Correlation functions with Karsten-Wilczek fermions

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Introduction

Minimally doubled fermions are a category of lattice fermion actions that retains ultralocality and a chiral symmetry while having the minimal number of degenerate flavours – two – which are considered as a degenerate doublet of quarks. Karsten-Wilczek fermions [1] are a type of minimally doubled fermions that retains two original poles of the naïve Dirac operator on exactly one Euclidean axis. Since the poles of minimally doubled Dirac operators generally lie at different points in the Brillouin zone, the hypercubic lattice symmetry is inevitably broken.

Karsten-Wilczek action at tree level

The Karsten-Wilczek fermion action reads (with Wilczek parameter ζ, $4\zeta^2 > 1$)

$$\begin{aligned} S^f[\psi,\bar\psi,U] = a^4 \sum_{n\in\Lambda} \sum_{\mu=0}^{3} \bar\psi_n \frac{1}{2a}\gamma^\mu \left(U_n^\mu \psi_{n+\hat\mu} - U_{n-\hat\mu}^{\mu\dagger}\psi_{n-\hat\mu}\right) + m_0\bar\psi_n\psi_n \\ + \sum_{j=1}^{3} \bar\psi_n \frac{i\zeta}{2a}\gamma^0 \left(2\psi_n - U_n^j\psi_{n+\hat\jmath} - U_{n-\hat\jmath}^{j\dagger}\psi_{n-\hat\jmath}\right) \end{aligned} \tag{1}$$

and the gauge action is assumed to be Wilson's plaquette action. The standard discrete symmetries of this action were reviewed first in [2] and read

- γ^5-hermiticity of the Dirac operator,
- spatial rotations,
- spatial reflections (parity $\widehat{P}$),
- the product of charge conjugation ($\widehat{C}$) and time reflection ($\widehat{\Theta}$).

Eq. (1) **varies** under **charge conjugation** ($\widehat{C}$) or **time reflection** ($\widehat{\Theta}$), since each transformation flips the sign of the Karsten-Wilczek term (lower line in eq. (1)). However, $\widehat{C\Theta}$ together is a **symmetry transform**. Moreover, the action is invariant [3] under local vector and axial transforms (the latter only for $m_0 = 0$)

$$\psi_n \to e^{+i\alpha_n^V}\psi_n,\ \bar\psi_n \to \bar\psi_n e^{-i\alpha_n^V}, \tag{2}$$

$$\psi_n \to e^{i\alpha_n^A\gamma^5}\psi_n,\ \bar\psi_n \to \bar\psi_n e^{+i\alpha_n^A\gamma^5}. \tag{3}$$

There is one further **unitary transform** $\widehat{T}$,

$$\psi_n \to \widehat{T}\psi_n = T_n\psi_n, \quad \bar\psi_n \to (\widehat{T}\bar\psi_n) = \bar\psi_n T_n, \qquad T_n \equiv i\gamma^0\gamma^5(-1)^{n_0}, \tag{4}$$

which leaves the upper line of eq. (1) invariant but flips the sign of the Karsten-Wilczek term. It is related to the remnant of the discrete subgroup of the $U(4)$-symmetry of the naïve Dirac action (cf. [4]) and permits two further symmetry transformations $\widehat{TC}$ and $\widehat{T\Theta}$. The latter has been pointed out as "mirror fermion symmetry" in [5]. Hence, there is **one further independent** non-standard discrete **symmetry** .

Counterterms

The action in eq. (1) requires inclusion of one relevant and two marginal counterterms that were perturbatively calculated in [3] at one-loop level,

$$S^3[\psi,\bar\psi] = c(g_0^2)\ a^4 \sum_{n\in\Lambda} \bar\psi_n \frac{i}{a}\gamma^0\psi_n \tag{5}$$

$$S^{4f}[\psi,\bar\psi,U] = d(g_0^2)\ a^4 \sum_{n\in\Lambda} \bar\psi_n \frac{1}{2a}\gamma^0 \left(U_n^0\psi_{n+\hat0} - U_{n-\hat0}^{0\dagger}\psi_{n-\hat0}\right), \tag{6}$$

$$S^{4g}[U] = d_p(g_0^2)\frac{\beta}{3}\sum_{n\in\Lambda}\sum_{j=1}^{3} \mathrm{ReTr}(1 - U_n^{j0}), \tag{7}$$

where $U_n^{\mu0}$ is the temporal plaquette at site n. In particular, eq. (5) flips its sign under $\widehat{C}$, $\widehat{\Theta}$ and $\widehat{T}$. Nevertheless, both eqs. (6) and (7) are invariant under $\widehat{C}$, $\widehat{\Theta}$ and $\widehat{T}$. The one-loop coefficients read

$$\begin{aligned} c_{1L}(g_0^2) &= -29.5320\, C_F\, b(g_0^2), & d_{1L}(g_0^2) &= -0.125540\, C_f\, b(g_0^2), \\ d_{P\,1L}(g_0^2) &= -12.69766\, C_2\, b(g_0^2), & b(g_0^2) &= g_0^2/(16\pi^2). \end{aligned} \tag{8}$$

A non-perturbative tuning scheme for $c(g_0^2)$ that minimises the anisotropy was presented in [6]. A new method is introduced below.

Higher order operators

Higher order operators are restricted by $\widehat{C\Theta}$- and chiral symmetry in particular. Dimension five continuum operators are collected in a table and referred to as $\mathcal{O}_{RC}$, where R and C are row and column indices.

$\bar\psi\ m_0^2\psi$	$\bar\psi\ m_0\not{D}\psi$	$\bar\psi\ m_0\gamma^0 D^0\psi$	$m_0 F^{\mu\nu}F^{\mu\nu}$	$m_0 F^{i0}F^{i0}$
$\bar\psi\ im_0\sum_{j=1}^{3}\Sigma^{0j}D^j\psi$	$\bar\psi\ i\sum_{j=1}^{3}\gamma^j F^{0j}\psi$	$\bar\psi\ i\{D^0,\not{D}\}\psi$	$\bar\psi\ i\,m_0 D^0\psi$	$\bar\psi\ i\,m_0^2\gamma^0\psi$
$\bar\psi\ i\gamma^0\sum_{j=1}^{3}D^jD^j\psi$	$\bar\psi\ i\gamma^0 D^0D^0\psi$	$\bar\psi\ i\gamma^0\sum_{j<k=1}^{3}\Sigma^{jk}F^{jk}\psi$	$\bar\psi\ i(\gamma^0\not{D}\not{D} + \not{D}\not{D}\gamma^0)\psi$	
$F^{kj}(D^kF^{0j})0$	$F^{0j}(D^0F^{0j})$	$*F^{j0}(D^0F^{j0})$	$*F^{jk}(D^0F^{jk})$	

a) Chiral symmetry prohibits $\mathcal{O}(am_0)$-corrections as in $\mathcal{O}_{11}$, …, $\mathcal{O}_{15}$.

b) Due to field equations: $2\mathcal{O}_{21} = \mathcal{O}_{22} + \mathcal{O}(a)$ and $\mathcal{O}_{21} = 0 + \mathcal{O}(a)$.

c) Field equations $\mathcal{O}_{23} = 2\mathcal{O}_{25} + \mathcal{O}(a)$ and $\mathcal{O}_{24} = -\mathcal{O}_{25} + \mathcal{O}(a)$ eliminate $\mathcal{O}_{23}$ and $\mathcal{O}_{24}$. $\mathcal{O}_{25}$ is absorbed into c of eq. (5).

d) $\mathcal{O}_{31}$ is the Karsten-Wilczek term's continuum form. Though it satisfies $2(\mathcal{O}_{31} + \mathcal{O}_{32}) + \mathcal{O}_{33} = \mathcal{O}_{34}$, no operator can be eliminated.

e) The field equation $\mathcal{O}_{34} = 2\mathcal{O}_{25} + \mathcal{O}(a)$ eliminates $\mathcal{O}_{34}$.

f) As $\mathcal{O}_{41}$ and $\mathcal{O}_{42}$ lack invariance under $\widehat{\Theta}$ but are $\widehat{C}$- and $\widehat{T}$-invariant and $\mathcal{O}_{43}$ and $\mathcal{O}_{44}$ lack invariance under parity, they are prohibited.

Hence, $\mathcal{O}_{32}$ and $\mathcal{O}_{33}$ are a complete set of independent additional terms.

Lattice operators

Lattice operators for $\mathcal{O}_{33}$ can be constructed using $\widehat{F}_n^{kl}$ as in the clover term for Wilson fermions. Variance under $\widehat{T}$ requires **next-to-next neighbour** terms in $\mathcal{O}_{32}$. The two additional independent dimension five lattice operators are chosen as:

$$\mathcal{O}_{32}: \quad S^{5t} = c_t(g_0)a^4\sum_{n\in\Lambda}\bar\psi_n\frac{1}{4a}\left(2\psi_n - U_n^0U_{n+\hat0}^0\psi_{n+2\hat0} - U_{n-\hat0}^{0\dagger}U_{n-2\hat0}^{0\dagger}\psi_{n-2\hat0}\right), \tag{9}$$

$$\mathcal{O}_{33}: \quad S^{5B} = c_B(g_0)a^4\sum_{n\in\Lambda}\bar\psi_n\sum_{j,k,l=1}^{3}\epsilon^{jkl}\gamma^5\gamma^j\widehat{F}_n^{kl}\psi_n. \tag{10}$$

The **gauge action** does not receive $\mathcal{O}(a)$-terms and retains separate **invariance** under $\widehat{C}$, $\widehat{\Theta}$ and $\widehat{T}$. This invariance obviously holds for any observables without valence quarks in quenched QCD.

Each **dimension five operator varies** under $\widehat{C}$, $\widehat{\Theta}$ and $\widehat{T}$ ($\widehat{C\Theta}$, $\widehat{TC}$ and $\widehat{T\Theta}$ are symmetries). Chiral symmetry prohibits chiral corrections to bare parameters of $\mathcal{O}(am_0)$.

Time reflection of correlation functions

Invariance of the Karsten-Wilczek action under either $\widehat{C\Theta}$- or $\widehat{T\Theta}$-transformations is reflected in invariance of the Hamiltonian $\widehat{H}$. A generic correlation function $\mathcal{C}(t_f - t_i)$ between initial and final states $\widehat{O}_i^\dagger|\Omega\rangle$ and $\widehat{O}_f^\dagger|\Omega\rangle$ using the transfer matrix $\widehat{U}(t) = e^{-\widehat{H}t}$ reads

$$\mathcal{C}(t_f - t_i) = \langle\Omega|\widehat{O}_f e^{-\widehat{H}(t_f-t_i)}\widehat{O}_i^\dagger|\Omega\rangle, \tag{11}$$

where $|\Omega\rangle$ is the invariant vacuum. In particular, $\widehat{C}|\Omega\rangle = |\Omega\rangle = \widehat{\Theta}|\Omega\rangle$. Due to invariance of $\widehat{H}$ under $\widehat{C\Theta}$, one has $\widehat{U}(t) = \widehat{C}^\dagger\widehat{\Theta}^\dagger\widehat{U}(-t)\widehat{C}\widehat{\Theta}$:

$$\mathcal{C}(t_f - t_i) = \langle\Omega|\widehat{O}_f\widehat{C}^\dagger\widehat{\Theta}^\dagger e^{-\widehat{H}(t_i-t_f)}\widehat{\Theta}\widehat{C}\widehat{O}_i^\dagger|\Omega\rangle. \tag{12}$$

If $\widehat{O}_f$ and $\widehat{O}_i$ both either commute or anticommute with $\widehat{C}$, then

$$\mathcal{C}(t_f - t_i) = \langle\Omega|\widehat{C}^\dagger\widehat{O}_f\widehat{\Theta}^\dagger e^{-\widehat{H}(t_i-t_f)}\widehat{\Theta}\widehat{O}_i^\dagger\widehat{C}|\Omega\rangle. \tag{13}$$

Insertion of $\mathbf{1} = \widehat{\Theta}\widehat{\Theta}^\dagger$ between vacuum states and operators yields

$$\begin{aligned} \mathcal{C}(t_f - t_i) &= \langle\Omega|\widehat{\Theta}^\dagger\widehat{\Theta}\widehat{O}_f\widehat{\Theta}^\dagger e^{-\widehat{H}(t_i-t_f)}\widehat{\Theta}\widehat{O}_i^\dagger\widehat{\Theta}^\dagger\widehat{\Theta}|\Omega\rangle & (14) \\ &= \langle\Omega|(\widehat{\Theta}\widehat{O}_f\widehat{\Theta}^\dagger)e^{-\widehat{H}(t_i-t_f)}(\widehat{\Theta}\widehat{O}_i^\dagger\widehat{\Theta}^\dagger)|\Omega\rangle = \widehat{\Theta}\mathcal{C}(t_f - t_i). & (15) \end{aligned}$$

This is **manifest $\widehat{\Theta}$-invariance** of $\widehat{C}$-invariant correlation functions. Since **dimension five operators explicitly break** each of $\widehat{C}$-, $\widehat{\Theta}$- and $\widehat{T}$-invariance, odd powers of these operators cancel and **discretisation effects are suppressed** to $\mathcal{O}(a^2)$ in $\widehat{C}$-invariant cases. Analogous results rely on $\widehat{T}$ instead of $\widehat{C}$ and $\widehat{O}_{i,f}$ that both either commute or anticommute with $\widehat{T}$. As **gluonic operators** commute with $\widehat{T}$, $\widehat{\Theta}$-invariance and $\mathcal{O}(a)$-suppression are apparent.

Caveat: Generic composite fields are not $\mathcal{O}(a)$-improved.

Decomposition of interacting fields

The free field has two components related by $\widehat{T}$ of eq. (4),

$$\psi_n = \sum_m g_{n,m}^\phi\phi_m + T_n g_{n,m}^\chi\chi_m,\ \bar\psi_n = \sum_m \bar\phi_m g_{m,n}^{\phi\dagger} + \bar\chi_m g_{m,n}^{\chi\dagger}T_n, \tag{16}$$

where two kernels $g_{m,n}^\phi$, $g_{m,n}^\chi$ implement an **energy smearing** (cf. [7]) and satisfy $g_{m,n}^\chi = T_m g_{m,n}^\phi T_n = \delta_{m,n} + \mathcal{O}(a)$. These two components lead in most correlation functions to **terms with alternating sign** similar to cases of naïve or staggered fermions. With interactions, the operator of eq. (5) cancels effects that would spoil the continuum limit. In the following, approximate tuning is assumed, where the **mismatch of tuning**, $\delta c = c - c(g_0^2)$, is considered small. A local field transform,

$$\psi_n \to \psi_n^c = \xi_n\psi_n,\ \bar\psi_n \to \bar\psi_n^c = \bar\psi_n\xi_n^*, \qquad \xi_n \equiv e^{i\varphi n_0},\ \varphi \equiv \frac{\delta c}{1+d}, \tag{17}$$

that modifies the temporal boundary condition (cf. [2]), **absorbs the mismatch** δc into the definition of the fields ψ^c, $\bar\psi^c$ at leading order. As the relevant operator of eq. (5) anticommutes with $\widehat{T}$, absorption of δc into two interacting components requires **opposite phases**,

$$\begin{aligned} \psi_n &= \sum_m \xi_n g_{n,m}^\phi[U]\phi_m^c + \xi_n^* T_n g_{n,m}^\chi[U]\chi_m^c \\ \bar\psi_n &= \sum_m \bar\phi_m^c g_{m,n}^{\phi\dagger}[U]\xi_n^* + \bar\chi_m^c g_{m,n}^{\chi\dagger}[U]T_n\xi_n \end{aligned}, \tag{18}$$

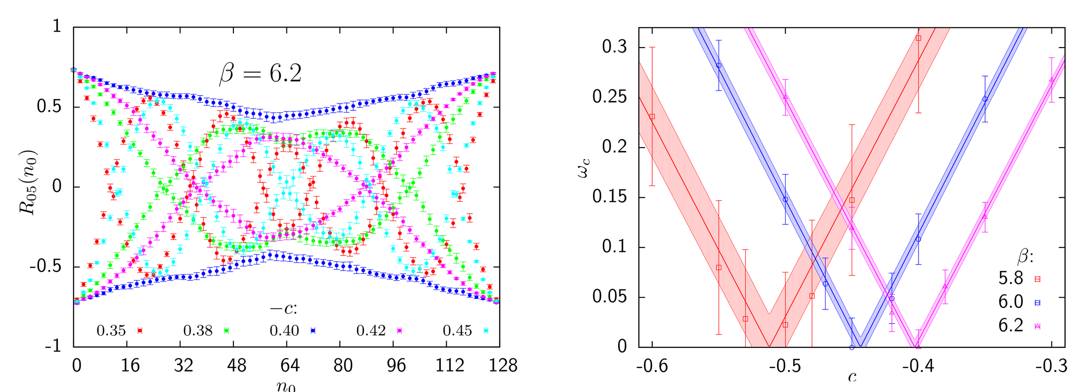
where kernels $g_{m,n}^\phi[U]$, $g_{m,n}^\chi[U]$ have been outfitted with Wilson lines. This definition yields a decomposition of local bilinears at leading order,

$$\begin{aligned} \bar\psi_n\mathcal{M}\psi_n = \xi_n^2\bar\chi_k g_{k,n}^{\chi\dagger}[U]T_n\mathcal{M}g_{n,l}^\phi[U]\phi_l + \xi_n^{*2}\bar\phi_k g_{k,n}^{\phi\dagger}[U]\mathcal{M}T_n g_{n,l}^\chi[U]\chi_l \\ + \bar\phi_k g_{k,n}^{\phi\dagger}[U]\mathcal{M}g_{n,l}^\phi[U]\phi_l + \bar\chi_k g_{k,n}^{\chi\dagger}[U]T_0\mathcal{M}T_0 g_{n,l}^\chi[U]\chi_l. \end{aligned} \tag{19}$$

Dirac structures of mixed (e.g. $\bar\phi\mathcal{M}T_0\chi$) and pure (e.g. $\bar\phi\mathcal{M}\phi$) terms differ by T_0. Phase factors $(-e^{\pm 2i\varphi})^{n_0}$ in mixed terms in eq. (19) generate **oscillations in correlation functions that depend on the mismatch** δc. Except for this δc-dependence, this pattern resembles naïve or staggered fermions. For Karsten-Wilczek fermions with mismatch δc, the frequency is shifted by $\omega_c = 2|\varphi|$ unless c is tuned properly. Since this derivation relies on kernels $g_{m,n}^{\phi,\chi}[U]$ that are not exactly known, simulations must put the conclusions to a test.

Oscillations in correlation functions

Good frequency resolution requires a long time direction. The frequency shift is studied on 10 quenched configurations of 128×24^3-lattices with pseudoscalar masses $r_0M_{55} \simeq 450\,\mathrm{MeV}$ and perturbatively tuned d.

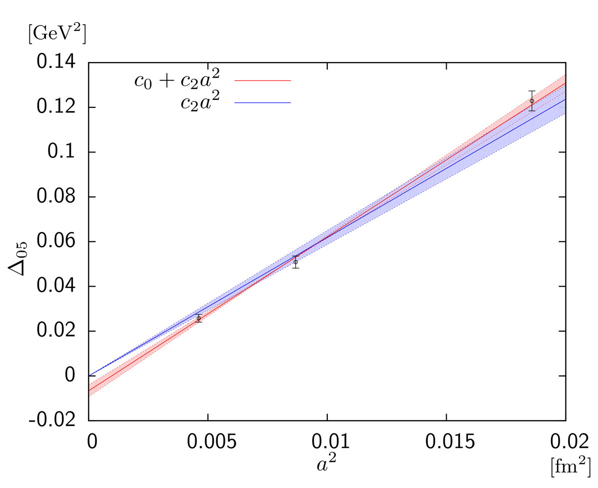


Since mixed terms in the γ^0-correlator $\mathcal{C}_{00}(n_0)$ generate oscillating pseudoscalar contributions, oscillations are apparent in the ratio $R_{05}(n_0) = \mathcal{C}_{00}(n_0)/\mathcal{C}_{55}(n_0)$. **Pseudoscalar mass splitting**,

$$\Delta_{05} = r_0^2(M_{00}^2 - M_{55}^2), \tag{20}$$

causes a small exponential decay that **broadens the peaks in frequency spectra** and causes a systematical error that dominates over the statistical error. The functional form of the frequency shift as $\omega_c \propto |c - c_0|$ agrees with expectations ($\omega_c = 2|\varphi| = 2|\frac{c-c(g_0^2)}{1+d}|$).

Pseudoscalar mass splitting

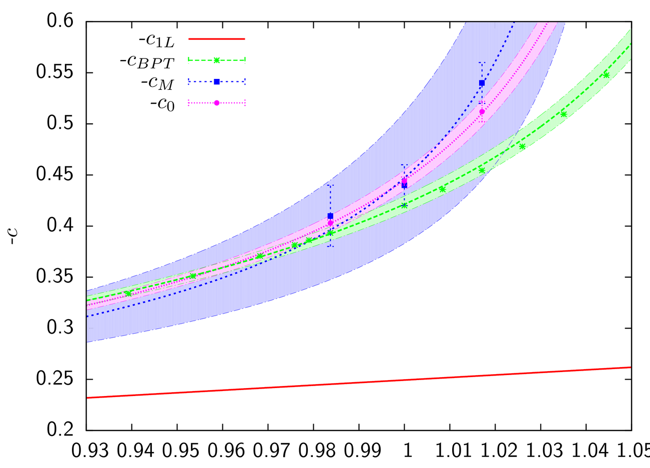


The **mass splitting** Δ_{05} within $250\mathrm{MeV} \lesssim M_{55} \lesssim 650\,\mathrm{MeV}$ is **nearly constant** and approaches the continuum limit as $\mathcal{O}(a^2)$. 200(100) quenched configurations on $48\times24^3(32^3)$-lattices with $\beta = 5.8, 6.0(6.2)$ are used.

Thus, the dominant systematic error of c_0 diminishes for finer lattices.

Non-perturbative tuning

Various methods for non-perturbative tuning of c are compared here. Interpolations use a Padé fit up to $\mathcal{O}(g_0^4)$ with $\mathcal{O}(g_0^2)$ fixed by eq. (8).



Fit parameters are N and D,

$$c(g_0^2) = \frac{c_{1L}(g_0^2) + N\,g_0^4}{1 + D\,g_0^2}, \tag{21}$$

The parameters for c_0 read

$$\begin{aligned} N_0 &= -0.2163(8) \\ D_0 &= -0.926(2) \end{aligned}. \tag{22}$$

c_0 defined by $\omega_c = 0$ and c_M defined by minimal mass anisotropy [6] **agree within errors**, but c_M has much larger uncertainties. For fine lattices ($a \le 0.06\,\mathrm{fm}$), c_{BPT} from boosted perturbation theory [8] is consistent. Thus, use of d_{BPT} appears justified as well.

Literature

[1] L. Karsten - Phys. Lett. B104, 315 (1981); F. Wilczek - PRL 59 (1987) 2397

[2] P.F. Bedaque, M.I. Buchoff, B.C. Tiburzi, A. Walker-Loud - Phys. Lett. B662 (2008), Phys. Rev. D78; PoS Lattice2008 068

[3] S. Capitani, M. Creutz, J.H.W., H. Wittig - JHEP 1009 (2010) 027, arXiv: 1006.2009

[4] L. Karsten, J. Smit - Nucl. Phys. B183 (1981) 103

[5] M. Pernici - Phys. Lett. B346 (1995) 99

[6] J.H.W., S. Capitani, H. Wittig - PoS Lattice2013 122, arXiv: 1312.0488

[7] B.C. Tiburzi - Phys. Rev. D82 (2010) 034511

[8] G.P. Lepage, P.B. Mackenzie - Phys. Rev. D48 (1993) 2250