van Baal's legacy: From renormalons to bions

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Some of the work presented here is done in collaboration with :
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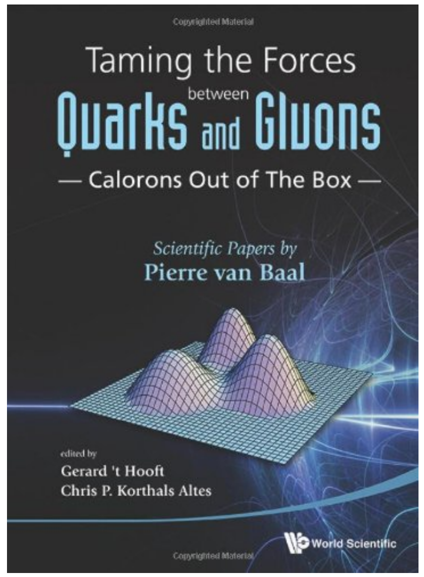
I met Pierre in person in May, 2008! (Picture is from 2002, stolen from a friend.)With his writings and especially his beautiful and insightful work on calorons, its monopole-instanton (fractional-instanton) constituents in 2006!



Since then, I have been developing ideas in this direction, hopefully, improving upon them. To me, he was one of the most influential **thinkers** in QCD.

Not knowing his condition, around 2007, I started to wonder, why this man, who wrote such brilliant papers was silent for some years.

Pierre has been taken away twice from us. After I met him, we became very close friends. With his wonderful sense of humor, he told me that “we met on his second Riemann sheet.”

Motivation: Can we make sense out of QFT? When is there a non-perturbative continuum definition of QFT? Picture from PierreFest2013

Dyson(50s), 't Hooft (77),

Today, I will tell you:

How a very deep problem (physical interpretation of IR-renormalons) that 't Hooft put forward in late 70s finds a resolution by invoking some ideas due to Pierre, physical principle of continuity, and a new mathematical formalism called resurgence theory!

YM/QCD on M4 and standard problems

1) Perturbation theory is an asymptotic *(divergent) expansion even after regularization and renormalization*. Is there a meaning to perturbation theory?

2) Invalidity of the semi-classical dilute instanton gas approximation on R4. DIG assumes inter-instanton separation is much larger than the instanton size, but the latter is a moduli, hence no meaning to the assumption.

3) ``Infrared embarrassment",e.g., large-instanton contribution to vacuum energy is IR-divergent, see Coleman's lectures.

4) A resolution of 2) was put forward by considering the theory in a small thermal box. But in the weak coupling regime, the theory always lands on the deconfined "regime". So, *no semi-classical approximation for the confined regime* until recently.

5) Incompatibility of large-N results with instantons.(better be so!)

6) The renormalon ambiguity, ('t Hooft,79), deeper, to be explained.

You may be surprised to hear that all of the above are interconnected according to the resurgence theory.

Since these are time-honored problems,
in order to say something new on them,
we must have both new physical perspective and
new mathematical tools. Here is my toolbox, two ideas
from physics and two from mathematics:

- Continuity
- (Reliable) Semi-classics
- Resurgence theory and Trans-series (Ecalle, 80s)
- Complex Morse Theory (or Picard-Lefschetz theory)

Simpler question: Can we make sense of the semi-classical expansion of QFT?

Argyres, MÜ,
Dunne, MÜ, 2012

$$f(\lambda\hbar) \sim \sum_{k=0}^{\infty} c_{(0,k)}\,(\lambda\hbar)^k + \sum_{n=1}^{\infty} (\lambda\hbar)^{-\beta_n}\, e^{-n\,A/(\lambda\hbar)} \sum_{k=0}^{\infty} c_{(n,k)}\,(\lambda\hbar)^k$$

pert. th. n-instanton factor pert. th. around n-instanton

All series appearing above are asymptotic, i.e., divergent as $c_{(0,k)} \sim k!$. The combined object is called trans-series following resurgence literature

Borel resummation idea: If $P(\lambda) \equiv P(g^2) = \sum_{q=0}^{\infty} a_q g^{2q}$ has convergent Borel transform

$$BP(t) := \sum_{q=0}^{\infty} \frac{a_q}{q!} t^q$$

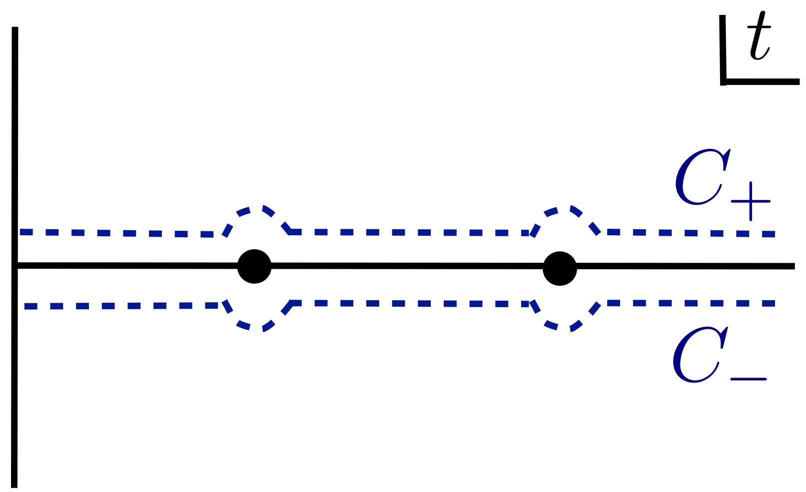
in neighborhood of $t = 0$, then

$$\mathbb{B}(g^2) = \frac{1}{g^2} \int_0^{\infty} BP(t) e^{-t/g^2}\, dt\ .$$

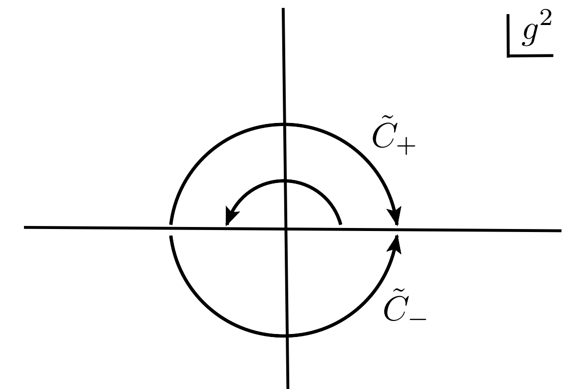
formally gives back $P(g^2)$, but is ambiguous if $BP(t)$ has singularities at $t \in \mathbb{R}^+$:

Borel plane and lateral (left/right) Borel sums

Directional (sectorial) Borel sum. $\mathcal{S}_\theta P(g^2) \equiv \mathbb{B}_\theta(g^2) = \frac{1}{g^2}\int_0^{\infty\cdot e^{i\theta}} BP(t)\, e^{-t/g^2}\, dt$



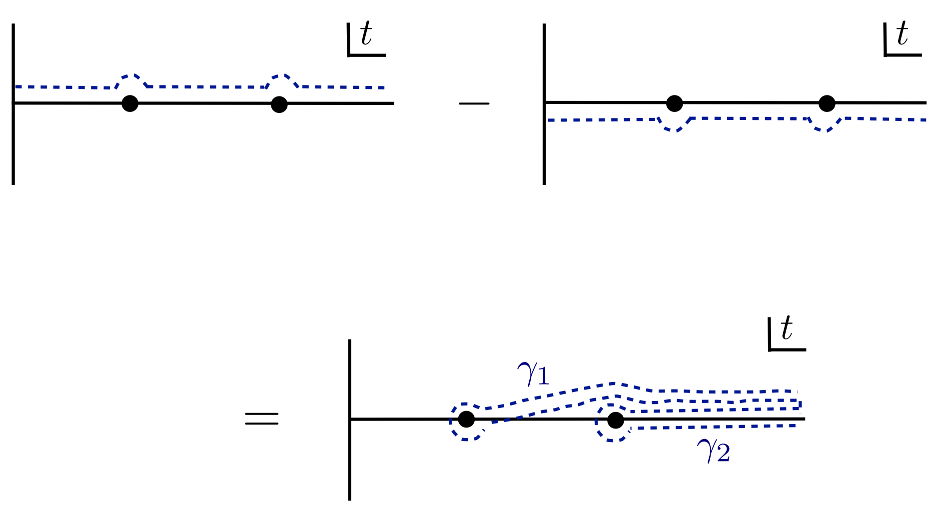
left and right Borel sums



Same structure by analytic continuation

$$\mathbb{B}_{0\pm}(|g^2|) = \mathrm{Re}\,\mathbb{B}_0(|g^2|) \pm i\,\mathrm{Im}\,\mathbb{B}_0(|g^2|), \qquad \mathrm{Im}\,\mathbb{B}_0(|g^2|) \sim e^{-2S_I} \sim e^{-2A/g^2}$$

The *non-equality* of the left and right Borel sum means the series is *non-Borel summable or ambiguous*. The ambiguity has the same form of a 2-instanton factor (not 1). The measure of ambiguity (Stokes automorphism/jump in g-space interpretation):



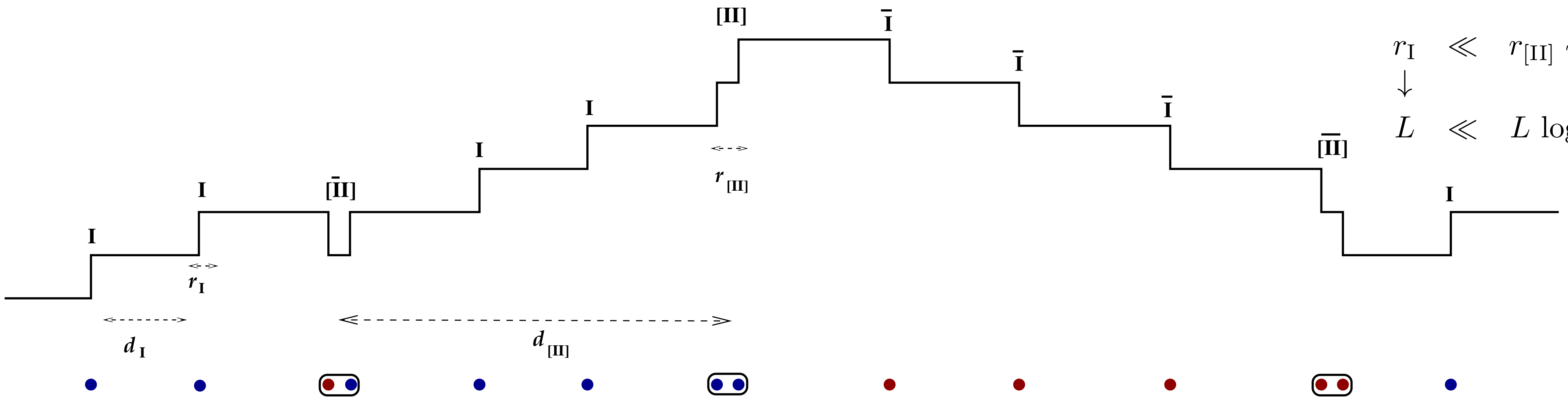
$$\mathcal{S}_{\theta+} = \mathcal{S}_{\theta-} \circ \underline{\mathfrak{S}}_\theta \equiv \mathcal{S}_{\theta-} \circ (\mathbf{1} - \mathrm{Disc}_{\theta-}),$$

$$\mathrm{Disc}_{\theta-}\,\mathbb{B} \sim e^{-t_1/g^2} + e^{-t_2/g^2} + \ldots \qquad t_i \in e^{i\theta}\mathbb{R}^+$$

Jean Ecalle, 80s

Bogomolny--Zinn-Justin (BZJ) prescription

Bogomolny-Zinn-Justin prescription in QM (80s): done for double well potential, but consider a periodic potential. Dilute instanton, molecular instanton gas.

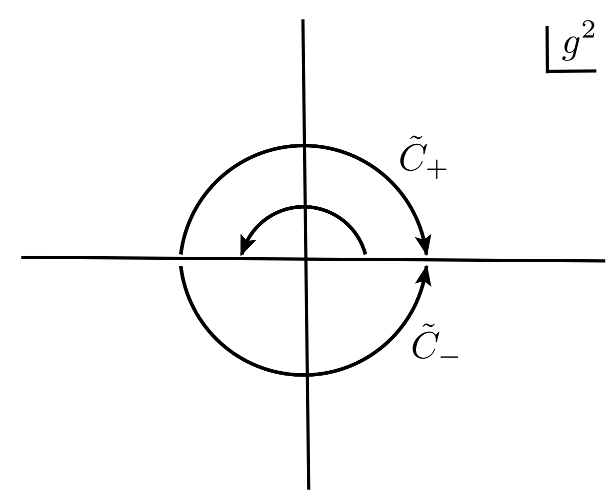


$$\begin{array}{ccccccc} r_{\mathrm{I}} & \ll & r_{[\mathrm{II}]} \sim \ell_{\mathrm{qzm}} & \ll & d_{\mathrm{I}} & \ll & d_{[\mathrm{II}]}, \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ L & \ll & L\,\log\left(\frac{1}{g^2}\right) & \ll & L\,e^{S_0} & \ll & L\,e^{2S_0}. \end{array}$$

How to make sense of topological molecules (or molecular instantons)? Why do we even need a molecular instanton? (Balitsky-Yung in SUSY QM, (86))

Naive calculation of I-anti-I amplitude: meaningless (why?) at $g^2 > 0$. The quasi-zero mode integral is dominated at small-separations where a molecular instanton is meaningless. Continue to $g^2 < 0$, evaluate the integral, and continue back to $g^2 > 0$: two fold-ambiguous!

$$[\mathcal{I}\bar{\mathcal{I}}]_{\theta=0^\pm} = \mathrm{Re}\,[\mathcal{I}\bar{\mathcal{I}}] + i\,\mathrm{Im}\,[\mathcal{I}\bar{\mathcal{I}}]_{\theta=0^\pm}$$

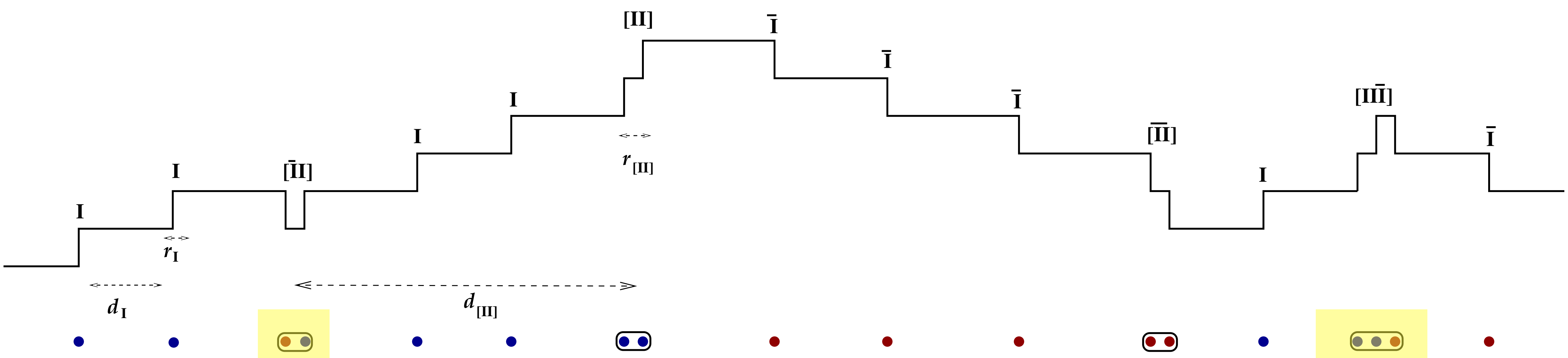


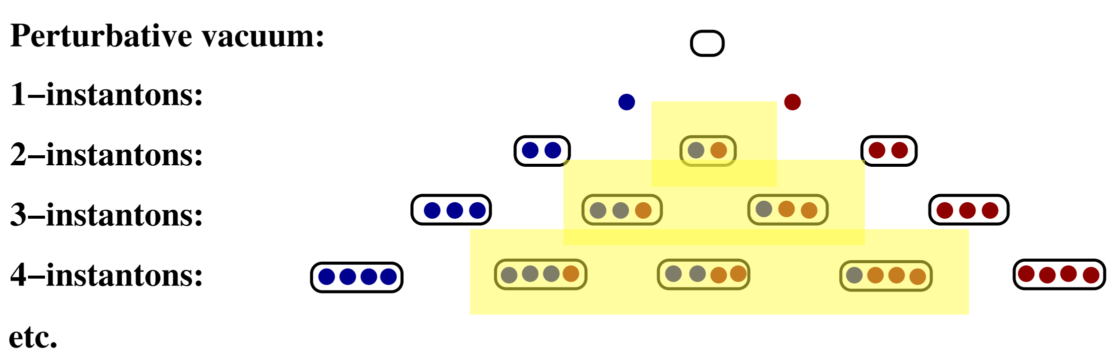
Why?: because we are on Stokes line, later....

Remarkable fact: Leading ambiguities cancel. “N.P. CONFLUENCE EQUATION”, elementary incidence of Borel-Ecalle summability which I will return:

$$\mathrm{Im}\,\mathbb{B}_{0,\theta=0^\pm} + \mathrm{Im}\,[\mathcal{I}\overline{\mathcal{I}}]_{\theta=0^\pm} = 0\ , \qquad \text{up to } O(e^{-4S_I})$$

The ambiguous topological configurations. All are non-BPS quasi-solutions!





Why is this happening? Zero-dim. prototype

Complex gradient-flow (or Picard-Lefschetz) equations.

$$Z^{0\mathrm{d}}(\lambda) = \frac{1}{\sqrt{\lambda}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx\, e^{-\frac{1}{2\lambda}\sin^2(x)}$$

Complexify everything, because thimbles lives in C!

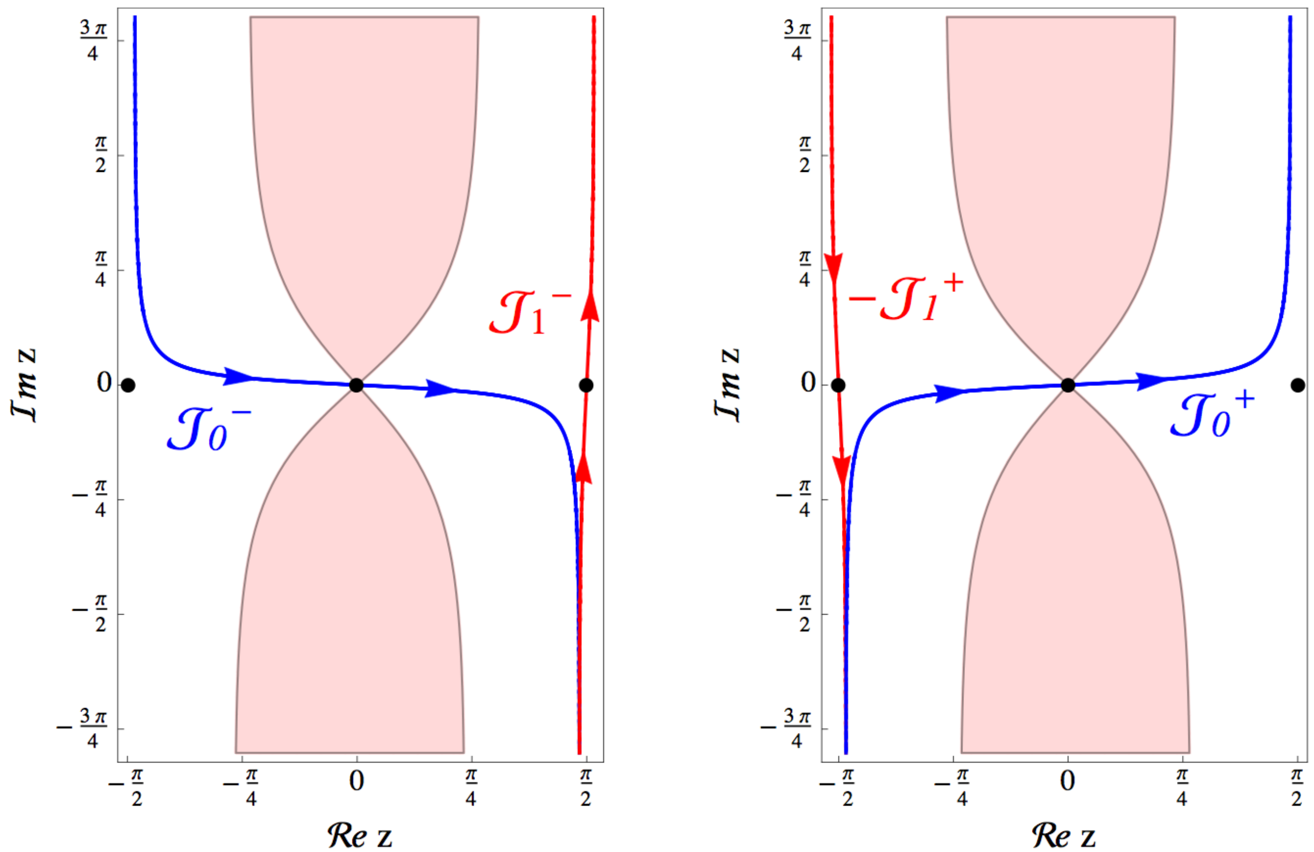


Figure 1. **Left:** Lefschetz thimbles at $\lambda = e^{i\theta}$ with $\theta = 0^-$: $\mathcal{J}_0 + \mathcal{J}_1$. **Right:** At $\theta = 0^+$. $\mathcal{J}_0 - \mathcal{J}_1$. We take $\theta = \mp 0.1$ to ease visualization.

$$\frac{dz}{dt} = -e^{-i\theta}\frac{\partial \bar{S}}{\partial \bar{z}}$$

$$\mathrm{Im} S(z)|_{\mathcal{J}_i} = \mathrm{Im} S(z_i),$$

$$I = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \longrightarrow \Sigma = \begin{cases} \mathcal{J}_0(0^-) + \mathcal{J}_1(0^-) \\ \mathcal{J}_0(0^+) - \mathcal{J}_1(0^+) \end{cases}$$

Borel sum = Integration over a thimble (down-ward manifold)

In the past, the interpretation was always obscure to me! This is a crystal clear interpretation!

Cancellation of ambiguity due to Jo-cycle (tail) jump and Stokes phenomenon

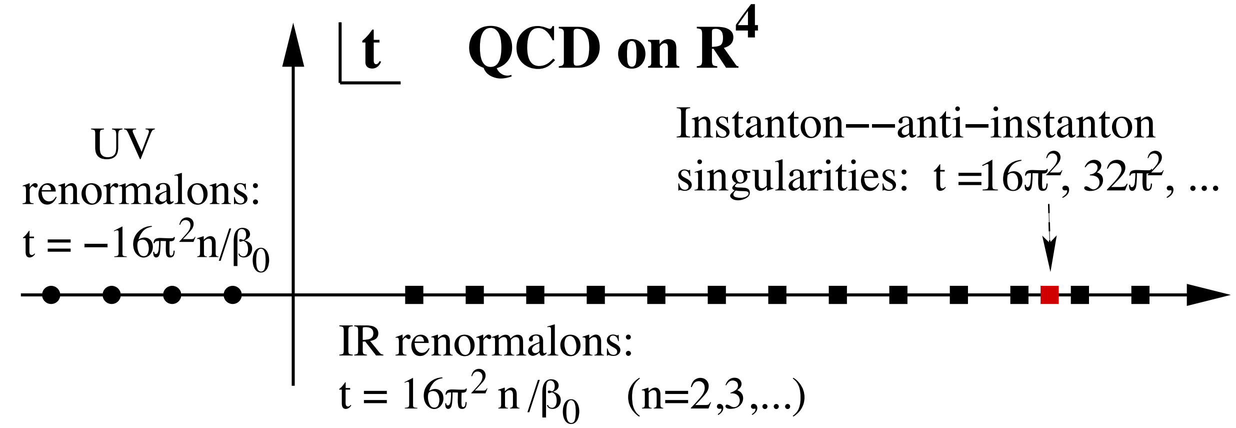
Can this work in QFT? QCD on R4 or CP(N-1) on R2?

't Hooft(79) :***No,*** *on R4,* Argyres, MÜ: ***Yes,*** *on R3 x S1,*
F. David(84), Beneke(93) : ***No,*** *on R2.* Dunne, MÜ: ***Yes***, *on R1 x S1*

Why doesn't it work, say for CP(N-1) on R2?

Instanton-anti-instanton contribution, calculated in some way, gives an $\pm i \exp[-2S_I]$.
Lipatov(77): Borel-transform BP(t) has singularities at $t_n = 2n\, g^2\, S_I$. (Modulo the standard IR problems with 2d instantons, also see Bogomolny-Fateyev(77)).

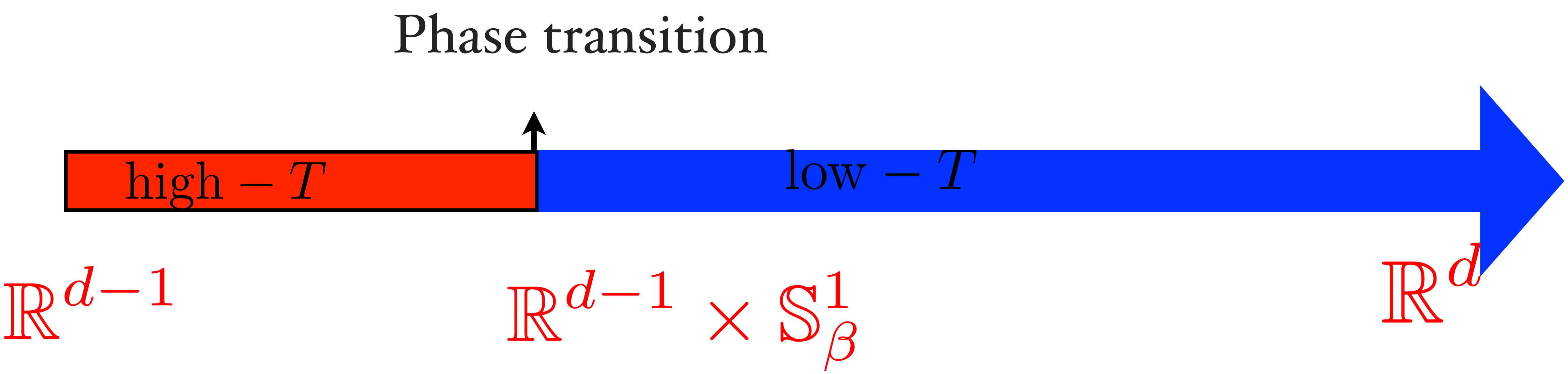
BUT, BP(t) has other (more important) singularities closer to the origin of the Borel-plane. (not due to factorial growth of number of diagrams!)



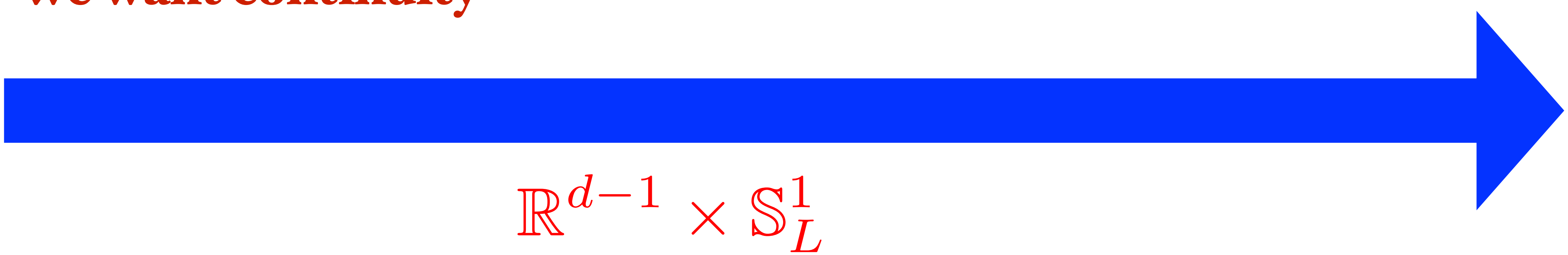
't Hooft called these IR-renormalon singularituies with the hope/expectation that they would be associated with a saddle point like instantons.
No such configuration is known!!

A real problem in QFT, means pert. theory, as is, ill-defined. How to cure starting from micro-dynamics?

The idea of continuity



We want continuity

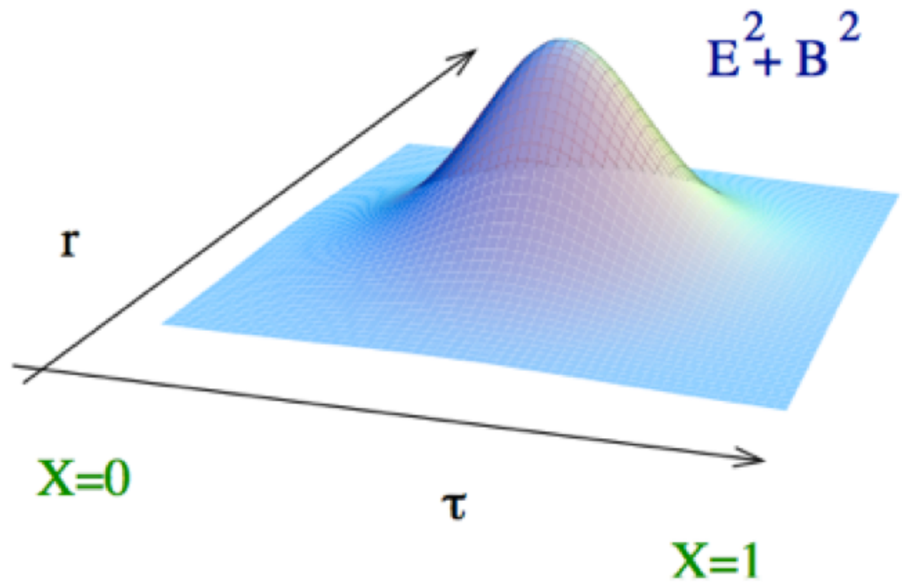


Thermal: Rapid crossover/phase transition at strong scale

Prevent both by using circle compactification, QCD(adj) with pbc, or double-trace deformation. (Yaffe, MU, Ogilvie, Myers, and others)

Periodic instantons (calorons)

Instanton solution in R^4 can be extended to solution on $R^3 \times S^1$



$$Q_{top} = \pm 1$$

$$P_\infty = 1 \quad Q_M^\alpha = 0$$

$SU(2)$ solution has $1 + 3 + 1 + 3 = 8$ bosonic zero modes

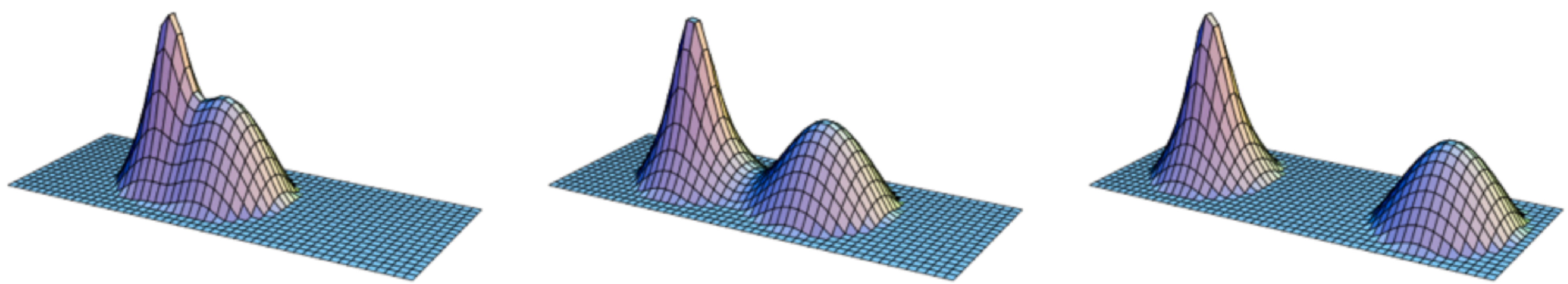
$$\int \frac{d\rho}{\rho^5} \int d^3x \, dx_4 \int dU \, e^{-2S_0} \qquad 2S_0 = \frac{8\pi^2}{g^2}$$

$4n_{adj}$ fermionic zero modes

$$\int d^2\zeta d^2\xi$$

Calorons at finite holonomy: monopole constituents

KvBLL (1998) construct calorons with non-trivial holonomy



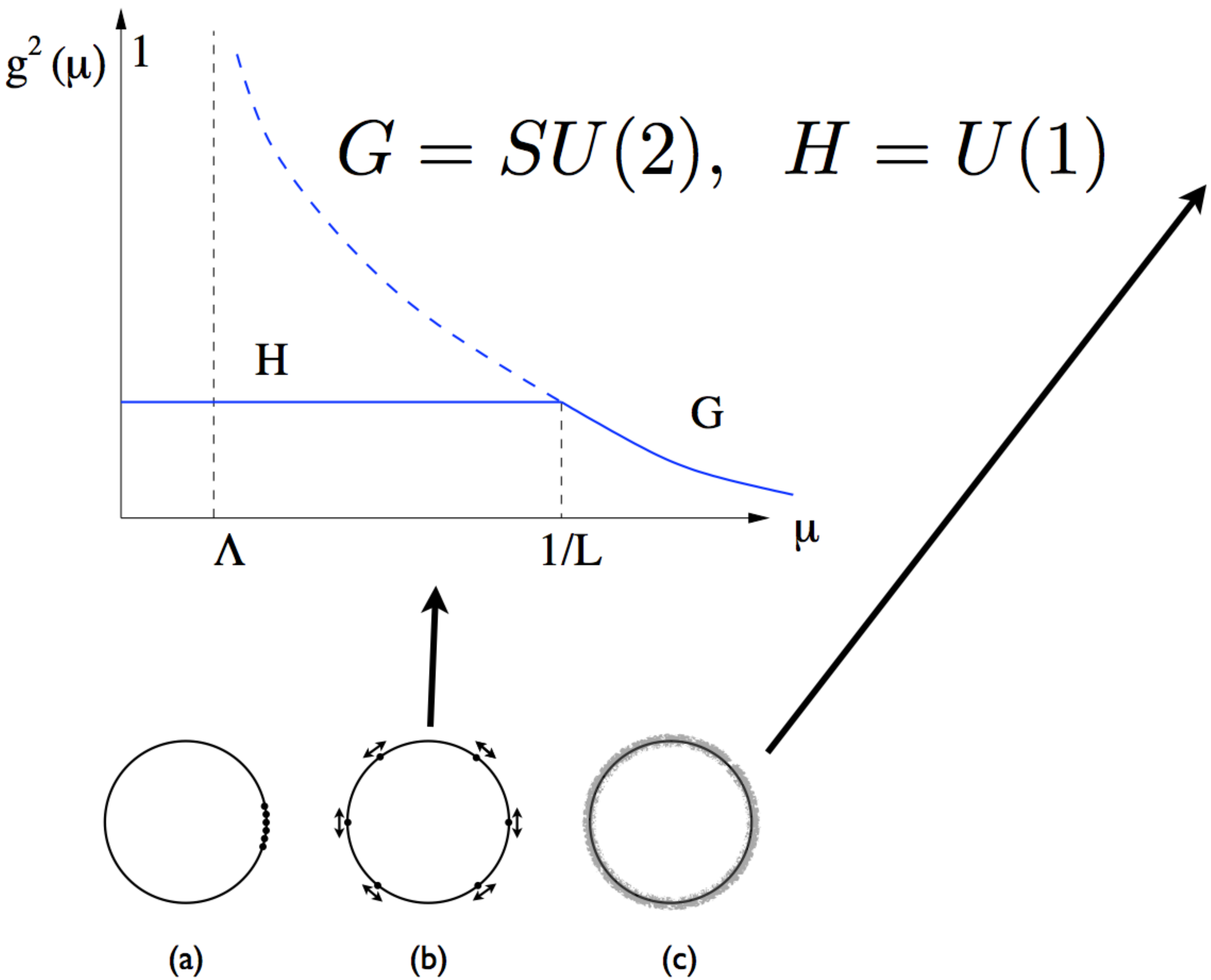
BPS and KK monopole constituents. Fractional topological charge, 1/2 at center symmetric point.

$2 \times (3+1) = 8$ bosonic zero modes, 2×2 fermionic ZM.

$$\int d\phi_1 \int d^3x_1 \int d^2\zeta \, e^{-S_1} \int d\phi_2 \int d^3x_2 \int d^2\xi \, e^{-S_2}$$

Monopole-instantons: **van Baal, Kraan, (97/98), Lee-Lu (98) Lee-Yi (97).**
One of the most important realization in NP-QCD!

Trivializing monopole-instantons



The cleverness of van Baal et.al.:
To realize mon. instantons in the strong coupling regime with a weak coupling intuition!

With deformations and pbc, this is now simpler. At the time, quite non-trivial task.

We can now also understand what the role of these monopole-instantons etc. in the calculable regime.
This is the recent progress. (2007-......)

IR in perturbation theory is a free theory of "photons". Is this perturbative fixed point destabilized non-perturbatively?

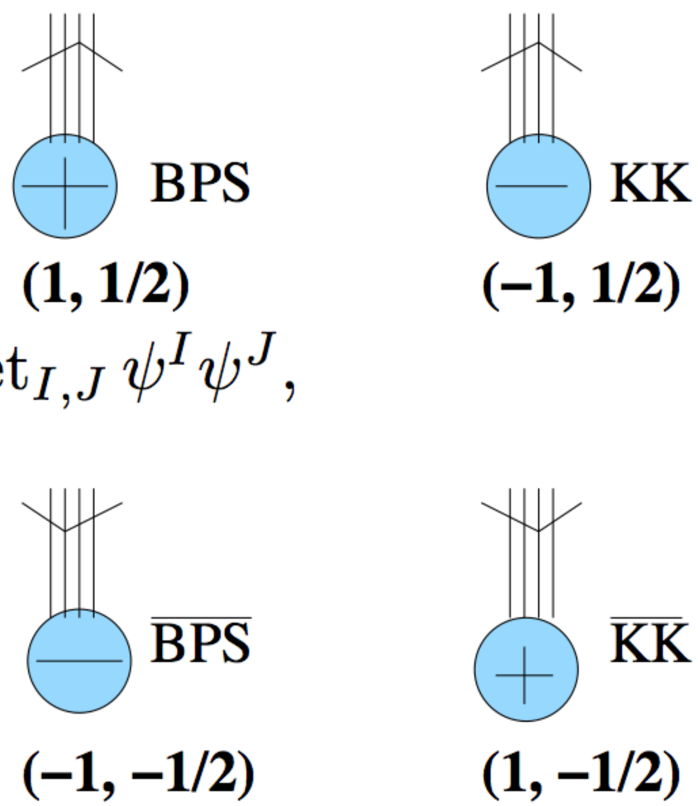
Topological excitations in QCD(adj), SU(2), Nf=2

MÜ 2007

$$\left(\int_{S^2} F,\ \int_{R^3\times S^1} F\tilde{F}\right)$$

index theorems
Callias 1978
E. Weinberg 1980
Nye-A.M.Singer, 2000
Poppitz, MU 2008
Atiyah-M.I.Singer 1975

Monopole-instantons



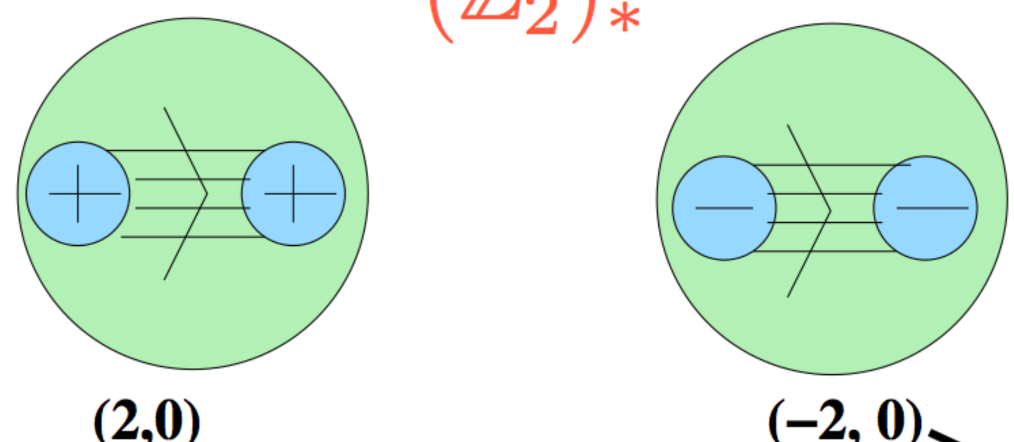
$e^{-S_0} e^{i\sigma} \det_{I,J} \psi^I \psi^J,$

$e^{-S_0} e^{i\sigma} \det_{I,J} \bar{\psi}^I \bar{\psi}^J$

Magnetic Bions

Mass gap for gauge fluctuations!

$(\mathbb{Z}_2)_*$



$$e^{-2S_0}\left(e^{2i\sigma} + e^{-2i\sigma}\right)$$

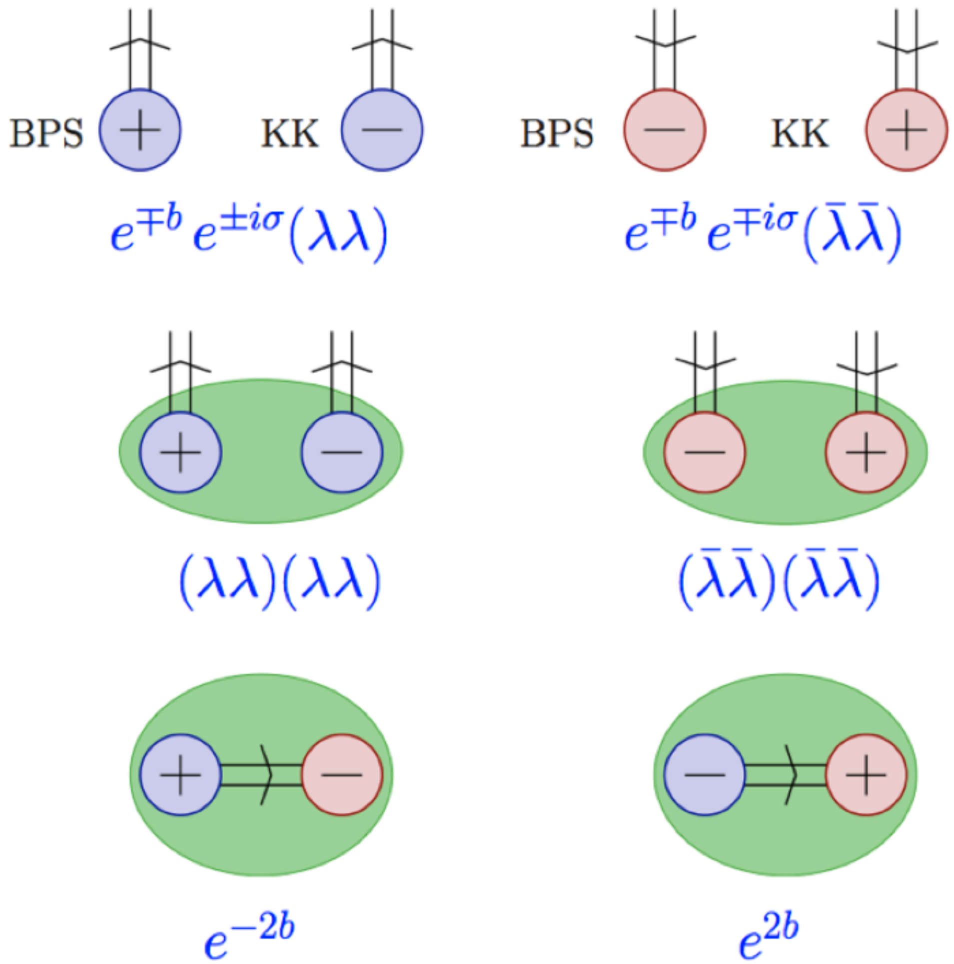
No net topological charge.

Discrete shift symmetry : $\quad \sigma \longrightarrow \sigma + \pi \qquad \psi^I \longrightarrow e^{i\frac{2\pi}{8}} \psi^I$

Crucial earlier work: van Baal, Kraan 97/98 and Lee, Lu, Yi, 97/98

Topological objects: Coupling to low energy fields

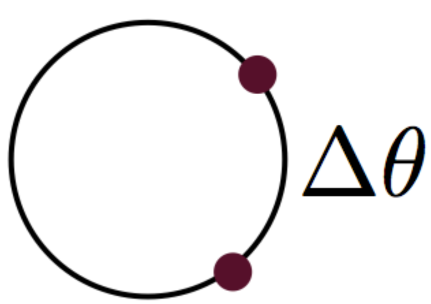
$$(Q_M, Q_{top}) = \left(\int_{S_2} B \cdot d\Sigma, \int_{R^3 \times S_1} F\tilde{F}\right)$$



monopoles
mass gap for fermions

instantons
Anomaly

Neutral bions
Center stabilizing NP-potential!



$$b = \frac{4\pi}{g^2}\Delta\theta$$

NP ambiguity in semi-classical expansion:
Disaster or blessing in disguise?

Naive calculation of typical neutral defect amplitude, as you may guess as per QM example, multi-fold ambiguous!

As it stands, this is a **disaster!** Semi-classical expansion at higher order is void of meaning! In QFT literature, people rarely discussed second or higher order effects in semi-classics, most likely, they thought no new phenomena would occur, and they would only calculate exponentially small subleading effects. **The truth is far more subtler!**

NP-ambiguity in PT Ambiguity in neutal-bions amplitude

$$0 = \mathrm{Im}\mathbb{B}_{[0,0]\pm} + \mathrm{Im}[\mathcal{B}_{ii}]_{\pm} \ , \qquad (\text{up to } e^{-4S_0}) \qquad \mathrm{YM, CP(N-1)}$$

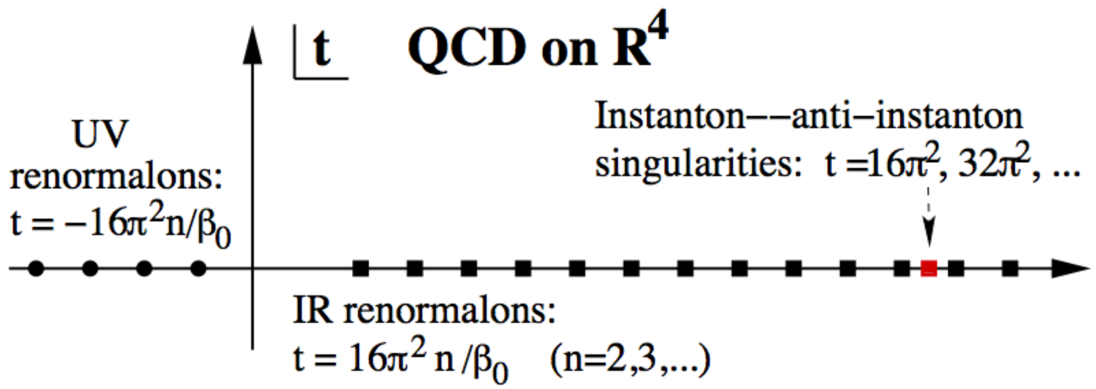
$$0 = \mathrm{Im}\mathbb{B}_{[0,0]\pm} + \mathrm{Im}[\mathcal{B}_{ij}\overline{\mathcal{B}}_{ij}]_{\pm} \ , \qquad (\text{up to } e^{-6S_0}) \qquad \mathrm{QCD(adj)}$$

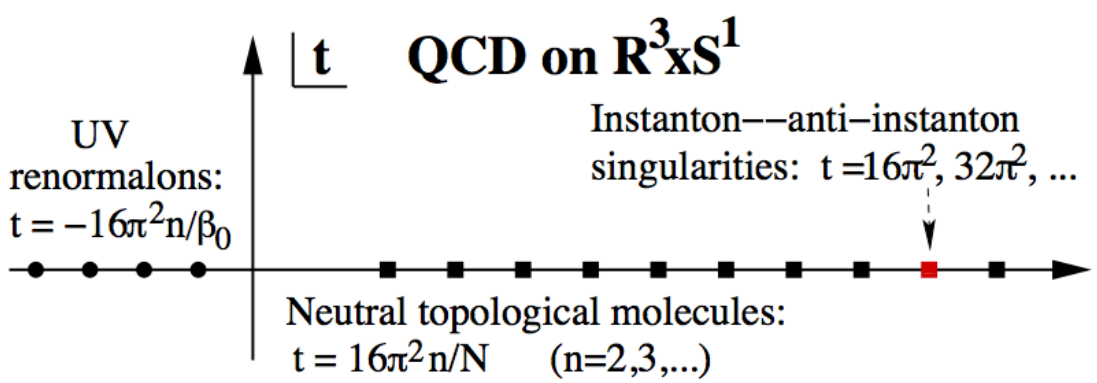
$$\mathrm{Im}[\mathcal{B}_{ii}]_{\pm} = \mathrm{Im}[\mathcal{M}_i\overline{\mathcal{M}}_i]_{\pm} \qquad \mathrm{Im}[\mathcal{B}_{ij}\overline{\mathcal{B}}_{ij}]_{\pm} = \mathrm{Im}[\mathcal{M}_i\overline{\mathcal{M}}_j\mathcal{M}_j\overline{\mathcal{M}}_i]_{\pm}$$

The ambiguities at order $\exp[-2S_I/N]$ cancel and
QFT is well-defined up to the ambiguities of order $\exp[-4S_I/N]$!
Ambiguities in the IR-renormalon territory as per 't Hooft, David, Beneke,....

Semi-classical renormalons as neutral bions

Claim (with Argyres in 4d) and (with Dunne in 2d): Neutral bions and neutral topological molecules are semi-classical realization of 't Hooft's elusive renormalons, and it is possible to make sense out of combined perturbative semi-classical expansion. We showed this only at leading (but most important) order. Subleading orders underway.





More than three decades ago, 't Hooft gave a famous set of (brilliant) lectures(79): *Can we make sense out of QCD?* He was thinking a non-perturbative continuum formulation. It seem plausible to me that in fact, we can, at least, in the semi-classical regime of QFT.

Picard-Lefschetz equations for YM theory (new)

Reminder: If S(A) is Chern-Simons functional in **3d,** the flow equations are **4d** instanton equations, crucial in an infinite dimensional version of Morse theory (i.e., Morse theory in field space.) This is crucial in Floer homology.
Relevant to Picard-Lefschetz equations is a complex version of CS-theory, which gives a complex generalization of 4d instanton equation:

$$\mathcal{F}_{\mu\nu} + e^{-i\theta} (\star \overline{\mathcal{F}})_{\mu\nu} = 0$$

MU, 05 in lattice-susy, hep-th/0603046
Kapustin-Witten 05 Geometric Langland, hep-th/0604151

In our case:

$$\frac{d\mathcal{A}^\mu}{dt} = -e^{-i\theta} \frac{\partial \bar{S}}{\partial \bar{\mathcal{A}}_\mu}$$

$$\mathcal{A}_\mu \in SL(N, \mathbb{C})$$

$$\frac{d\mathcal{A}^\mu}{dt} = -e^{-i\theta} \bar{\mathcal{D}}_\nu \bar{\mathcal{F}}^{\nu\mu}$$

Fixed points of flow are monopole-instantons, bions, etc.

Attach a down-ward flow manifold to each one of the critical point. It is plausible that in the semi-classical regime, this provides a (homology) cycle/Lefschetz thimble decomposition of the space of fields. The integrations over the homology cycles are finite by construction.

Due to certain properties of these PDEs, **plausibly,** this may provide a finite definition of gauge theory. (work in progress).