Isospin Effects by Mass Reweighting

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Mass Reweighting

Why?

- Tuning of quark masses,
  e.g., $m_s$ in a 2+1 simulation, isospin splitting, . . .
- Quark mass dependence
→ for small corrections: applicable and cheaper than new simulation

How?

- rewrite determinant: using pseudofermion integral
- using stochastic estimation

Observable:

$$\langle O \rangle_W = \langle OW \rangle/\langle W \rangle = \langle O\tilde{W} \rangle$$

with the corrections introduced by the mass reweighting factor of $n_f$-flavors
$$W = \prod_{i=1}^{n_f} [\det D(m_{new,i})/ \det D(m_{old,i})]$$

and normalized factor $\tilde{W} = W/\langle W \rangle$

One-flavor integral:

$$\frac{1}{\det M} = \int \mathcal{D}[\eta] \exp\{-\eta^\dagger M \eta\} \rightarrow \frac{1}{N_\eta} \sum_{i=1}^{N_\eta} e^{-\eta_i^\dagger (M-1) \eta_i}$$

holds for $\lambda(M + M^\dagger) > 0$  

[J.F., Knechtli, Leder (2013)]
Outline

Ensembles : CLS - $n_f = 2$ - Wilson $\mathcal{O}(a)$ improv. fermions with $m_{ud} = m_u = m_d$

<table>
<thead>
<tr>
<th>Name</th>
<th>$a$ [fm]</th>
<th>$m_\pi$ [MeV]</th>
<th>$N_{cnfg}$</th>
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<tr>
<td>A5</td>
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<td>330</td>
<td>202</td>
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<tr>
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<td>580</td>
<td>100</td>
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<tr>
<td>D5</td>
<td>“</td>
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<td>99</td>
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<td>“</td>
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<td>“</td>
<td>190</td>
<td>37</td>
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<tr>
<td>O7</td>
<td>0.049</td>
<td>270</td>
<td>49</td>
</tr>
</tbody>
</table>

[Fritzsch et al. (2012)]

https://twiki.cern.ch/twiki/bin/view/CLS/WebHome

- still improving the statistics
→ RESULTS are Preliminary

Goals:
- scaling of the fluctuations
- extract up- and down-quark mass
Isospin Reweighting

**In general:** lattice simulations are done in the Isospin symmetric limit in the light quark sector

\[ m_u = m_{ud} = m_d \]

**Idea:** using mass reweighting to introduce Isospin breaking effects with

\[ m_u + m_d = \text{const} \]

\[ m_u = m_{ud} - 0.5 \cdot \Delta m_{ud} \quad \leftrightarrow \quad m_{ud} \quad \rightarrow \quad m_{ud} + 0.5 \cdot \Delta m_{ud} = m_d \]

**Isospin reweighting factor:**

\[ W_{iso} = \frac{\det D(m_u) \det D(m_d)}{\det D(m_{ud})^2} \]

**fluctuations:**

stochastic fluctuations (expanding in \( \Delta m_{ud} \)):

\[ \sigma_{st}^2(N_{inv}) \sim \frac{\Delta m_{ud}^4}{N_{inv}} \text{Tr} \left( \frac{1}{(DD^\dagger)^2} \right) + O(\Delta m_{ud}^6) \]

ensemble fluctuations (expanding in \( \Delta m_{ud} \)) :

\[ \sigma_{ens}^2 = \Delta m_{ud}^4 \text{var} \left( \text{Tr} \left[ \frac{1}{D^2} \right] \right) + O(\Delta m_{ud}^6) \]

Cost given by :

\[ \frac{\sigma_{st}^2(N_{inv})}{\sigma_{ens}^2} \sim 10\% \]
Stochastic fluctuations

Stochastic fluctuations (expanding in $\Delta m_{ud}$):

$$\sigma_{st}^2(N_{inv}) \sim \frac{\Delta m_{ud}^4}{N_{inv}} \text{Tr} \frac{1}{(DD^\dagger)^2} + \mathcal{O}(\Delta m_{ud}^6)$$

$$\Rightarrow \text{chiral perturbation theory: } \text{Tr} \frac{1}{(DD^\dagger)^2} \propto \frac{\Sigma V}{m_R^3}$$

$$\sigma_{st}^2 \approx k_{st} \frac{\Delta m_{ud}^4 V}{N_{inv} m_R^3} \frac{1}{r_0^3}$$

with $k_{st} = 2.9(0.5) \times 10^{-11}$ and $q = -2.71(6)$
Ensemble fluctuations (expanding in $\Delta m_{ud}$):

$$\sigma_{\text{ens}}^2 = \Delta m_{ud}^4 \text{ var} \left( \text{Tr} \left[ \frac{1}{D^2} \right] \right) + \mathcal{O}(\Delta m_{ud}^6)$$

$$\Rightarrow \text{behavior of } \text{var} \left( \text{Tr} \left[ \frac{1}{D^2} \right] \right) \text{ unknown}$$

numerical observe: $\rightarrow$ weak volume dependence $V^q$ with $q < 1$

$\Rightarrow$ a good fit: $q = 0.25$

$$\sigma_{\text{ens}}^2 \approx k_{\text{ens}} \frac{\Delta m_R^4 \sqrt[4]{V}}{m_R^4} \frac{1}{r_0}$$

$y = k_{\text{ens}} \ast (m_R a)^q$

with $k_{\text{ens}} = 31(39) \ast 1e^{-5}$

and $q = -3.96(14)$
Scaling of Fluctuations

Stochastic fluctuations

\[ \sigma_{st}^2 \approx k_{st} \frac{\Delta m_R^4 V}{N_{inv} m_R^3} \frac{1}{r_0^3} \]

Cost given by:

\[ \sigma_{st}^2(N_{inv})/\sigma_{ens}^2 \sim \frac{k'_{st}}{k'_{ens}} \frac{(LMPS)^2 L}{N_{inv} \cdot r_0^2} \]

with \( \frac{k'_{st}}{k'_{ens}} = 1e - 3 \)

10% @G8 \( (m_\pi = 190 \text{ MeV}, \ a = 0.066 \text{ fm}) \) : \( N_{inv} \approx 200 \)

Ensemble fluctuations

\[ \sigma_{ens}^2 \approx k_{ens} \frac{\Delta m_R^4 \sqrt[4]{V}}{m_R^4} \frac{1}{r_0} \]
Fixing the bare mass parameter

Using physical ratios built from meson masses and decay constants to fix the bare mass parameters $\kappa_s, \kappa_d$ and $\kappa_u$:

\[ R_1 = \frac{m_{K^0}^2 + m_{K^\pm}^2}{(f_{K^0} + f_{K^\pm})^2}, \quad R_2 = \frac{m_{K^0}^2 - m_{K^\pm}^2}{m_{K^0}^2 + m_{K^\pm}^2} \]

and

\[ R_3 = \frac{m_{\pi^\pm}^2}{(f_{K^0} + f_{K^\pm})^2} \]

Strategie:
Use $R_1$ and $R_2$ to fix $\kappa_s$ and $(\Delta m_{ud})_{bare}$ and extrapolate in $R_3$ towards physical light quark masses.
Quark masses

By fixing $R_1$ and $R_2$ and using the $m_{PCAC}$ mass:

\[ m_{ud}(R_3) \approx a_1 R_3 \quad \text{and} \quad \Delta m_{ud}(R_3) \approx b_0 + b_1 R_3 \]

with $R_3 \propto m_{ud}$ in $\chi$pt at LO

result:
\[ \Rightarrow \] relative high precision with low statistics

future:
\[ \Rightarrow \] increasing statistics and including other ensembles

with $m_{ud} = 3.21(14)$ MeV and $\Delta m_{ud} = 2.48(10)$ MeV
(at finite lattice $a = 0.066$ [fm])
Conclusion

Results:

- costs: increases with $L$ and is around 200 inversions for G8
- fixing condition suitable to extract light quark masses
- relative good results with small statistics

Prospects:

- improving statistics
- including QED-effects
Covariance of $\tilde{W}$ with $f_{PP}$

$b(t)$ measured on F7:

\[ b(t) = \text{cov}(C(t), \tilde{W})/\langle C(t) \rangle \]

with $\langle C(t) \rangle = f_{PP}(t)$