Lattice Formulations of Supersymmetric Gauge Theories with Matter Fields

Anosh Joseph

John von Neumann Institute for Computing (NIC), DESY

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Supersymmetric Yang-Mills Theories

Supersymmetric Yang-Mills (SYM) theories exhibit many interesting properties

- Confinement
- Spontaneous chiral symmetry breaking
- Strong coupling - weak coupling duality
- Electric-magnetic duality
- ...

Interesting relations between 16 supercharge YM theory and string theory

AdS/CFT (gauge/gravity) correspondence

Many interesting features are at strong coupling

Need non-perturbative definition of SYM: lattice construction

SUSY on the Lattice: Exact Lattice SUSY

Preserve a subset of SUSY algebra exactly on the lattice.

Recent reviews: J. Giedt, [Int. J. Mod. Phys. A **21**, 3039 (2006)],
S. Catterall, D. B. Kaplan, M. Ünsal, [Phys. Rept. 484, 71 (2009)],
A.J., [Int. J. Mod. Phys. A 26, 5057 (2011)]

Two approaches. **(i)** Topological Twisting. [Catterall, Sugino, Kawamoto, Matsuura, Giedt, ...]

Inspired by techniques in topological field theory.

E. Witten [Commun. Math. Phys. 117 (1988) 353]

(ii) Orbifolding/deconstruction.

[Kaplan, Ünsal, Cohen, Damgaard, Matsuura, Giedt, ...]

Inspired by the method of "Deconstruction" by Arkani-Hamed, Cohen, Georgi (AHCG).

N. Arkani-Hamed, A. G. Cohen, H. Georgi [Phys. Rev. Lett. 86 (2001) 4757]

Topologically Twisted $\mathcal{N} = (2, 2)$ SYM

Dimensional reduction of $4d$ $\mathcal{N} = 1$ SYM theory.

Symmetry group: $\mathrm{SO}(4)_{\mathrm{E}} \times \mathrm{U}(1)$ in $4d$

Symmetry group in $2d$:

$$\mathrm{G} = \mathrm{SO}(2)_{\mathrm{E}} \times \mathrm{SO}(2)_{\mathrm{R}_1} \times \mathrm{U}(1)_{\mathrm{R}_2}.$$

$\mathrm{SO}(2)_{\mathrm{E}}$: Euclidean Lorentz symmetry,
$\mathrm{SO}(2)_{\mathrm{R}_1}$: rotational symmetry along reduced dimensions,
$\mathrm{U}(1)_{\mathrm{R}_2}$: chiral $\mathrm{U}(1)$ symmetry of the original theory.

Twist gives:

$$\mathrm{SO}(2)' = \mathrm{Diag.\ Subgroup}\ \Big[\mathrm{SO}(2)_{\mathrm{E}} \times \mathrm{SO}(2)_{\mathrm{R}_1}\Big].$$

B-model twist (or self-dual twist)

Topologically Twisted $\mathcal{N} = (2,2)$ SYM [contd.]

Fermions: Twisted fermions (p-forms) η, ψ_μ, $\chi_{\mu\nu}$.

$$\Psi_{\alpha i} = \Big[\eta I + \psi_\mu \gamma_\mu + \tfrac{1}{2}\chi_{[12]}(\gamma_1\gamma_2 - \gamma_2\gamma_1)\Big]_{\alpha i}$$

$\alpha(= 1, 2)$: Lorentz spinor index, $i(= 1, 2)$: flavor index.

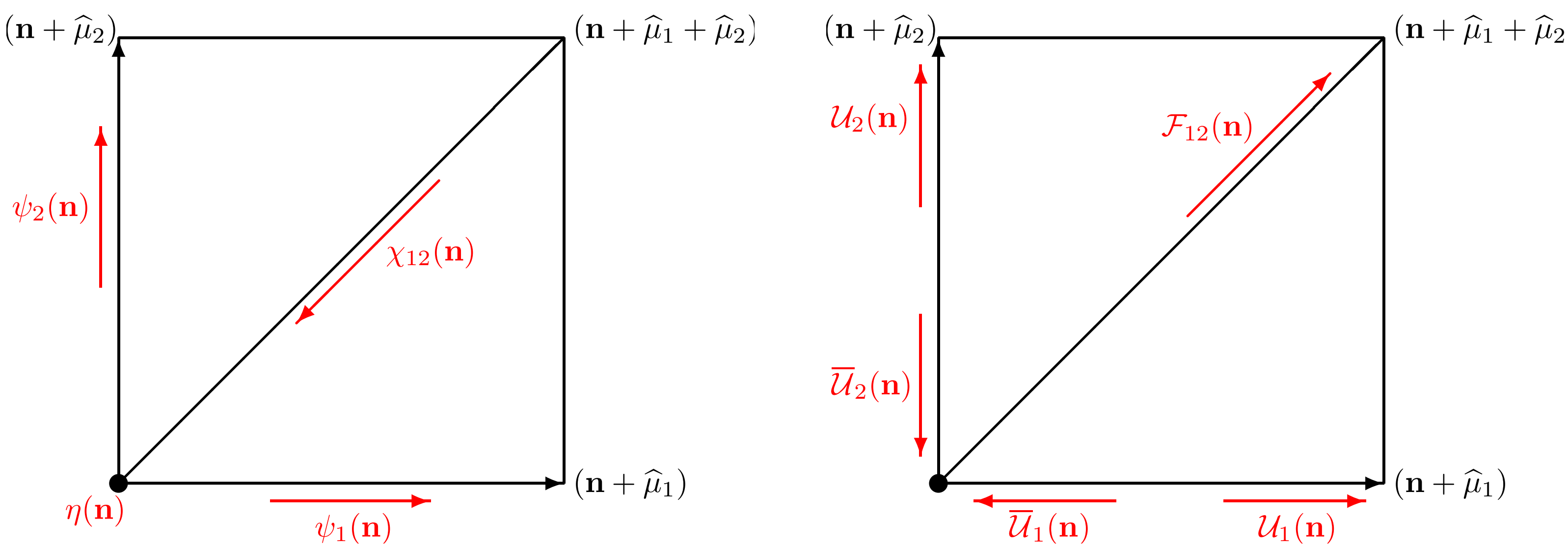
Bosons: Under $\mathrm{SO}(2)'$, gauge field and scalars transform as vectors.

$$A_\mu = (A_1, A_2), \qquad B_\mu = (s_1, s_2)$$

Can combine them to form complexified gauge fields

$$\mathcal{A}_\mu = A_\mu + iB_\mu \text{ and } \overline{\mathcal{A}}_\mu = A_\mu - iB_\mu.$$

Topologically Twisted $\mathcal{N} = (2,2)$ SYM [contd.]



The unit cell of $\mathcal{N} = (2,2)$ lattice SYM

Topologically Twisted $\mathcal{N} = (2,2)$ SYM [contd.]

Supercharges: Also transform as p-forms

$$\mathcal{Q}_{\alpha i} = \Big[\mathcal{Q} I + \mathcal{Q}_\mu \gamma_\mu + \frac{1}{2}\mathcal{Q}_{[12]}(\gamma_1\gamma_2 - \gamma_2\gamma_1)\Big]_{\alpha i}$$

Twisted $\mathcal{N} = (2,2)$ SUSY algebra:

$$\{\mathcal{Q}, \mathcal{Q}\} = 0, \quad \{\mathcal{Q}_\mu, \mathcal{Q}_\nu\} = 0, \quad \{\tilde{\mathcal{Q}}, \tilde{\mathcal{Q}}\} = 0, \quad \{\mathcal{Q}, \tilde{\mathcal{Q}}\} = 0,$$

$$\{\mathcal{Q}, \mathcal{Q}_\mu\} = P_\mu, \quad \{\tilde{\mathcal{Q}}, \mathcal{Q}_\mu\} = \epsilon_{\mu\nu} P_\nu$$

$$\tilde{\mathcal{Q}} \equiv \epsilon_{\mu\nu}\mathcal{Q}_{\mu\nu}$$

Topologically Twisted $\mathcal{N} = (2, 2)$ SYM [contd.]

Scalar supersymmetry $\mathcal{Q}$ is nilpotent (similar to BRST charge):

$$\mathcal{Q}^2 = 0$$

Does not generate any translations on the lattice.

$\Longrightarrow$ Can have lattice realization of a subalgebra of the twisted SUSY algebra

$$\mathcal{Q}\mathcal{A}_\mu = \psi_\mu, \qquad \mathcal{Q}\psi_\mu = 0,$$
$$\mathcal{Q}\overline{\mathcal{A}}_\mu = 0, \qquad \mathcal{Q}\chi_{\mu\nu} = -[\overline{\mathcal{D}}_\mu, \overline{\mathcal{D}}_\nu],$$
$$\mathcal{Q}\eta = d, \qquad \mathcal{Q}d = 0.$$

We have $\mathcal{D}_\mu = \partial_\mu + \mathcal{A}_\mu$, $\overline{\mathcal{D}}_\mu = \partial_\mu + \overline{\mathcal{A}}_\mu$. d: auxiliary field.

Topologically Twisted $\mathcal{N} = (2,2)$ SYM [contd.]

Action has $\mathcal{Q}$-exact form:

$$S = \mathcal{Q}\ \tfrac{1}{g^2} \int \mathrm{Tr}\ \left(\chi_{\mu\nu}\mathcal{F}_{\mu\nu} + \eta[\overline{\mathcal{D}}_\mu, \mathcal{D}_\mu] - \tfrac{1}{2}\eta d\right)$$

After $\mathcal{Q}$-variation:

$$S = \frac{1}{g^2} \int \mathrm{Tr}\ \left(-\overline{\mathcal{F}}_{\mu\nu}\mathcal{F}_{\mu\nu} + \frac{1}{2}[\overline{\mathcal{D}}_\mu, \mathcal{D}_\mu]^2 - \chi_{\mu\nu}\mathcal{D}_{[\mu}\psi_{\nu]} - \eta\overline{\mathcal{D}}_\mu\psi_\mu\right)$$

It is $\mathcal{Q}$-supersymmetric: $\mathcal{Q}S = 0$.

$2d\ \mathcal{N} = (2,2)$ **SYM on the Lattice**

Discretization on a hypercube. Fields live on on sites, links and plaquettes.

Bosons $\to$ complexified Wilson links: $\mathcal{A}_\mu \to \mathcal{U}_\mu$

$$\mathcal{D}^{(+)}_\mu f_\nu(\mathbf{n}) = \mathcal{U}_\mu(\mathbf{n}) f_\nu(\mathbf{n} + \widehat{\boldsymbol{\mu}}_\mu) - f_\nu(\mathbf{n})\mathcal{U}_\mu(\mathbf{n} + \widehat{\boldsymbol{\mu}}_\nu),$$

$$\mathcal{D}^{(-)}_\mu f_\mu(\mathbf{n}) = \mathcal{U}_\mu(\mathbf{n}) f_\mu(\mathbf{n}) - f_\mu(\mathbf{n} - \widehat{\boldsymbol{\mu}}_\mu)\mathcal{U}_\mu(\mathbf{n} - \widehat{\boldsymbol{\mu}}_\mu).$$

P. H. Damgaard and S. Matsuura [Phys. Lett. B **661**, 52 (2008)]

Use of forward and backward difference operators

$\Longrightarrow$ Solutions of lattice theory map one-to-one with that of continuum theory.

Fermion doubling problems evaded.

T. Banks, Y. Dothan, D. Horn [Phys. Lett. B117, 413 (1982)]

$2d\ \mathcal{N} = (2,2)$ SYM on the Lattice [contd.]

$\mathcal{Q}$ transformations on the lattice:

$$\mathcal{Q}\mathcal{U}_\mu(\mathbf{n}) = \psi_\mu(\mathbf{n}) \qquad \mathcal{Q}\psi_\mu(\mathbf{n}) = 0$$

$$\mathcal{Q}\overline{\mathcal{U}}_\mu(\mathbf{n}) = 0 \qquad \mathcal{Q}\chi_{\mu\nu}(\mathbf{n}) = -\overline{\mathcal{D}}_\mu^{(+)}\overline{\mathcal{U}}_\nu(\mathbf{n})$$

$$\mathcal{Q}\eta(\mathbf{n}) = d(\mathbf{n}) \qquad \mathcal{Q}d(\mathbf{n}) = 0$$

Only $\mathcal{Q}$ is unbroken by discretization.

Bosons and fermions are interchanged at the same place on the lattice.

$$S = \frac{1}{g_{\mathrm{LAT}}^2}\sum_{\mathbf{n}} \mathrm{Tr}\,\Big(\mathcal{D}_\mu^{(+)}\mathcal{U}_\nu(\mathbf{n})\Big)^\dagger\Big(\mathcal{D}_\mu^{(+)}\mathcal{U}_\nu(\mathbf{n})\Big) + \frac{1}{2}\Big(\mathcal{D}_\mu^{\dagger(-)}\mathcal{U}_\mu(\mathbf{n})\Big)^2$$
$$- \chi_{\mu\nu}(\mathbf{n})\mathcal{D}_{[\mu}^{(+)}\psi_{\nu]}(\mathbf{n}) - \eta(\mathbf{n})\mathcal{D}_\mu^{\dagger(-)}\psi_\mu(\mathbf{n}).$$

Including Matter: Fundamental/Higher Reps

Motivations

- SQCD on the lattice (Pure SYM + fundamental matter)
- Exploring technicolor models w/ 2-index matter
- Such constructions appear in string theory: orbifolding/orientifolding

  SU(N) $\rightarrow$ SO(2N) or Sp(2N)

- Corrigan-Ramond limit:

  QCD(AS) $N_c = 3$: 2-index = (anti-)fundamental

Including Matter: Fundamental/Higher Reps [contd.]

- Action of the theory including matter

$$S = S_{\rm SYM} + S_{\rm matter} + S_{\rm potential}$$

- Lattice construction of $S_{\rm SYM}$ - same as discussed before.
- How to construct terms with $S_{\rm matter}$ in a lattice compatible way?
- Clever trick by Matsuura

  - To formulate $\mathcal{N} = (2, 2)$ lattice theories with fundamental matter.

  S. Matsuura [JHEP **0807**, 127 (2008)]

- Can be extended to $3d$ and other matter representations.

Including Matter: Fundamental/Higher Reps [contd.]

- How to construct them?
- Start from a theory with more susy and in one higher dimension

  Examples: $\mathcal{N} = 4$ SYMs in $3d$ or $4d$ ($\mathcal{Q} = 8$ or 16).

- Twist the theory $\rightarrow$ Blau-Thompson or Marcus twist.

  M. Blau and G. Thompson [Nucl. Phys. B **492**, 545 (1997)]

  N. Marcus [Nucl. Phys. B **452**, 331 (1995)]

- Dimensionally reduce to a theory with adjoint matter

  E.g.: for 8 supercharge theory $3d \rightarrow 2d$:

  $\mathcal{N} = (2, 2)$ SYM: $(\mathcal{A}_a, \overline{\mathcal{A}}_a, \eta, \psi_a, \chi_{ab})$,

  Adjoint Matter: $(\phi, \overline{\phi}, \overline{\eta}, \overline{\psi}_a, \overline{\chi}_{ab})$.

Including Matter: Fundamental/Higher Reps [contd.]

- Elevate the theory to a $2d$ quiver $\mathcal{N} = (2,2)$ gauge theory.

  Gauge group $U(N_1) \times U(N_2)$.

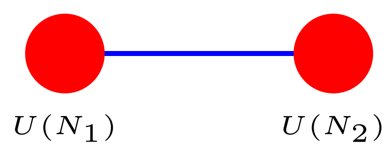
  One can take $N_1 =$ color, $N_2 =$ flavor.

- Change the representation of matter to bi-fundamental.

  Not in conflict with SUSY.

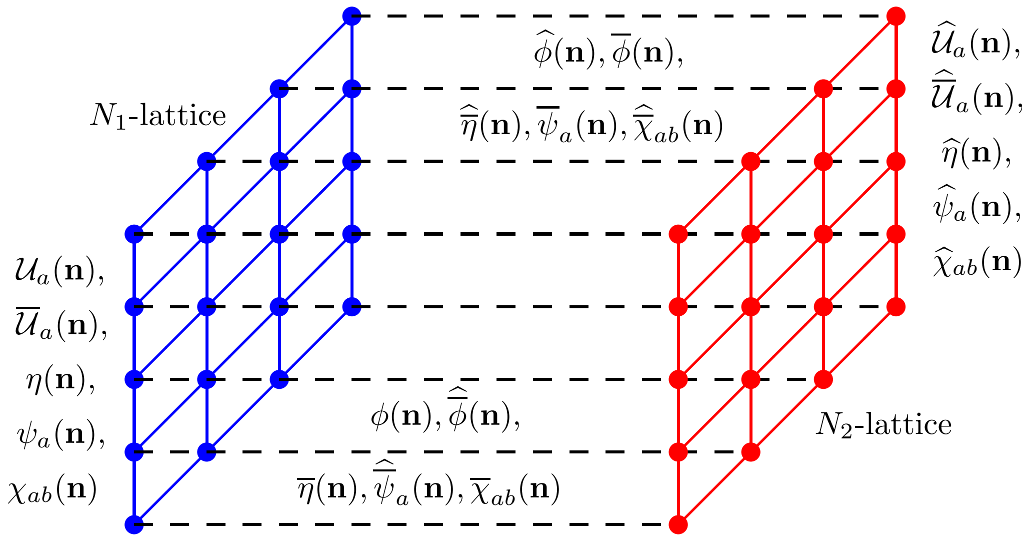
  Action: $S = S^{\rm SYM}_{(\mathbf{adj},\mathbf{1})} + S^{\rm SYM}_{(\mathbf{1},\mathbf{adj})} + S^{\rm matter}_{(\Box,\overline{\Box})} + S^{\rm matter}_{(\overline{\Box},\Box)}$

  Quiver diagram:



Including Matter: Fundamental/Higher Reps [contd.]

Two lattice spacetimes. Connected by bi-fundamental matter fields



$\mathcal{N} = (2,2)$ $U(N_1) \times U(N_2)$ quiver lattice gauge theory with bi-fundamental matter.

Including Matter: Fundamental/Higher Reps [contd.]

- Truncate one side of the quiver lattice gauge theory.

  Freeze $U(N_2)$ theory.

- Make fields decorated with hats non-dynamical by hand.

  Un-gauging one node of the quiver gives $U(N_2)$ flavor symmetry.

- Resultant theory: A $2d$ $\mathcal{N} = (2, 2)$ lattice gauge theory

  with fundamental matter and N_2 flavor symmetry.

Lattice construction of $3d$ $\mathcal{N} = 4$ theory with fundamental matter also possible

A. J. [JHEP **09**, 046 (2013)]

Can be extended to theories with matter in higher (2-index) reps:
$\Psi^{(ij)}$, $\Psi^{[ij]}$ etc., with SU, SO, Sp gauge groups.

A. J. [arXiv:1403.4390, To appear in JHEP]

Conclusions

- SUSY lattices possible for certain classes of SYM theories

  $\mathcal{Q} = 4, 8, 16$ in $d \leq 4$.

  More on $\mathcal{Q} = 16$, $d = 4$ SYM:
  See talks by Catterall, Giedt and Schaich.

- Lattice constructions are:

  local, gauge-invariant, doubler-free and exact-supersymmetric.

- Non-perturbative explorations, for SYM theories with matter:

  fundamental, 2-index,...

- Room for new nonperturbative explorations.