiv:1110.6365]
Step 1: Determination of $\Pi(q^2)$

The vacuum polarization tensor can be computed by

$$\Pi_{\mu\nu}(q^2) = \int d^4x e^{iqx} \langle J_\mu(x) J_\nu(0) \rangle.$$ 

From Euclidean invariance and current conservation one finds

$$\Pi_{\mu\nu}(q^2) = (g_{\mu\nu}q^2 - q_\mu q_\nu) \Pi(q^2).$$ 

![Diagram of vacuum polarization](image.png)

$\Pi(q^2)$ on F7, $m_\pi = 277\text{MeV}$, $a = 0.063\text{fm}$
Step 2: Methods to compute the Adler function

**Fit to \( \Pi(q^2) \)**

Fit an ansatz to \( \Pi(q^2) \), and compute the derivative of the fit function. We use the Padé ansatz

\[
\Pi_{fit}(q^2) = \Pi(0) - q^2 \left( \frac{a_1}{q^2 + b_1} + \frac{a_2}{q^2 + b_2} \right),
\]

\[
\frac{d}{dq^2} \Pi_{fit}(q^2) = - \frac{a_1 b_1}{(b_1 + q^2)^2} - \frac{a_2 b_2}{(b_2 + q^2)^2}.
\]


**Numerical derivative of \( \Pi(q^2) \)**

We use linear fits with varying ranges to approximate the derivative of \( \Pi(q^2) \).
Step 2: Procedures for the numerical derivative

**Procedure I**

- At each $q^2$ perform a linear fit

$$\Pi_{fit}^{[\ell]}(q^2) = a_\ell + b_\ell q^2,$$

- Repeat these fits for several fit ranges $\epsilon \in [0.1, 1.0]\text{GeV}^2$,

- Search for a region in $\epsilon$ where variations in $b_\ell$ are small.

**$\Pi(q^2)$ on F7, $m_\pi = 277\text{MeV}, a = 0.063\text{fm}$**
Step 2: Procedures for the numerical derivative

**Procedure II**

- at each \(q^2\) we fit the two functions

  \[
  \Pi^{[l]}_{fit}(q^2) = a_l + b_l \ln (q^2), \\
  \Pi^{[q]}_{fit}(q^2) = a_q + b_q \ln (q^2) + c_q (\ln (q^2))^2,
  \]

- repeat these fits for several fit ranges \(\epsilon \in [0.1, 1.0] \text{GeV}^2\),
- apply cuts to the fits, such as removing fits with a large curvature \(c_q\),
- from the fits that survive pick the result, where the coefficients \(b_l\) and \(b_q\) are similar.

\[\Pi(q^2)\] on F7, \(m_\pi = 277\text{MeV}, a = 0.063\text{fm}\]
1. The Adler function

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5. Summary and Outlook
Results for the Adler function on F7: Procedure I

\[ D(q^2) \text{ on F7, } m_\pi = 277\text{MeV, } a = 0.063\text{fm (ud)} \]
Results for the Adler function on F7: Procedure II

\[ D(q^2) \text{ on F7, } m_\pi = 277\text{MeV, } a = 0.063\text{fm (ud)} \]
Comparison of the different methods on F7

\[ D(q^2) \text{ on F7, } m_\pi = 277\text{MeV, } a = 0.063\text{fm (ud)} \]
Comparison of the different methods on F7

$D(q^2)$ on F7, $m_\pi = 277\text{MeV}$, $a = 0.063\text{fm (ud)}$

Deriv. of P[1,2]-fit to VPF-data

- proc. I
- proc. II

$q^2$ [GeV$^2$]
Comparison with the mixed representation method

For the mixed representation method see [arXiv:1306.2532], and the talk by Anthony Francis.
The Adler function

Results for the Adler function

Combined fits to the Adler function

Alternative method to determine $a_\mu$

Summary and Outlook
Step 3: Combined fits

To determine the Adler function at the physical point and in the continuum we apply combined fits of the type:

\[ D(q^2) = A(q^2)(1 + B(a, q) + C(m_\pi, q^2)) \]

- The momentum dependence is described by the derivative of Padé-approximants

\[
A_{[12]}(q^2) = q^2 \left( \frac{a_1 b_1}{(b_1 + q^2)^2} + \frac{a_2 b_2}{(b_2 + q^2)^2} \right), \\
A_{[22]}(q^2) = q^2 \left( \frac{a_1 b_1}{(b_1 + q^2)^2} + \frac{a_2 b_2}{(b_2 + q^2)^2} + a_0 \right).
\]

- The lattice spacing dependence is given by

\[
B_1(a, q) = c_1(aq) + c_2(4\pi f_K a), \quad B_2(a, q) = c_1 (aq)^2 + c_2 (4\pi f_K a)^2.
\]

- The light quark mass dependence is

\[
C(m_\pi, q^2) = d_1 \frac{m_\pi^2 - (m_\pi^{\text{exp}})^2}{d_2 + q^2}
\]
Results for $m_\pi \approx 270\text{MeV}$ for P[2,2], $O(a^2)$, $N_f = 2$

![Graph](image-url)
Results for $\beta = 5.30$, $a = 0.063\text{fm}$, $N_f = 2$

$D(q^2)$

$D(m_{\pi}^2)$

- E5, $m_{\pi} = 456\text{MeV}$
- F6, $m_{\pi} = 325\text{MeV}$
- F7, $m_{\pi} = 277\text{MeV}$
- G8, $m_{\pi} = 193\text{MeV}$

$q^2 = 1.0\text{GeV}^2$
$q^2 = 3.0\text{GeV}^2$
Results for $\beta = 5.30$, $a = 0.063$ fm for $P[2,2]$, $O(a^2)$, $N_f = 2$
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Step 4: Determination of $a_H^{HLO}$

We can use the results from the combined fits to determine $a_H^{HLO}$:

$$a_H^{HLO} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dq^2 f(q^2) 4\pi^2 \hat{\Pi}(q^2),$$

$$f(q^2) = \frac{m_\mu^2 q^2 Z^3 (1 - q^2 Z)}{1 + qm_\mu^2 q^2 Z^2},$$

$$Z = -\frac{q^2 - \sqrt{q^4 + 4m_\mu^2 q^2}}{2m_\mu^2 q^2}.$$

where we insert the coefficients of the combined chiral and continuum fits of the Adler function to determine the $q^2$-behaviour of $\hat{\Pi}(q^2)$:

$$\hat{\Pi}(q^2) \rightarrow \hat{\Pi}_{12}(q^2) = -q^2 \left(\frac{a_1}{b_1 + q^2} + \frac{a_2}{b_2 + q^2}\right) \text{ or}$$

$$\rightarrow \hat{\Pi}_{22}(q^2) = -q^2 \left(\frac{a_1}{b_1 + q^2} + \frac{a_2}{b_2 + q^2} + a_0\right).$$
Preliminary results for $a_\mu^{HLO}$ from combined fits, $N_f = 2$

$P[1,2]$

$P[2,2]$

$(m_\pi < 400\text{MeV})$

$P[1,2]$

$(m_\pi < 400\text{MeV})$

$O(a^2)$

$O(a)$
1. The Adler function

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5. Summary and Outlook
From the vacuum polarization we can obtain the Adler function with different methods, which agree within errors for a large range of $q^2$, and are similar to the mixed representation method, cf. talk by Anthony Francis on Monday (1D).

We presented a method to extract the Adler function in the continuum and at the physical point using a combined fit for the light quark sector.

From the extrapolated Adler function we can extract the hadronic contribution to the anomalous magnetic moment of the muon.

We plan to extend the analysis to the already available strange and charm data.

E. Shintani is currently investigating AMA for the vacuum polarization.

In the future we will also investigate methods which make use of the moments of $\Pi(q^2)$ to compute $\alpha^{HLO}_{\mu}$ [arXiv:1403.1778, arXiv:1406.4671].
Thank you for your attention.
Backup
Preliminary results for $a_{\mu}^{HLO}$ for $P[2,2]$, $N_f = 2$
Low $q^2$—region of $\Pi(q^2)$

$\Pi(q^2)$ on F7, $m_\pi = 277\text{MeV}$, $a = 0.063\text{fm}$