How to reduce O(a^2) lattice artefacts in gradient flow observables?

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Topics

1. Motivation, definition of $\langle E(t,x)\rangle$
2. Anatomy of tree level $\mathrm{O}(a^2)$ effects
3. GF coupling in finite volume with twisted periodic b.c.'s
4. Scaling test in pure SU(3) gauge theory
5. Symanzik $\mathrm{O}(a^2)$ improvement
6. Classical expansion of the flow equation
7. Conclusions

Gradient flow & lattice artefacts

Flow equation:

$$\partial_t B_\mu(x,t) = D_\nu G_{\nu\mu}(x,t), \qquad B_\mu(x,0) = A_\mu(x)$$

Many uses (Lüscher '10–'14), in particular

- Scale setting & definitions of finite volume running couplings, all based on the action density observable

$$\langle E(t,x)\rangle, \qquad E(t,x) = -\tfrac{1}{2}\,\mathrm{tr}\left\{G_{\mu\nu}(x,t)G_{\mu\nu}(x,t)\right\}$$

- plenty of choice for lattice discretization of action, observable, gradient flow.
- leading cutoff effects are $\mathrm{O}(a^2)$ but can be quite large!

$\Rightarrow$ use Symanzik $\mathrm{O}(a^2)$ improvement to reduce the leading effects; important e.g. for precision study of running coupling.

Anatomy of tree-level $\mathrm{O}(a^2)$ effects in $\langle E(t)\rangle$ I

$$
\begin{aligned}
t^2\langle E(t,x)\rangle &= g^2\int_{-\pi/a}^{\pi/a}\mathrm{d}^4p\,\mathrm{tr}\left[K^{(\mathrm{o})}_{\mu\nu}(p,0)\bar{D}_{\mu\nu}(p,\lambda,\alpha)\right],\\
\bar{D}_{\mu\nu}(p,\lambda,\alpha) &= (\mathrm{e}^{-tK^{(\mathrm{f})}(p,\alpha)})_{\mu\rho}\left(K^{(\mathrm{a})}(p,\lambda)^{-1}\right)_{\rho\sigma}(\mathrm{e}^{-tK^{(\mathrm{f})}(p,\alpha)})_{\sigma\nu},
\end{aligned}
$$

- λ,α: gauge fixing parameters for the action and flow equation, respectively.
- Observable, gradient flow and action characterized by kernels $K_{\mu\nu}(p)$ of "free lattice actions":

$$
\begin{aligned}
S^{(\mathrm{a,o,f})} &= \frac{1}{2}\int_{-\pi/a}^{\pi/a}\mathrm{d}^4p\,A^b_\mu(-p)K^{(\mathrm{a,o,f})}_{\mu\nu}(p,\lambda)A^b_\nu(p)+\mathrm{O}(A^3),\\
K^{(\mathrm{a,o,f})}_{\mu\nu}(p,\lambda) &= K^{\mathrm{cont}}_{\mu\nu}(p,\lambda)+a^2R^{(\mathrm{a,o,f})}_{\mu\nu}(p,\lambda)+\mathrm{O}(a^4)\\
K^{\mathrm{cont}}_{\mu\nu}(p,\lambda) &= p^2\delta_{\mu\nu}+(\lambda-1)p_\mu p_\nu
\end{aligned}
$$

Anatomy of tree-level $\mathrm{O}(a^2)$ effects in $\langle E(t,x)\rangle$ II

Extend momentum integrals to infinity, then evaluate traces:

$$\langle E(t,x)\rangle = \frac{3g^2}{16\pi^2 t^2}\left\{1+\frac{a^2}{t}\left[\left(d_1^{(\mathrm{o})}-d_1^{(\mathrm{a})}\right)J_{4,-2}+\left(d_2^{(\mathrm{o})}-d_2^{(\mathrm{a})}\right)J_{2,0} + d_1^{(\mathrm{f})}J_{4,0}+d_2^{(\mathrm{f})}J_{2,2}\right]\right\}$$

where

$$J_{n,m}=\frac{t^{(m+n)/2}\int_p \mathrm{e}^{-2tp^2}p^n p^m}{\int_p \mathrm{e}^{-2tp^2}}, \qquad p^n\Big|_{n>0}=\sum_\mu p_\mu^n, \quad p^{-n}=1/p^n$$

All momentum integrals can be easily evaluated:

$$J_{2,0}=1, \quad J_{2,2}=3/2, \quad J_{4,0}=3/4, \quad J_{4,-2}=1/2$$

Anatomy of tree-level $\mathrm{O}(a^2)$ effects in $\langle E(t,x)\rangle$ III

$$\langle E(t,x)\rangle = \frac{3g^2}{16\pi^2 t^2}\left\{1 + \frac{a^2}{t}\left(d^{(\mathrm{o})} + d^{(\mathrm{a})} + d^{(\mathrm{f})}\right) + \mathrm{O}(a^4)\right\}.$$

For each d-coeffcient we may choose Wilson-plaquette, Lüscher-Weisz or the clover kernel or combinations thereof. We find the relation

$$d^{(\mathrm{f})} = 3d^{(\mathrm{a})} = -3d^{(\mathrm{o})},$$

and

$$d^{(\mathrm{o})} = \begin{cases} -\frac{1}{24}, & & \text{plaquette (pl)}, \\ \frac{1}{72}, & \left(= -\frac{1}{24} - \frac{2}{3}c_1\right) & \text{Lüscher-Weisz (lw)}, \\ -\frac{5}{24}, & & \text{Clover (cl)}. \end{cases}$$

in agreement with Nogradi's calculation (cf. previous talk)

Anatomy of tree-level $O(a^2)$ effects in $\langle E(t,x)\rangle$ IV

- A single term $d^{\text{total}} a^2/t$ needs to be cancelled $\Rightarrow$ many possibilities, e.g. τ-shift (cf. talk by A. Hasenfratz), linear combinations of plaquette and clover observables, ...
- Naive expectation: $O(a^2)$ improvement obtained by using
  - an $O(a^2)$ improved action, e.g. Lüscher-Weisz (LW)
  - the gradient of an $O(a^2)$ improved action for the flow, e.g Lüscher-Weisz (LW) flow (a.k.a. Symanzik flow) ;
  - a discretized observable free of $O(a^2)$ effects; we consider 2 options:
    1. $E(t,x)$ from the LW action density, $E_{\text{lw}}(t,x)$;
    2. $E(t,x)$ as a linear combination

$$4/3 E_{\text{pl}}(t,x) - 1/3 E_{\text{cl}}(t,x).$$

  - Result (for both definitions of the observable)

$$d^{\text{total}} = -\frac{1}{24} \neq 0 \quad \Rightarrow \quad \text{incomplete } O(a^2) \text{ improvement!}$$

GF coupling in finite volume with twisted periodic b.c.'s

- fix the relation between flow time t and space-time volume L^4 by choosing a value for $c = \sqrt{8t}/L$.
- The trace algebra same as before, however: the numbers $J_{n,m}$ become functions of c!

$\Rightarrow$ more conditions for improvement: each coefficient must vanish separately!

- Cannot be satisfied with LW/Symanzik type flow, need to be more general: include bent rectangles/chairs with coefficient c_2 & define the Zeuthen flow

$$c_0 = 1, \qquad c_1 = -1/12, \qquad c_2 = 1/24$$

$\Rightarrow$ complete tree-level $\mathrm{O}(a^2)$ improvement!

- We checked that the connected 2-point function

$$a^4 \sum_x \left[\langle E(t,x)E(s,y)\rangle - \langle E(t,x)\rangle\langle E(s,y)\rangle\right]$$

is tree-level $\mathrm{O}(a^2)$ improved.

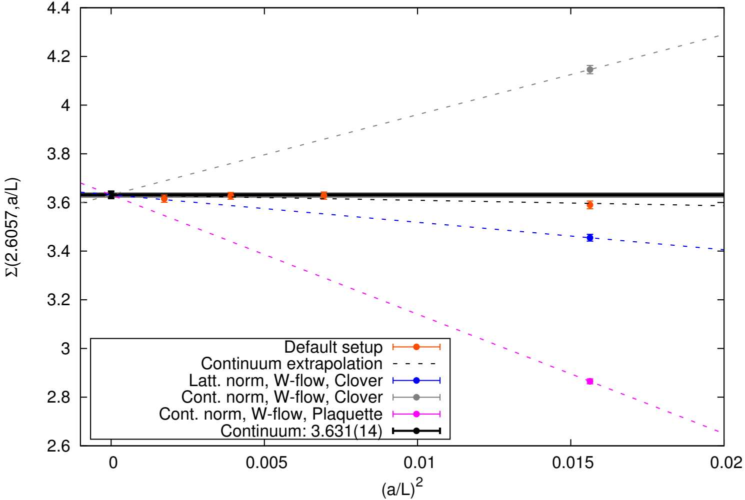
Scaling test in pure SU(3) gauge theory

- GF coupling with SF b.c.'s (Fritzsch & Ramos '13); use only magnetic components, set $x_0 = T/2$, and $T = L$, $c = \sqrt{8t}/L$

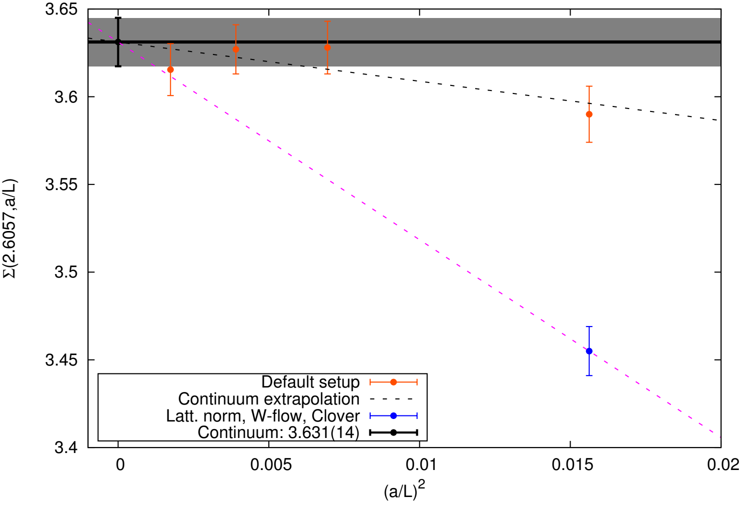
$$-\frac{1}{2}\langle \operatorname{tr} G_{kl}(x,t)G_{kl}(x,t)\rangle|_{x_0=T/2} = \mathcal{N}(c, a/L)g^2_{\mathrm{GF}}(L)$$

- use various discretizations of the coupling, both with $\mathcal{N}(c, a/L)$ and with $\mathcal{N}(c, 0)$.
- O(a) effects from boundaries are negligible at chosen parameters.
- Step-scaling functions at $u = 2.6057$ for lattice sizes $L/a = 8, 12, 16, 24$ with scale factor $s = 2$
- Observe strong reduction of lattice artefacts!

Scaling test in pure SU(3) gauge theory (preliminary)



Scaling test in pure SU(3) gauge theory (preliminary)



<u>Question:</u> did we correctly implement improvement à la Symanzik?

Symanzik O(a^2) improvement of flow observables I

Following Lüscher & Weisz: use $4+1$ dimensional local theory:

$$
\begin{aligned}
S \quad = \quad & S_{\rm lat}[U;\{c_1,c_2,c_3\}] + a^4 \int_0^\infty \mathrm{d}t\, a^4 \sum_x \mathrm{tr}\Big(L_\mu(x,t) \\
& \times \Big\{ (\partial_t V_\mu(x,t))\, V_\mu(x,t)^\dagger + \partial_{x,\mu}\left(g_0^2 S_{\rm lat}[V]\right) \Big\} \Big)
\end{aligned}
$$

Expectation (follows reasoning by M. Lüscher in fermionic case):

- classical nature of the flow equation: expect no bulk O(a^2) counterterms.
- Hence all O(a^2) counterterms must be localized at the 4-d boundary: look for $d=6$ local fields, polynomial in the fundamental fields and their derivatives.
- The basis can be reduced by use of the flow equation!

Symanzik O(a^2) improvement of flow observables II

Candidate counterterms:

- The same counterterms as for the usual 4-d action $\Rightarrow$ parameterize by an offset $c_i \to c_i + \Delta c_i, (i = 1, 2, 3)$ of the terms in the general 4-dimensional action.
- New dimension 6 terms:
  1. $\partial_t \,\mathrm{tr}\,\{G_{\mu\nu}(x,t) G_{\mu\nu}(x,t)\}|_{t=0}$,
  2. $\mathrm{tr}\,\{L_\mu(x,t) \partial_t B_\mu(x,t)\}|_{t=0}$,
  3. $\mathrm{tr}\,\{L_\mu(x,t) D_\nu G_{\nu\mu}(x,t)\}|_{t=0}$.

  Use the flow equation:

$$\int \mathrm{d}^4 x \, \partial_t \,\mathrm{tr}\,\{G_{\mu\nu} G_{\mu\nu}\}\Big|_{t=0} = -4 \int \mathrm{d}^4 x \;\mathrm{tr}\,\{D_\mu G_{\mu\nu} D_\rho G_{\rho\nu}\}\Big|_{t=0}$$

$\Rightarrow$ modifies Δc_2

- last 2 terms related by flow equation $\Rightarrow$ stay e.g. with the last term only.

Symanzik O(a^2) improvement of flow observables III

- The Δc_i offsets cannot be arbitrary as this would ruin O(a^2) improvement of 4d observables!
- The Δc_2 term is the free parameter in the Lüscher-Weisz 1-parameter family of on-shell improved theories.
- Can choose $\Delta c_2 = 0$.

$\Rightarrow$ Only a single counterterm remains, choose

$$\int \mathrm{d}^4 x \ \mathrm{tr}\left\{L_\mu(x,t) D_\nu G_{\nu\mu}(x,t)\right\}|_{t=0}$$

- Previous analysis implies that this counterterm vanishes at tree level!

Classical expansion of flow equation

- Lattice flow equation:

$$a^2\left(\partial_t V_\mu(x,t)\right)V_\mu(x,t)^\dagger = -\partial_{x,\mu}\left(g_0^2 S_{\mathrm{lat}}[V]\right), \qquad V_\mu(x,0)=U_\mu(x)$$

- The $\mathrm{O}(a^2)$ term for the LW flow has a simple structure

$$\partial_t B_\mu = D_\nu G_{\nu\mu} - \tfrac{1}{12}a^2 D_\mu^2 D_\nu G_{\nu\mu} + O(a^3)$$

- This suggests a simple modification of the lattice flow equation (Zeuthen flow v2.0):

$$a^2\left(\partial_t V_\mu(x,t)\right)V_\mu(x,t)^\dagger = -\left(1+\tfrac{1}{12}a^2\nabla_\mu^*\nabla_\mu\right)\partial_{x,\mu}\left(g_0^2 S_{\mathrm{lat}}[V]\right)$$

- This removes <u>all</u> $\mathrm{O}(a^2)$ effects from the flow equation!

<u>Q:</u> How does the Zeuthen flow v1.0 compare to v2.0 (work in progress)?

Conclusions

- Investigation of tree-level $O(a^2)$ effects in gradient flow observables involving $E(x,t)$.
- Zeuthen flow: eliminates $O(a^2)$ tree-level effects from the flow equation, in all cases considered.
- Quenched scaling test: remaining cutoff effects very small!

$\Rightarrow$ complete solution for *any* flow observable?

- Symanzik $O(a^2)$ improvement in $4+1$ dimensional set-up: a single counterterm remains, vanishes at tree level.
- Classical expansion of flow equation and observables:
  1. Classical $O(a^2)$ improvement of $E(x,t)$ easily achieved.
  2. Classical expansion of the flow equation suggests a modification to the LW/Symanzik flow to remove all $O(a^2)$ effects.

$\Rightarrow$ expect: once classically $O(a^2)$ improved both the flow observable and the flow equation are $O(a^2)$ improved to all orders in the coupling g !