A HYBRID STRATEGY FOR COMPUTING $a_\mu^{LO,HVP}$ ON THE LATTICE

Report on M. Golterman, KM, S. Peris: PRD88 (2013) 114508 [GMP13] and arXiv:1405.2389 [GMP14]

+ ongoing work with Jamie Hudspith, Randy Lewis, Antonin Portelli, ...

Lattice 2014, Columbia, June 23, 2014

OUTLINE

- *Low-Q^2 systematics problem for typical fits to lattice polarization function data*

- *Reliable forms for low-Q^2 fitting and a hybrid strategy for $a_\mu^{LO,HVP}$*

- *Exploratory implementation on existing RBC/UKQCD ensembles with low-Q^2 contributions via, e.g., the HPQCD time-moment approach*

GENERALITIES

- $\hat{\Pi}(Q^2) \equiv \Pi(Q^2) - \Pi(0)$, the subtracted EM current polarization function

- $a_\mu^{LO,HVP}$ from Euclidean Q^2 integral representation

$$
\begin{aligned}
a_\mu^{\rm LO,HVP} &= -4\alpha^2 \int_0^\infty dQ^2\, f(Q^2)\, \hat{\Pi}(Q^2) \\
f(Q^2) &= m_\mu^2 Q^2 Z^3(Q^2) \frac{1 - Q^2 Z(Q^2)}{1 + m_\mu^2 Q^2 Z^2(Q^2)} \\
Z(Q^2) &= \left(\sqrt{(Q^2)^2 + 4m_\mu^2 Q^2} - Q^2\right) / (2m_\mu^2 Q^2)
\end{aligned}
$$

- NOTE: Peaked at **much** lower Q^2 than analogous dispersive representation as weighted integral over bare $e^+e^- \to hadrons$ cross-sections

- The once-subtracted dispersion relation for $\Pi(Q^2)$

$$\hat{\Pi}(Q^2) \qquad = -Q^2 \int_{4m_\pi^2}^{\infty} ds \, \frac{\rho(s)}{s(s+Q^2)} \, ,$$

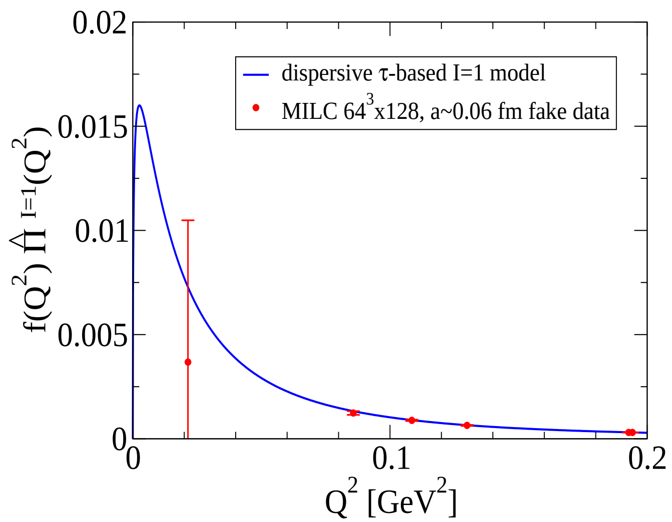
  - ▷ $\rho(s) \geq 0 \;\Rightarrow$ all derivatives monotonic in Q^2; smaller curvature for $\hat{\Pi}$ at large Q^2 than small Q^2

  - ▷ $\Rightarrow$ potential systematic bias at low-Q^2 when fitting lattice data with many low-error large-Q^2 points and only a few large-error low-Q^2 points

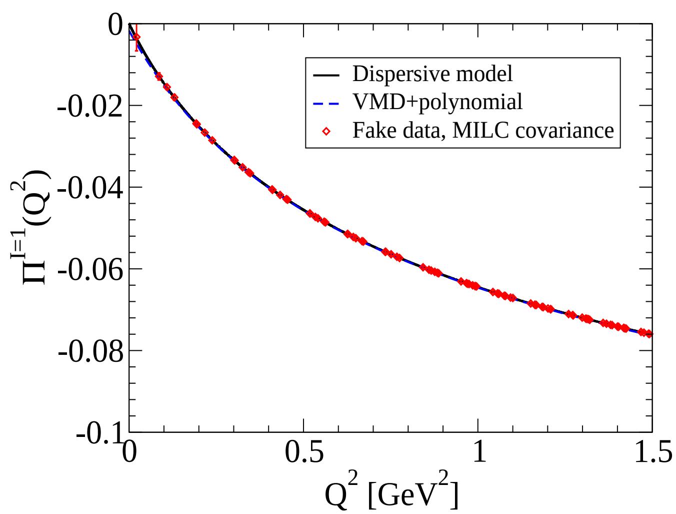
  - ▷ High precision hadronic τ decay data for $\rho^{I=1}(s)$ allows construction of extremely physical "dispersive model" for $\hat{\Pi}^{I=1}(Q^2)$, quantitative examination of systematics for any given fit strategy [GMP13]

More on the low-Q^2 systematic problem

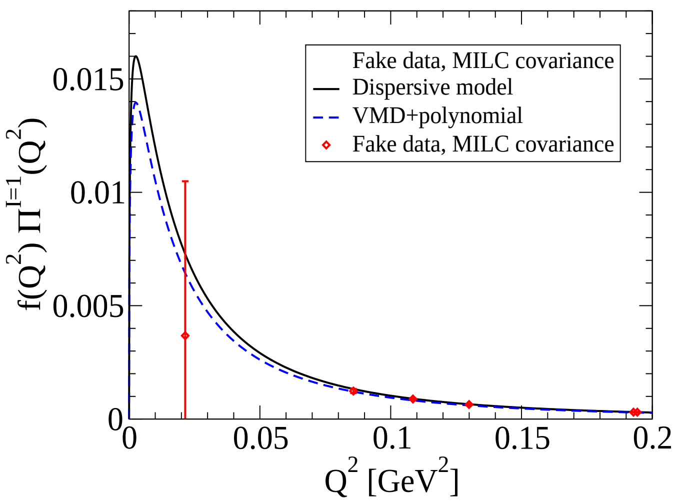
- Integrand peaked at *very* low Q^2 ($\sim m_\mu^2/4$ for I=1)
- Sparse low-Q^2 coverage, especially with PBCs



- Is the in-principle problem a real one?
    - $*$ E.g., good quality VMD+polynomial fit to fake data (underlying dispersive model + MILC covariance), fit interval $0 < Q^2 < 1\ GeV^2$



$*$ The resulting integrand in the low-Q^2 region, c.f. the underlying dispersive model version



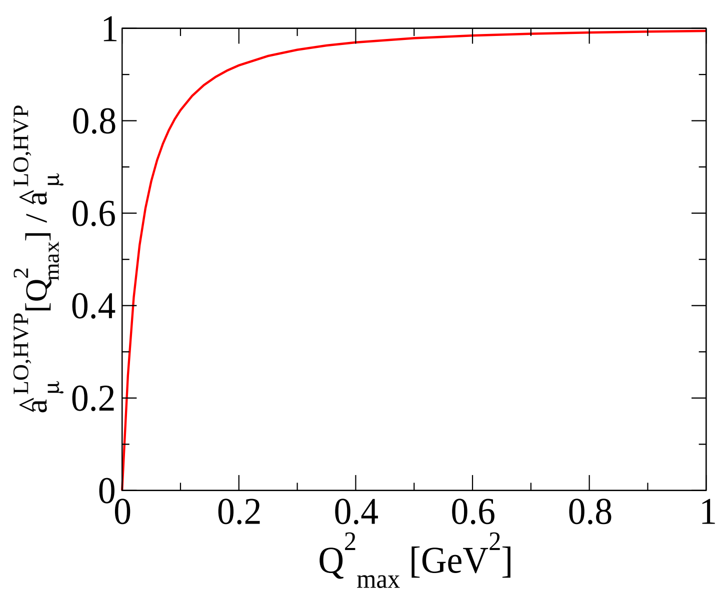
$a_\mu^{LO,HVP}$ "pull" (deviation from exact model value in units of nominal fit error) ~ 18

* $*$ Problem worsens with more high-Q^2 fit points
  * Eg., $[2,2]$ $\Pi^{I=1}(Q^2)$ Pade fit: $pull = 0.5$ for $0 \rightarrow 1\ GeV^2$ fit window $\rightarrow$ 4 for $0 \rightarrow 1.5\ GeV^2$
  * Similarly for $[3,2]$ Pade fit: $pull = 0.5$ on $0 \rightarrow 1\ GeV^2 \rightarrow 1.8$ on $0 \rightarrow 1.5\ GeV^2$

- Lessons
  - Better focus on low-Q^2 region needed
  - Systematic problem makes testing proposed fit strategies, e.g., using dispersive model, crucial

LOW-Q^2 CONTRIBUTIONS AND A HYBRID STRATEGY

- Accumulation of $a_\mu^{LO,HVP}[0 \leq Q^2 \leq Q^2_{max}] \equiv a_\mu^{LO,HVP}[Q^2_{max}]$ wrt Q^2_{max}



- More than 80% of $a_\mu^{LO,HVP}$ accumulated below $Q^2 = 0.1\ GeV^2$, more than 90% below $Q^2 = 0.2\ GeV^2$

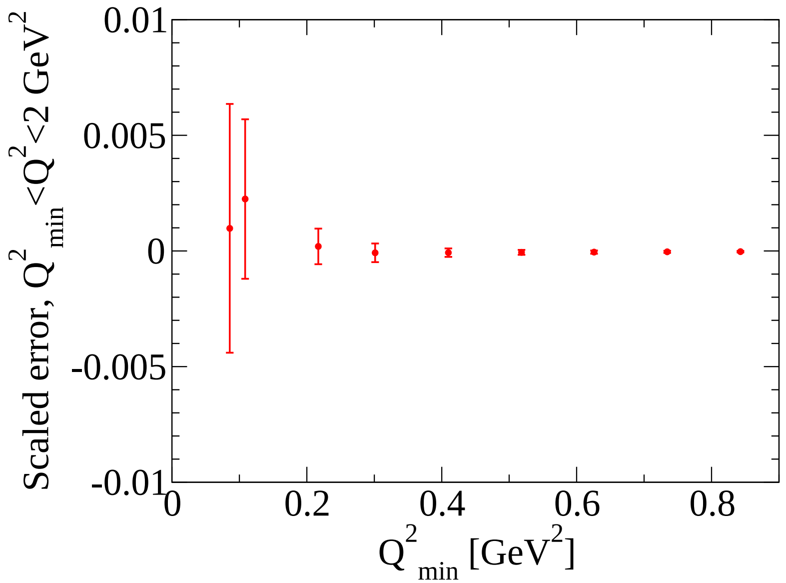
  - ▷ Required tolerance on $< 10{-}20\%$ contribution above $\sim 0.1 - 0.2\ GeV^2 << $ on low-Q^2 contribution

  - ▷ bulk of contribution from low enough Q^2 that low-order Pade, conformal variable polynomial, ChPT representations likely to suffice

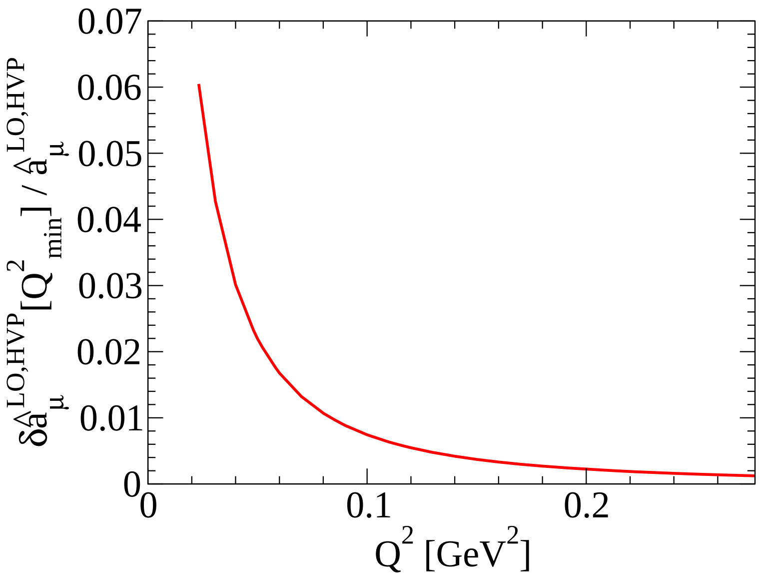
- Suggests hybrid strategy: low-Q^2 contributions by low-Q^2-tailored (Pade, conformal polynomial, ChPT) representations, high-Q^2 by direct numerical integration

- Direct numerical integration above $Q^2 \sim 0.1-0.2\ GeV^2$
  - ▷ Trapezoid rule errors for $Q^2 > Q^2_{min}$ data
    - ◦ Statistical
    - ◦ Systematic (trapezoid rule approximation)
    - ◦ Uncertainty on $\Pi(0)$ entering $\hat{\Pi}(Q^2)$
  - ▷ Investigate using fake data from $I = 1$ dispersive model, MILC $64^3 \times 144$ $a \sim 0.06\ fm$ covariances
  - ▷ FIGURES: errors on $a_\mu^{LO,HVP}[Q^2 > Q^2_{min}]$, as a fraction of $a_\mu^{LO,HVP}$

Systematic and statistical errors on the trapezoid rule evaluation as fractions of $a_\mu^{LO,HVP}$



Impact of an uncertainty $\delta\Pi^{I=1}(0) = 0.001$ on $\hat{a}_\mu^{\mathrm{LO,HVP}}[Q^2 > Q^2_{min}]/a_\mu^{LO,HVP}$



- Successful strategies for low-Q^2 $a_\mu^{LO,HVP}[Q^2 < Q^2_{match}]$

  ▷ Low-order Pades [Aubin, Blum, Golterman, Peris, PRD86 (2012) 054509; HPQCD, 1403.1778]

  ▷ Low-degree polynomials in the conformal variable, $w(z)$, $z = \frac{Q^2}{4m_\pi^2}$, $w(z) = \frac{1-\sqrt{1+z}}{1+\sqrt{1+z}}$

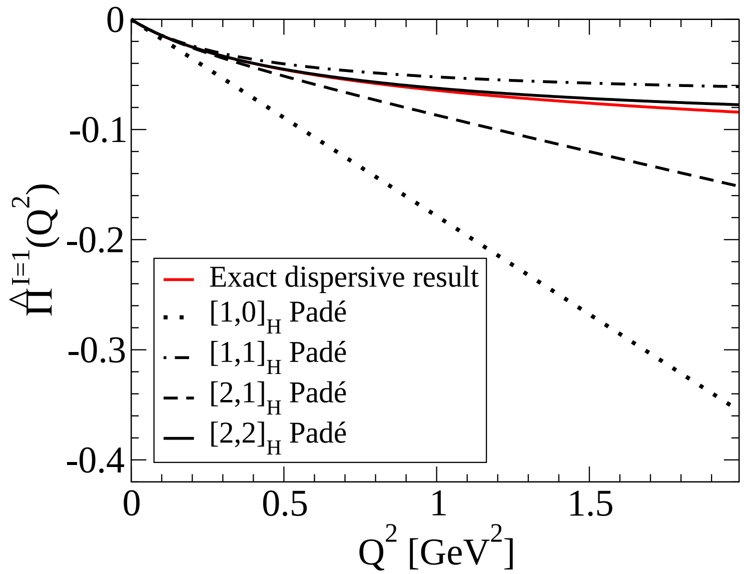
  ▷ Appropriately supplemented NNLO ChPT

- Will illustrate fixing parameters from derivatives of dispersive $\Pi^{I=1}(Q^2)$ wrt Q^2 at $Q^2 = 0$ [GMP14] (potentially determinable on lattice via HPQCD time-moment strategy or discrete-difference version thereof)

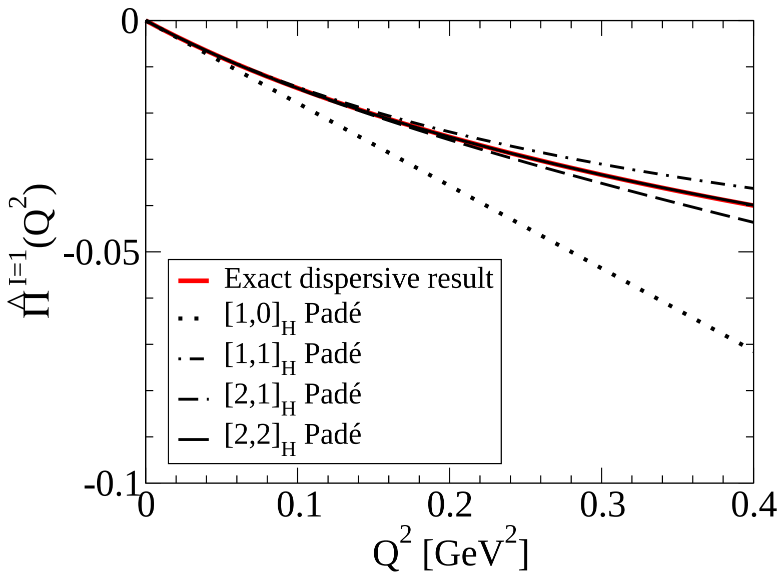
- **Low-order Pades**

  ▷ How low is low?

  Pades c.f. dispersive model $0 < Q^2 < 2\ GeV^2$



Pades c.f. dispersive model $0 < Q^2 < 0.4\ GeV^2$

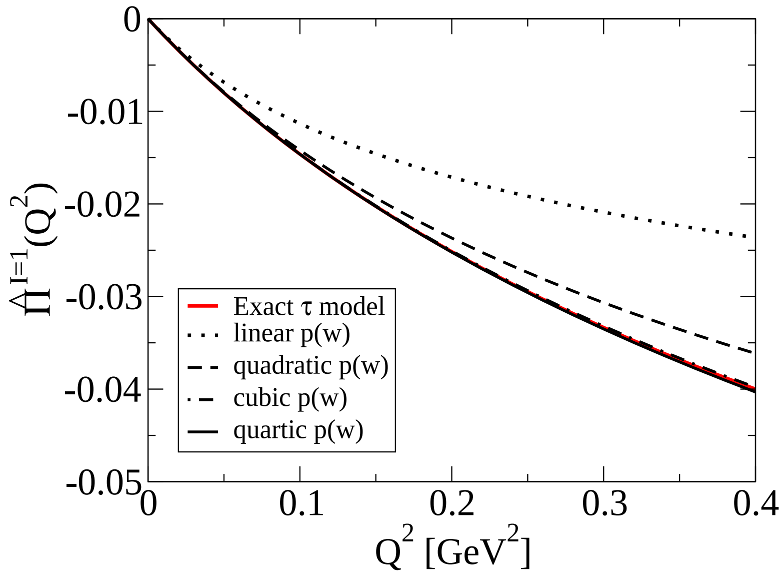


▷ Low-order Pades: summary

- With coefficients from $Q^2 = 0$ derivatives/time moments
  - ∗ $[1,1]$ Pade (up to t^6 moments only) yields sub-1% $a_\mu^{LO,HVP}[0 < Q^2 < Q^2_{match}]$ systematic error up to $Q^2_{match} > 0.2\ GeV^2$
  - ∗ $[2,2]$ Pade (up to t^{10} moment) needed for sub-1% systematic for $Q^2_{match} \sim 2\ GeV^2$ and above
- ALTERNATE IMPLEMENTATION: Pade coefficients also determinable from fit to sufficiently accurate data in interval 0.1 to 0.2 GeV^2 [GMP14]

- **Low-degree polynomials in w**
  - ▷ How low is low?

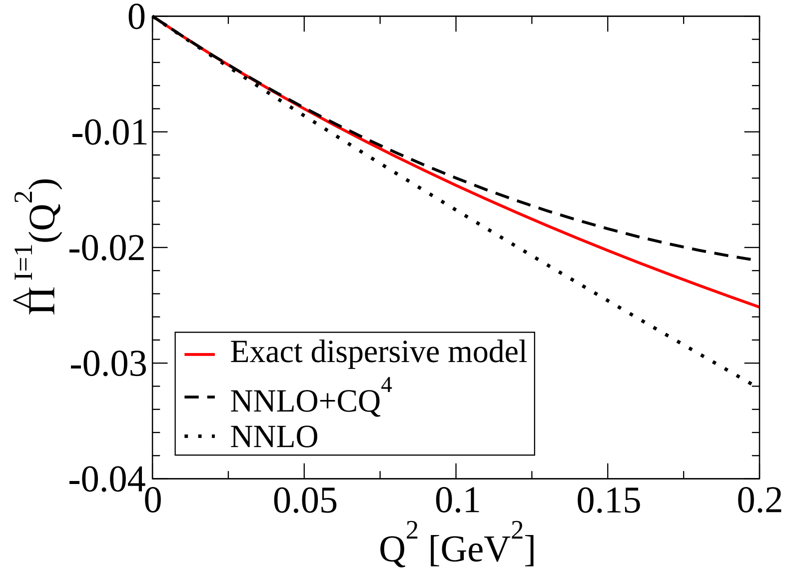
$$P(w) = \sum_{k=1}^{N} c_k\, w^k \text{ c.f. dispersive model,}$$
$$N = 1, \cdots, 4,\ 0 < Q^2 < 0.4\ GeV^2$$



▷ **Low-order conformal polynomials: summary**

- With coefficients from $Q^2 = 0$ derivatives/time moments
  - ∗ Quadratic/cubic $w(Q^2)$: sub-1% systematic error on $a_\mu^{LO,HVP}[0 < Q^2 < Q^2_{match}]$ for Q^2_{match} up to $\sim 0.15\ GeV^2$/well beyond 0.2 GeV^2
  - ∗ Disadvantage c.f. Pades: no underlying theorem ensuring continuing improved convergence with increasing order at low order
- ALTERNATE IMPLEMENTATION: $P(w)$ coefficients, c_k, determinable from fit to sufficiently accurate data in interval 0.1 to 0.2 GeV^2 [GMP14]

- **Supplemented NNLO ChPT**
  - ▷ Supplemented NNLO ChPT c.f. dispersive model $0 < Q^2 < 0.2\ GeV^2$



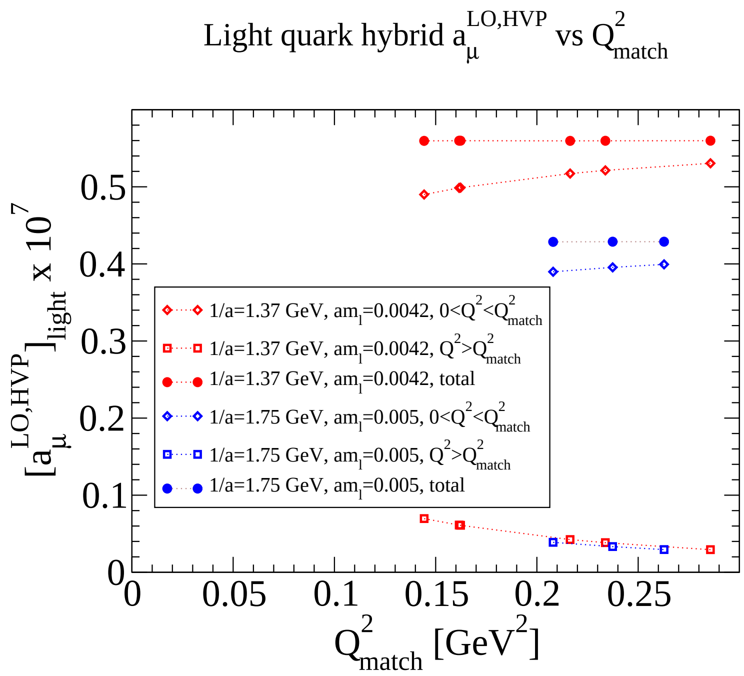
- ▷ Sub-1% $a_\mu^{LO,HVP}[0 < Q^2 < Q^2_{match}]$ systematic error up to and somewhat above $Q^2_{match} \sim 0.1\ GeV^2$, but deteriorates above this

- ▷ Alternative strategy of fitting required LECs in interval $0.1 - 0.2\ GeV^2$ not successful at the sub-1% level (additional curvature contributions beyond phenomenological NNNLO CQ^4 term)

- ▷ ChPT thus usable primarily with time-moments, and as cross-check on other low-Q^2 methods

A FEW PRELIMINARY RESULTS

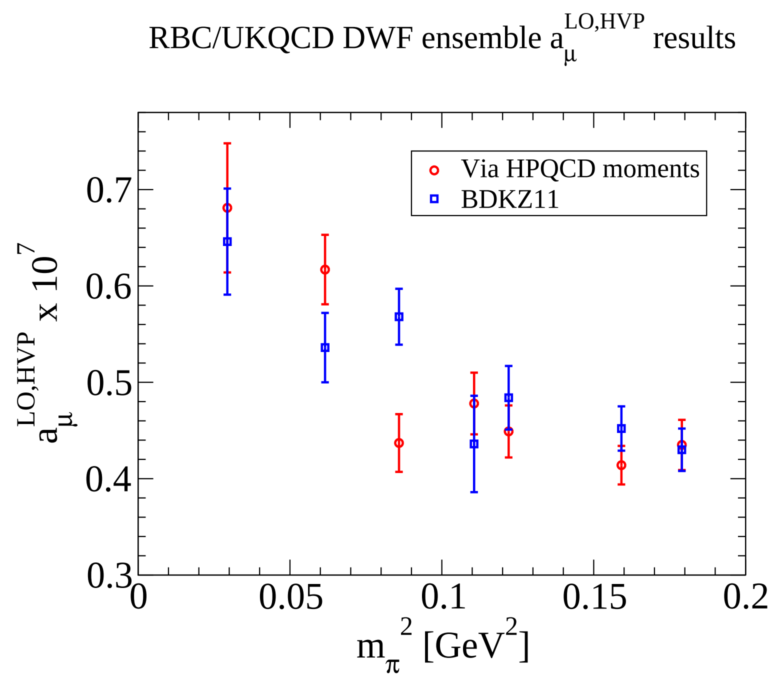
- Excellent hybrid stability with Q^2_{match} for both light, strange (connected) contributions, e.g., the light case:



- Strange contribution from RBC/UKQCD DWF data, via HPQCD time-moments
  - $\triangleright$ Convergence already by $[1,1]$ Pade, as seen by HPQCD
  - $\triangleright$ Pade poles lie on cut as required (self-consistency check)
  - $\triangleright$ "Alternation" of $[1,0]$, $[1,1]$, $[2,1]$, $[2,2]$ Pades as required
  - $\triangleright$ $\left[a_\mu^{LO,HVP}\right]_s = (52.4 \pm 2.1) \times 10^{-10}$, c.f. $(53.4 \pm 0.6) \times 10^{-10}$ HPQCD, $(53 \pm 3) \times 10^{-10}$ ETM

- Preliminary results on the light contribution, RBC/UKQCD DWF data, HPQCD moments
  - ▷ Errors much larger than for strange
  - ▷ Spurious poles for $[2,2]$ Pade
  - ▷ Poles properly on cut for $[1,1]$, $[2,1]$ Pades
  - ▷ "Alternation" of lower $[1,0]$, $[1,1]$, $[2,1]$ Pades as required
  - ▷ Discrete-difference moments also being studied

▷ **VERY PRELIMINARY** time-moment results c.f. two-vector-meson ansatz fits to same data [Boyle, Del Debbio, Kerrane, Zanotti PRD85 (2012) 074504]



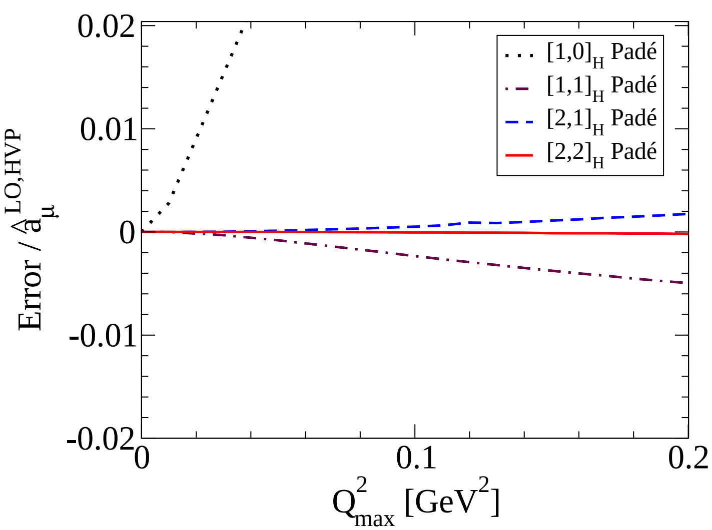
CONCLUSIONS

- Fitting over sizable Q^2 range VERY dangerous for $a_\mu^{LO,HVP}$
- Hybrid strategy viable with low-order Pades, low-degree conformal polynomials, supplemented NNLO ChPT for low-Q^2 contributions
- Excellent stability of hybrid evaluation wrt Q^2_{match}
- No blowing up of errors in hybrid strategy

BACKUP SLIDES

$a_\mu^{LO,HVP}[0 < Q^2 < Q^2_{match}]$ systematic error vs. Q^2_{match} for low-order Pades

- $0 < Q^2_{match} < 0.2\ GeV^2$



- $0 < Q^2_{match} < 2\ GeV^2$