π-π Scattering with $N_f = 2 + 1 + 1$ Twisted Mass Fermions

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Motivation

- Scattering lengths are fundamental quantities in QCD and ingredients for EFTs and interesting for nuclear physics.

- Most particles in the hadron spectrum are resonances described by their mass and decay width.

- Some states, e.g., the roper resonance, are not even qualitatively described by naive quark model.

- Need non-perturbative method from first principles \( \rightarrow \) lattice QCD.
Scattering at low Energies

- Some interesting scattering channels:
  - $\pi-\pi$ scattering for $I = 0, 1, 2 \rightarrow$ e.g. the $\rho$ meson
  - $K-\pi$ scattering for $I = 1/2, 3/2 \rightarrow$ e.g. the $K^*(892)$ and $\kappa$ meson
  - $D$-meson scattering $\rightarrow X, Y, Z$ states (talk by Liuming Liu on Friday)

- At low energies details of potentials are not important for scattering

- In the partial wave expansion of the scattering process, only the lowest partial waves contribute, here only $s$-wave
The easiest possible scattering to calculate is $\pi-\pi$ scattering with $I = 2$ and pions at rest $\rightarrow$ no disconnected contributions

The scattering phase-shift $\delta_s$ can be related to the scattering length $a_s$

$$\lim_{k \to 0} k \cot(\delta_s(k)) = -\frac{1}{a_s}$$

Lüscher\(^1\): two particles in a box cause energy shift due to interaction

Energy shift $\delta E$ is related to the scattering length of the particles

$$\delta E^{I=2}_{\pi\pi} = -\frac{4\pi a^{I=2}_{\pi\pi}}{m_{\pi} L^3} \left\{ 1 + c_1 \frac{a^{I=2}_{\pi\pi}}{L} + c_2 \left(\frac{a^{I=2}_{\pi\pi}}{L^2}\right)^2 \right\} + O(L^{-6})$$

Laplacian Heaviside Smearing

- Fermion smearing: \( \tilde{\psi}(n) = S(n, m)\psi(m) \) with \( S = \Theta(\sigma^2_s + \Delta) \)

- Heaviside function: \( \Theta(x) \)

- Laplace operator: \( \Delta \)

- Cutoff for spectrum of \( \Delta \): \( \sigma^2_s \)

- Decomposition into eigenvalues \( \Lambda_{\Delta} = \text{diag}(\lambda_1, \ldots, \lambda_{\Delta}) \):

\[
\Delta = V_{\Delta}^\dagger \Lambda_{\Delta} V_{\Delta} \quad \rightarrow \quad S = V_{\Delta}^\dagger \Theta(\sigma^2_s + \Lambda_{\Delta}) V_{\Delta} = V_{s}^\dagger V_{s}
\]

- \( V_s \) contains \( N_v \) lowest eigenvectors which are used as sources

- Inversions are stored in perambulator: \( V_s^\dagger \Omega^{-1} V_s \)

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Stochastic LapH$^3$

- Introduce $N_R$ random vectors, $\rho$, in $T$, $D$ and $V_s$
  
  $E(\rho) = 0$ and $E(\rho \rho^\dagger) = 1$

- $\rho$ must be different for each quark line to avoid bias

- Dilution of random vectors, $P^{(b)}\rho$, to zero many off-diagonal elements
  
  $P^{(b)}$ dilution matrix, $N_D$ number of dilution vectors

- Statistical errors of correlation functions
  
  - Random vectors $\propto \frac{1}{\sqrt{N_R}}$
  
  - Dilution vectors $\propto \frac{1}{N_D}$

  $\Rightarrow$ Find balance between $N_R$ and $N_D$ for best signal in dependence of number of inversions, $N_I$

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Simulation Details

- We want all-to-all propagators for everything because:
  - Fierz rearrangement
  - Twisted mass: \( D_u^{-1} = \left[ \gamma_5 D_d^{-1} \gamma_5 \right]^\dagger \)
  - Same perambulators for connected and disconnected diagrams
  - Temporal extent not too large: \( T = 48, 64, 96 \)
    - block dilution in time with size 2 or 3
  \( \Rightarrow \) - interlace dilution in LapH-space with size 4 or 6
    - full dilution in Dirac space

- Number of random vectors: 5
- Number of eigenvectors: \( L = 24 : 120; \ L = 32 : 220; \ L = 48 : 660 \)
Software Details

- **Petsc** and **Slepc** for eigenvector computation
  - Lanczos with thick restart
  - Chebyshev acceleration
  - 3 steps of 3-dim HYP smearing in Laplace operator
- **Eigen** for all matrix related computations
- tmLQCD library with CG and EigCG for inversions
  - CG on GPUs with MPI and OpenMP for $L = 24/32$
  - EigCG on Juqueen with MPI and OpenMP for $L = 32/48$
Overview over Ensembles

Ensembles are generated by the European Twisted Mass Collaboration\textsuperscript{4}

<table>
<thead>
<tr>
<th>name</th>
<th>$L_s$</th>
<th>$L_t$</th>
<th>$a m_\pi$</th>
<th>$a f_\pi$</th>
<th># conf</th>
<th>$m_\pi$ [MeV]</th>
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<td>0.06451</td>
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<td>0.06791</td>
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<tr>
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<td>0.07926</td>
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<td>510</td>
</tr>
</tbody>
</table>

Lattice spacings: $A = 0.086$ fm, $B = 0.082$ fm, $D = 0.062$ fm

\textsuperscript{4}R. Baron et al., PoS LATTICE 2010, 123 (2010) and R. Baron et al., JHEP 1006, 111 (2010)
Thermal States

\[ am_{\text{eff}}[C_{\pi}] \]

\[ am_{\text{eff}}[C_{\pi}^2] \]

\[ am_{\text{eff}}[C_{\pi\pi}] \]

\[ 6 \ 8 \ 10 \ 12 \ 14 \ 16 \ 18 \ 20 \]

\[ x_0/a \]
Removal of Thermal states\textsuperscript{5}

Taking the ratio: \[
\frac{C_{\pi\pi}(t)}{C_{\pi}^2(t)} \propto \exp(-\delta E_{\pi\pi}^I t)
\]

→ Thermal states do not cancel in the ratio

Use derivative method

\[
R(t + \frac{1}{2}) = \frac{C_{\pi\pi}(t) - C_{\pi\pi}(t + 1)}{C_{\pi}^2(t) - C_{\pi}^2(t + 1)}
\]

\[
= A \left( \cosh(\delta E_{\pi\pi}^I t') + \sinh(\delta E_{\pi\pi}^I t') \coth(2m_\pi t') \right)
\]

with \( t' = t + \frac{1}{2} - \frac{T}{2} \)

Extract \( \delta E_{\pi\pi}^I = 2 \) by fitting \( R(t + \frac{1}{2}) \)

The ratio $R(t + 1/2)$
Overall data

The diagram shows a scatter plot of $m_\pi a_{\pi\pi}$ vs. $m_\pi/f_\pi$. The data points are labeled with different symbols and colors corresponding to various datasets:

- **NA48/2 (2010)**
- **A40, L=20**
- **A-ens., L=24**
- **A-ens., L=32**
- **D45, L=32**
- **B55, L=32**

The y-axis represents $m_\pi a_{\pi\pi}$, while the x-axis represents $m_\pi/f_\pi$. The data points are distributed across the graph, indicating the range of values for each dataset.
Overall data - comparison to $N_f = 2$
Dependence on fit range

- Fitrange: \( t_{\text{start}} - 22.5 \)

A60, 100 measurements

\[
\begin{array}{c}
\begin{array}{ccc}
15 & 14 & 13 \\
12 & 11 & 10 \\
-0.23 & -0.22 & -0.21 \\
\end{array}
\end{array}
\]

A100, 300 measurements

\[
\begin{array}{c}
\begin{array}{ccc}
15 & 14 & 13 \\
12 & 11 & 10 \\
-0.31 & -0.30 & -0.29 \\
\end{array}
\end{array}
\]
Volume Effects on A40

A40, L=20, 24, 32
Luescher formular A40.32

$\delta E_{\pi\pi}^{I=2}$ vs $1/L$

B. Knippschild (HISKP)
Conclusions and outlook

First attempt to extract scattering parameters with $N_f = 2 + 1 + 1$ twisted mass fermions

Test of viability of stochastic LapH method on large lattices

Go on to larger lattices: $L = 48$ and smaller pion masses $\to$ approaching the physical point

Include momenta and displacements

Closer investigation of systematic effects - might become quite demanding

Investigation of other scattering processes ...
Thank you!