Improved gradient flow for step scaling function and scale setting

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The gradient flow coupling
Step Scaling
Scale setting
Perturbative improvement
Conclusion

**SPOILER**

**β function, 4 flavors**

\[
\left(g^2(sL) - g^2(L)\right)/\log(s^2)
\]

- **4 flavors**
- non-perturbative
- 2-loop
- 1-loop

**8 flavors**

- Preliminary
- 2-loop
- 1-loop

- **8 flavor**
- non-perturbative

- **Scale setting 2+1+1 HISQ**

- **g_{IRFP} in 12 flavors**

- \[c = 0.2, \tau = 0.04\]

\[
g^2(L) = \frac{a^2}{t_0}
\]

- \[\tau_0 = -0.18\]

- m1
- m2
- m3
The gradient flow coupling

Popular new running coupling

\[ g_{GF}^2(\mu = \frac{1}{\sqrt{8t}}) = \frac{1}{N} t^2 \langle E(t) \rangle, \quad E(t) = -\frac{1}{2} G_{\mu \nu}^2 \]

- easy to measure with small systematical errors
- appropriate both for scale setting and step scaling function
- but \( g_{GF}^2(\mu, a) \) can have significant cut-off corrections
The gradient flow coupling

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$t$-shift improved $\tilde{g}_{GF}^2(\mu)$: simple modification that can remove cut-off effects (1404.0984 and in prep)
T-SHIFT IMPROVED GRADIENT FLOW

Define the t-shifted coupling as

\[ \tilde{g}^2_{GF}(\mu = \frac{1}{\sqrt{8t}}, a) = \frac{1}{\mathcal{N}} t^2 \langle E(t + a^2 \tau_0) \rangle, \quad a^2 \tau_0 \ll t \]

In the continuum limit \( a \to 0 \) limit \( \tilde{g}^2_{GF}(\mu) \to g^2_{GF}(\mu) \)
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Why would this help? Three ways of looking at it:

1. \( \langle E(t) \rangle \to \langle E(t + a^2 \tau_0) \rangle \)
   replaces \( E(t) \) with a smeared operator
   \( \to \) smearing tends to remove lattice artifacts
2. \( t + a^2 \tau_0 \to t \) removes initial flow time artifacts
3. The shift can remove \( O(a^2) \) terms
T-SHIFT IMPROVED GRADIENT FLOW

Expand the t-shifted coupling

\[ \tilde{g}_{GF}^2(\mu = \frac{1}{\sqrt{8t}}, a) = \frac{1}{N} t^2 \langle E(t + a^2 \tau_0) \rangle, \quad a^2 \tau_0 \ll t \]

in \( a^2 \tau_0 \)

\[ \tilde{g}_{GF}^2(\mu, a) = g_{GF}^2(\mu, a) + a^2 \tau_0 \frac{d}{dt} \langle t^2 \langle E(t) \rangle \rangle + \ldots \]

\[ g_{GF}^2(\mu, a) = g_{GF}^2(\mu) + a^2 C + \ldots \]

If \( C = -\tau_0 \frac{d}{dt} \langle t^2 \langle E(t) \rangle \rangle \) the \( O(a^2) \) corrections are removed

\[ \tilde{g}_{GF}^2(\mu, a) = g_{GF}^2(\mu) + O(a^4, a^2 \log^n(a)) \]
Yet an other way to look at the t-shifted coupling

\[ \tilde{g}^2_{GF}(t, a) = \frac{1}{\mathcal{N}} t^2 \langle E(t + a^2 \tau_0) \rangle = g^2_{GF}(t + a^2 \tau_0)(1 + \frac{a^2 \tau_0}{t})^{-2} \]

\((1 + a^2 \tau_0/t)^{-1}\) term gives tree-level corrections while

\[ g^2_{GF}(t+a^2 \tau_0) = g^2_{GF}(t) + \frac{a^2 \tau_0}{t} t \frac{dg^2_{GF}}{dt} + \cdots = g^2_{GF}(t) + \frac{a^2 \tau_0}{t} b_0 g^4_{GF}(t) + \cdots \]

gives 1-loop corrections. If

- the tree level corrections are small
- or removed analytically

the \(\tau_0\) shift can give 1-loop improvement!
T-SHIFT IMPROVED GRADIENT FLOW

\[ \tilde{g}_{GF}^{2}(\mu = \frac{1}{\sqrt{8t}}, a) = \frac{1}{N} t^2 \langle E(t + a^2 \tau_0) \rangle, \]

- For full \( O(a^2) \) improvement \( \tau_{opt} \) must depend on both the bare and renormalized couplings
  → might mean no predictive power
- If the tree-level corrections are small, \( \tau_0 = \text{const} \) can give 1-loop improvement
- Every \( \tau_0 \) value is correct - some are just better
  → comparing different \( \tau_0 \) values is a good consistency check
The gradient flow coupling

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\( N_f = 4 \)

Test case: step scaling function with 4 flavor staggered fermions

- Set \( \mu = (cL)^{-1}, c = 0.25 \)
- Define discrete \( \beta \) function with scale change \( s = 1.5 \)

\[
\beta_{\text{lat}}(g^2_{\text{GF}}; s; a) = \frac{\tilde{g}^2_{\text{GF}}(L; a) - \tilde{g}^2_{\text{GF}}(sL; a)}{\log(s^2)}
\]

Continuum extrapolation:

\( g^2(L) = 2.2 \)

Cut-off corrections with our action are small

All \( \tau_0 \) shifts predict the same continuum value \( \rightarrow \) consistency check!
$N_f = 4$

Tree-level perturbative (1406.0827) vs t-shift improvement:

$$g^2_{GF} = 2.2$$

PT corrections remove $O(a^2)$ terms,
$N_f = 4$

Tree-level perturbative (1406.0827) vs t-shift improvement:

$g_{GF}^2 = 2.2$

$g_{GF}^2 = 4.8$

PT corrections remove $O(a^2)$ terms,

at stronger coupling PT overshoots

The continuum extrapolations both for improved and unimproved gradient flow are consistent
$N_f = 4$

**Discrete $\beta$ function**

$$\frac{(g^2(sL) - g^2(L))/\log(s^2)}{g_c}$$

- **4 flavors**
- non-perturbative
- 2-loop
- 1-loop

**Conclusion**

Close agreement with 2-loop perturbative value

$$\tau_0 = -0.02 - 0.0$$ in the investigated $g_{GF}^2$ range
$N_f = 8$

Expected to be chirally broken but very strongly coupled

Very different from 2-loop perturbative

$\tau_0 = 0.0 - 0.04$ with 1x nHYP
$\tau_0 = 0.12 - 0.20$ with 2x nHYP
t-shift optimization is essential
\[ g_{GF}^2(L) \] versus \( \beta \) bare coupling shows crossings
- does that imply an IRFP?

Only if the crossings survive the continuum limit!
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- does that imply an IRFP?

Only if the crossings survive the continuum limit!
$N_f = 12$

Zoom in:

This is special: other published step scaling function studies of $N_f = 12$ do not see crossings, they identify an IRFP by extrapolating from the weak coupling side.
The gradient flow coupling

$N_f = 12$

Take the continuum limit of the crossings:

$\langle L \rangle = \langle sL \rangle \implies \langle L; s \rangle = \langle \rangle$

c=0.2, s=2

optimization is essential,

$\tau_{\text{opt}} \approx 0.04$
\( N_f = 12 \)

Combine \( s = 4/3, 3/2 \) and 2 with common \( \tau_0 = 0.04 \)

All scale factors predict \( g_\star^2(L) \approx 6.2 \) with no (apparent) dependence on the lattice spacing

Extrapolating \( g_\star^2(L) \) is more reliable than the \( \beta \) function
$N_f = 12$

Results are similar with $c = 0.25, 0.3$ Larger $c$ gives stronger $g_\star^2(L)$ and has increased statistical errors, but t-shift improvement works the same.

$c = 0.25, s = 2$

$\tau_{opt} \approx 0.06$

Preliminary
The modified gradient flow coupling $\tilde{g}_{GF}^2$ can be used to define improved $t_0$, $\omega_0$ scales

$$t^2 \langle E(t + a^2 \tau_0) \rangle|_{t=t_0} = 0.3$$
Optimizing Scale Setting

The modified gradient flow coupling $\tilde{g}_{GF}^2$ can be used to define improved $t_0, w_0$ scales

$$t^2\langle E(t + a^2\tau_0) \rangle|_{t=t_0} = 0.3$$

If $\tilde{g}_{GF}^2$ has no lattice artifacts, the definition

$$t^2\langle E(t + a^2\tau_0) \rangle|_{t=t_1} = 0.35$$

will predict a consistent scale, i.e. $t_0/t_1$ is independent of the lattice spacing - just like $r_0$ and $r_1$

(Assuming finite volume effects can be neglected.)
HISQ 2+1+1

Test: Symanzik flow data on HISQ 2+1+1 configurations \(^1\)
\[ \sqrt{t_0/t_1} \text{ vs } a^2/t_0 \text{ for } m_s/m_l = 5, 10, 27 \]

without t-shift

Quadratic + quartic \(a^2\) dependence?

\(^1\)Thanks N. Brown for sharing the MILC gradient flow data
HISQ 2+1+1

Test: Symanzik flow data on HISQ 2+1+1 configurations

\[ \sqrt{t_0/t_1} \text{ vs } a^2/t_0 \text{ for } m_s/m_l = 5, 10, 27 \]

\[ \tau_0 = 0.0 \]

\[ \tau_0 = -0.18 \]

Quadratic + quartic \( a^2 \) dependence?

The coarsest \( a \approx 0.15 \text{fm} \) set is (probably) not in the \( O(a^2) \) scaling regime!

\[ ^1 \text{Thanks N. Brown for sharing the MILC gradient flow data} \]
HISQ 2+1+1

There is nothing special about $t_0$ or $t_1$: $\tilde{g}^2_{GF}$ vs $t/t_0$ should be independent of the lattice spacing if there are no cut-off effects $^2$

$$\tau_0 = 0.0$$

$$\tau_0 = -0.18$$

A: $a \approx 0.15\text{fm}$; B: $a \approx 0.12\text{fm}$; C: $a \approx 0.06\text{fm}$; D: $a \approx 0.06\text{fm}$;

$^2$Assuming $t$ is large to avoid gradient flow integration artifacts but small enough to minimize finite volume effects
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**HISQ 2+1+1**

**Compare** $t_0$ and $r_1$:

- $\tau_0 = 0.0$
- $\tau_0 = -0.18$

Just like before:
- without t-shift improvement lattice artifacts mask that the coarsest set is not in the $O(a^2)$ scaling regime
- With t-shift the lattice scale is predicted better than 1% with $\tau_{opt}$ predicted using $t_0/t_1$
Perturbative vs T-shift Improvement

How does tree-level perturbative improvement compare with t-shift improvement?

- For the HISQ action tree-level perturbative improvement helps large $g^2$, $t$ region but not small.
- This could be different for other actions
CONCLUSION

t-shift gradient flow improvement is a simple yet powerful method

- It is easy to implement and can give 1-loop improvement
- In step scaling function studies extrapolation to the continuum limit is possible even at strong running coupling
- In scale setting the optimal $\tau_0$ parameter can be found by comparing two configuration sets
- t-shift improved coupling can reveal non $O(a^2)$ scaling violations that are hidden otherwise

Application for walking coupling in $N_f = 4 + 8$ flavor system check out the poster by O. Witzel