Improved gradient flow for step scaling function and scale setting

Anna Hasenfratz
University of Colorado Boulder

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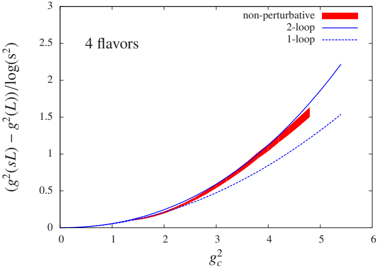
In collaboration with

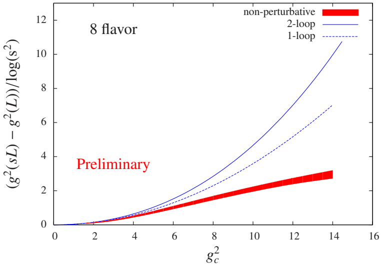
A. Cheng, Y. Liu, G. Petropoulos, D. Schaich, A. Veernala

SPOILER

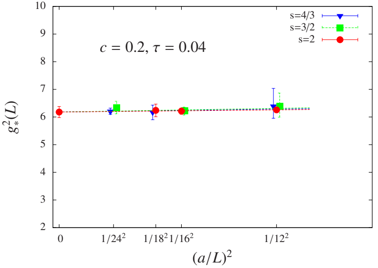
β function, 4 flavors



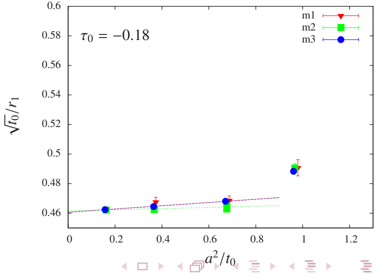
8 flavors



g^2_{IRFP} in 12 flavors



Scale setting 2+1+1 HISQ



THE GRADIENT FLOW COUPLING

Popular new running coupling

$$g^2_{GF}\left(\mu = \frac{1}{\sqrt{8t}}\right) = \frac{1}{\mathcal{N}} t^2 \langle E(t) \rangle, \qquad E(t) = -\frac{1}{2} G^2_{\mu\nu}$$

- easy to measure with small systematical errors
- appropriate both for scale setting and step scaling function
- but $g^2_{GF}(\mu, a)$ can have significant cut-off corrections

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t-shift improved $\tilde{g}^2_{GF}(\mu)$: simple modification that can remove cut-off effects (1404.0984 and in prep)

T-SHIFT IMPROVED GRADIENT FLOW

Define the t-shifted coupling as

$$\tilde{g}^2_{GF}(\mu = \frac{1}{\sqrt{8t}}, a) = \frac{1}{\mathcal{N}} t^2 \langle E(t + a^2\tau_0) \rangle, \qquad a^2\tau_0 \ll t$$

In the continuum $a \to 0$ limit $\tilde{g}^2_{GF}(\mu) \to g^2_{GF}(\mu)$

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Why would this help? Three ways of looking at it:

1. $\langle E(t)\rangle \to \langle E(t + a^2\tau_0)\rangle$
   replaces $E(t)$ with a smeared operator
   $\to$ smearing tends to remove lattice artifacts
2. $t + a^2\tau_0 \to t$ removes initial flow time artifacts
3. The shift can remove $\mathcal{O}(a^2)$ terms

T-SHIFT IMPROVED GRADIENT FLOW

Expand the t-shifted coupling

$$\tilde{g}^2_{GF}\left(\mu = \frac{1}{\sqrt{8t}}, a\right) = \frac{1}{\mathcal{N}} t^2 \langle E(t + a^2\tau_0)\rangle, \qquad a^2\tau_0 \ll t$$

in $a^2\tau_0$

$$\begin{aligned} \tilde{g}^2_{GF}(\mu, a) &= g^2_{GF}(\mu, a) + a^2\tau_0 \frac{d}{dt}\left(t^2\langle E(t)\rangle\right) + \dots \\ g^2_{GF}(\mu, a) &= g^2_{GF}(\mu) + a^2\mathcal{C} + \dots \end{aligned}$$

If $\quad \mathcal{C} = -\tau_0 \frac{d}{dt}\left(t^2\langle E(t)\rangle\right) \quad$ the $\mathcal{O}(a^2)$ corrections are removed

$$\tilde{g}^2_{GF}(\mu, a) = g^2_{GF}(\mu) + \mathcal{O}(a^4, a^2 \log^n(\mathrm{a}))$$

T-SHIFT IMPROVED GRADIENT FLOW

Yet an other way to look at the t-shifted coupling

$$\tilde{g}^2_{GF}(t,a) = \frac{1}{\mathcal{N}} t^2 \langle E(t + a^2\tau_0)\rangle = g^2_{GF}(t + a^2\tau_0)\left(1 + \frac{a^2\tau_0}{t}\right)^{-2}$$

$(1 + a^2\tau_0/t)^{-1}$ term gives tree-level corrections while

$$g^2_{GF}(t+a^2\tau_0) = g^2_{GF}(t) + \frac{a^2\tau_0}{t}\, t\frac{dg^2_{GF}}{dt} + \cdots = g^2_{GF}(t) + \frac{a^2\tau_0}{t}\, b_0 g^4_{GF}(t) + \ldots$$

gives 1-loop corrections. If

- the tree level corrections are small
- or removed analytically

the τ_0 shift can give 1-loop improvement!

T-SHIFT IMPROVED GRADIENT FLOW

$$\tilde{g}^2_{GF}\left(\mu = \frac{1}{\sqrt{8t}}, a\right) = \frac{1}{\mathcal{N}} t^2 \langle E(t + a^2\tau_0) \rangle,$$

- For full $\mathcal{O}(a^2)$ improvement $\tau_{\rm opt}$ must depend on both the bare and renormalized couplings
  - $\rightarrow$ might mean no predictive power
- If the tree-level corrections are small, $\tau_0 = \text{const}$ can give 1-loop improvement
- Every τ_0 value is correct - some are just better
  - $\rightarrow$ comparing different τ_0 values is a good consistency check

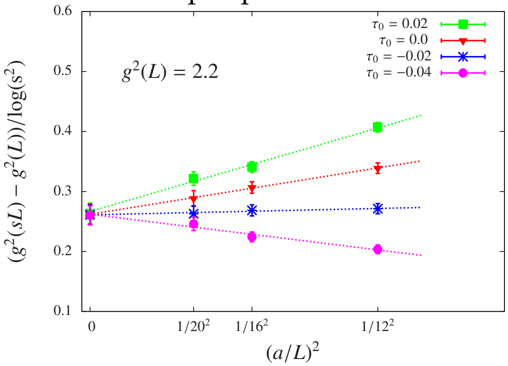
$N_f = 4$

Test case: step scaling function with 4 flavor staggered fermions

- Set $\mu = (cL)^{-1}$, $c = 0.25$
- Define discrete β function with scale change $s = 1.5$

$$\beta_{\text{lat}}(g^2_{GF}; s; a) = \frac{\tilde{g}^2_{GF}(L;a) - \tilde{g}^2_{GF}(sL;a)}{\log(s^2)}$$

Continuum exptrapolation:



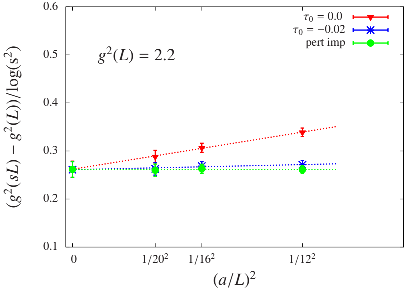
Cut-off corrections with our action are small

All τ_0 shifts predict the same continuum value $\rightarrow$ consistency check!

$N_f = 4$

Tree-level perturbative (1406.0827) vs t-shift improvement:

$g^2_{GF} = 2.2$



PT corrections remove $\mathcal{O}(a^2)$ terms,

$N_f = 4$

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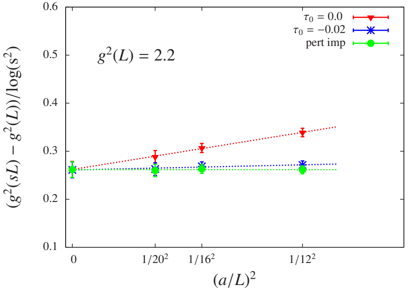
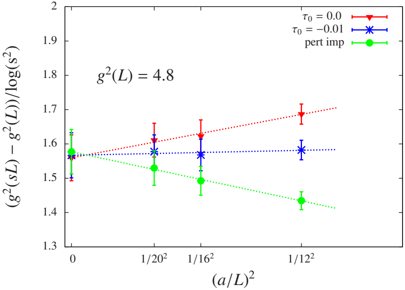


$g^2_{GF} = 4.8$

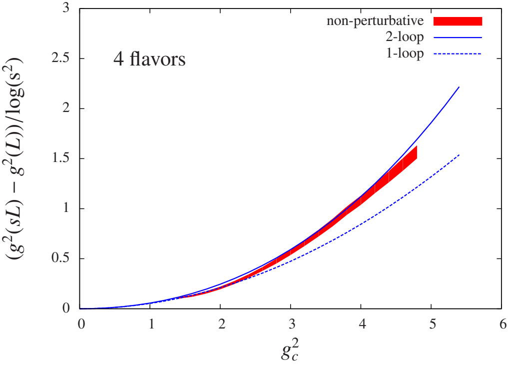


PT corrections remove $\mathcal{O}(a^2)$ terms, at stronger coupling PT overshoots

The continuum extrapolations both for improved and unimproved gradient flow are consistent

$N_f = 4$

Discrete β function

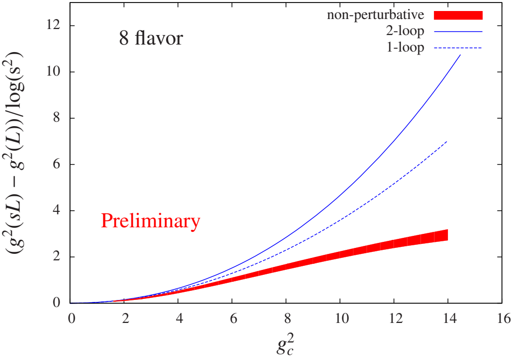


Close agreement with 2-loop perturbative value

$\tau_0 = -0.02 - 0.0$ in the investigated g^2_{GF} range

$N_f = 8$

Expected to be chirally broken but very strongly coupled

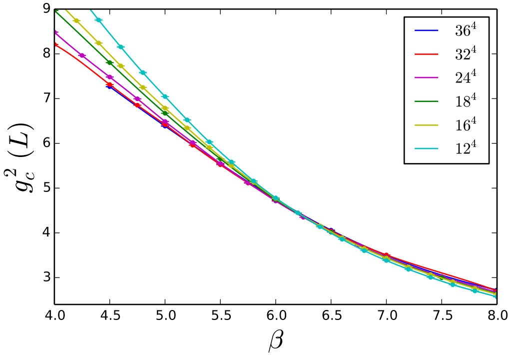


Very different from 2-loop perturbative

$\tau_0 = 0.0 - 0.04$ with 1x nHYP
$\tau_0 = 0.12 - 0.20$ with 2x nHYP
t-shift optimization is essential

$N_f = 12$

$g^2_{GF}(L)$ versus β bare coupling shows crossings
- does that imply an IRFP?



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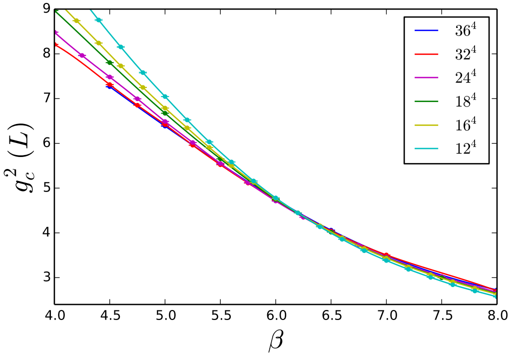
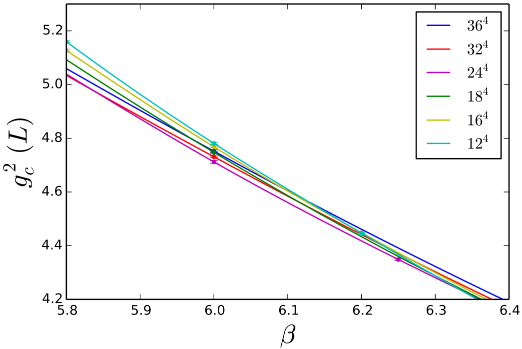


Only if the crossings survive the continuum limit!

$N_f = 12$

Zoom in:



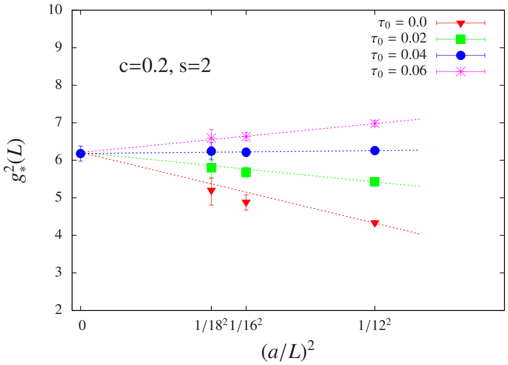
Only if the crossings survive the continuum limit!

This is special: other published step scaling function studies of $N_f = 12$ do not see crossings, they identify an IRFP by extrapolating from the weak coupling side.

$N_f = 12$

Take the continuum limit of the crossings:

$$g^2_{GF}(L) = g^2_{GF}(sL) \implies g^2_\star(L;s) = g^2_{GF}(L)$$



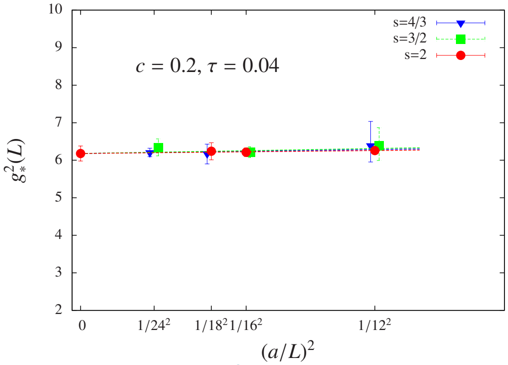
$c = 0.2, s = 2$

optimization is essential,

$\tau_{opt} \approx 0.04$

$N_f = 12$

Combine $s = 4/3, 3/2$ and 2 with common $\tau_0 = 0.04$

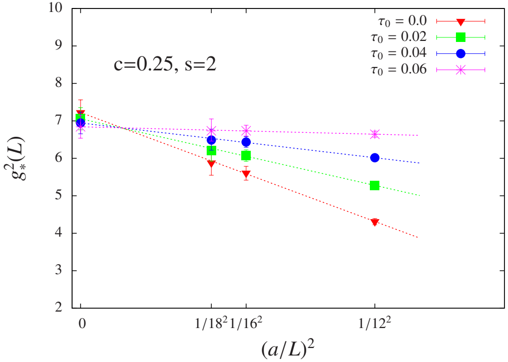


All scale factors predict $g^2_\star(L) \approx 6.2$ with no (apparent) dependence on the lattice spacing

Extrapolating $g^2_\star(L)$ is more reliable than the β function

$N_f = 12$

Results are similar with $c = 0.25, 0.3$ Larger c gives stronger $g_\star^2(L)$ and has increased statistical errors, but t-shift improvement works the same



$c = 0.25, s = 2$

$\tau_{\text{opt}} \approx 0.06$

Preliminary

Optimizing scale setting

The modified gradient flow coupling $\tilde{g}^2_{GF}$ can be used to define improved t_0, w_0 scales

$$t^2 \langle E(t + a^2\tau_0)\rangle|_{t=t_0} = 0.3$$

Optimizing scale setting

The modified gradient flow coupling $\tilde{g}^2_{GF}$ can be used to define improved t_0, w_0 scales

$$t^2 \langle E(t + a^2\tau_0)\rangle|_{t=t_0} = 0.3$$

If $\tilde{g}^2_{GF}$ has no lattice artifacts, the definition

$$t^2 \langle E(t + a^2\tau_0)\rangle|_{t=t_1} = 0.35$$

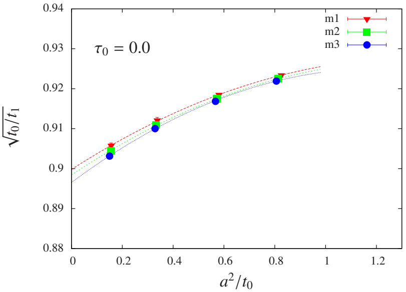
will predict a consistent scale, i.e. t_0/t_1 is independent of the lattice spacing - just like r_0 and r_1
(Assuming finite volume effects can be neglected.)

HISQ 2+1+1

Test: Symanzik flow data on HISQ 2+1+1 configurations [1]

$\sqrt{t_0/t_1}$ vs a^2/t_0 for $m_s/m_l = 5, 10, 27$

without t-shift



Quadratic + quartic a^2 dependence?

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[1] Thanks N. Brown for sharing the MILC gradient flow data

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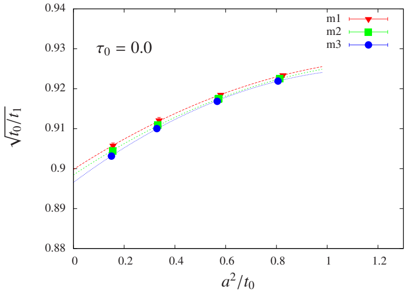

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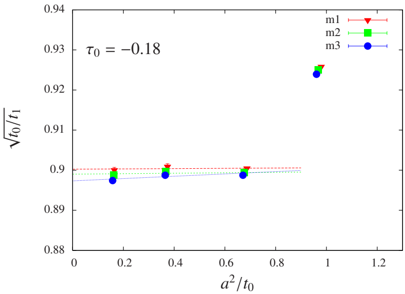

The coarsest $a \approx 0.15$fm set is (probably) not in the $\mathcal{O}(a^2)$ scaling regime!

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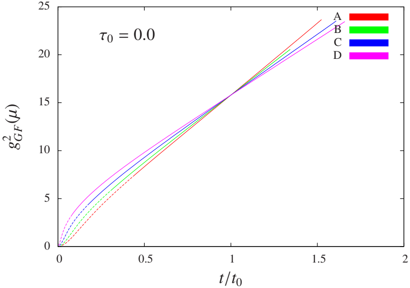
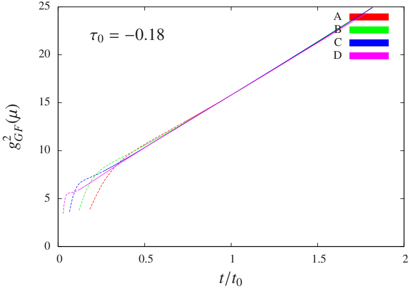
HISQ 2+1+1

There is nothing special about t_0 or t_1:
$\tilde{g}^2_{GF}$ vs t/t_0 should be independent of the lattice spacing if there are no cut-off effects [2]

$\tau_0 = 0.0$



$\tau_0 = -0.18$



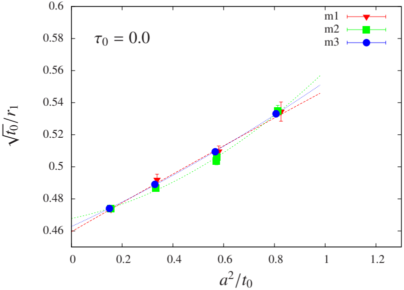
A: $a \approx 0.15$fm; B: $a \approx 0.12$fm; C: $a \approx 0.06$fm; D: $a \approx 0.06$fm;

[2] Assuming t is large to avoid gradient flow integration artifacts but small enough to minimize finite volume effects

HISQ 2+1+1

Compare t_0 and r_1:

$\tau_0 = 0.0$



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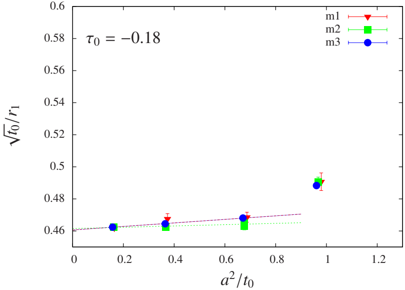


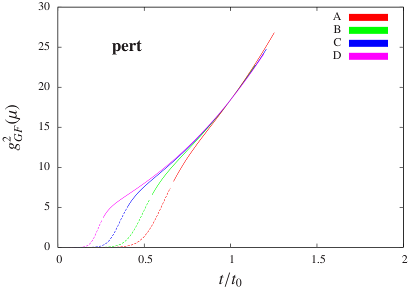
Just like before:

- without t-shift improvement lattice artifacts mask that the coarsest set is not in the $\mathcal{O}(a^2)$ scaling regime
- With t-shift the lattice scale is predicted better than 1% with τ_{opt} predicted using t_0/t_1

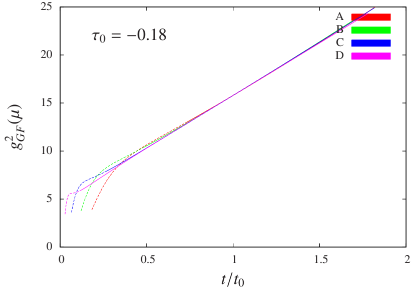
PERTURBATIVE VS T-SHIFT IMPROVEMENT

How does tree-level perturbative improvement compare with t-shift improvement?

perturbative



t-shift $\tau_0 = -0.18$



- For the HISQ action tree-level perturbative improvement helps large g^2, t region but not small.
- This could be different for other actions

Conclusion

t-shift gradient flow improvement is a simple yet powerful method

- It is easy to implement and can give 1-loop improvement
- In step scaling function studies extrapolation to the continuum limit is possible even at strong running coupling
- In scale setting the optimal τ_0 parameter can be found by comparing two configuration sets
- t-shift improved coupling can reveal non $\mathcal{O}(a^2)$ scaling violations that are hidden otherwise

Application for walking coupling in $N_f = 4 + 8$ flavor system check out the poster by O. Witzel