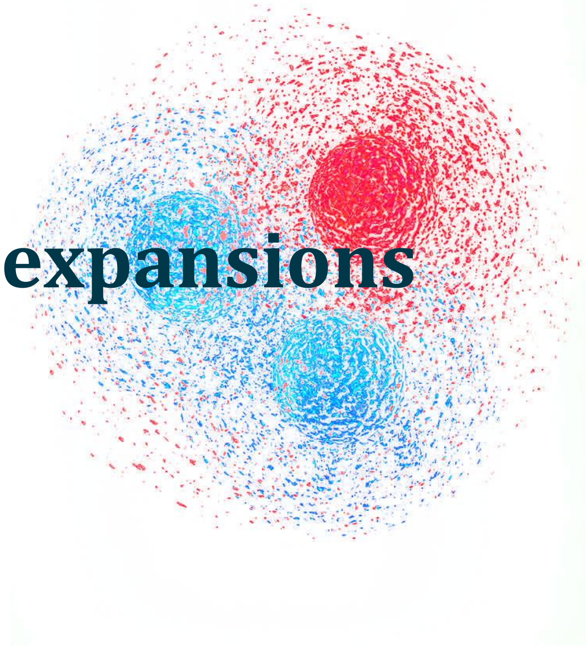
Locally smeared operator product expansions



Chris Monahan

with Kostas Orginos

College of William and Mary/JLab

Operator mixing on the lattice

Rotational symmetry broken on the lattice to cubic symmetry

1. operators mix under renormalisation on the lattice

2. power divergent mixing between operators of different mass dimension

Operator mixing on the lattice

Rotational symmetry broken on the lattice to cubic symmetry

1. operators mix under renormalisation on the lattice

2. power divergent mixing between operators of different mass dimension

for example

twist expansion of parton distribution functions

(Twist-2) operator mixing on the lattice

Parton distribution functions reflect internal structure of nucleons

- defined on the light-cone
- Mellin moments of parton distribution functions
  ~ matrix elements of "twist" (dimension - spin) operators
- twist-2 operators dominate in Bjorken limit

$$\overline{q}\,\gamma_{\{\mu_1} D_{\mu_2} \ldots D_{\mu_n\}}\, q$$

- power divergent mixing

e.g. $$\overline{q}\,\gamma_\mu D_\nu D_\nu\, q \sim \frac{1}{a^2}\,\overline{q}\,\gamma_\mu\, q$$

- limits lattice calculations to first four moments

Monahan and Orginos, poster at this conference

Operator mixing on the lattice

Rotational symmetry broken on the lattice to cubic symmetry

1. operators mix under renormalisation on the lattice

2. power divergent mixing between operators of different mass dimension

Smearing partially restores rotational symmetry

reduces operator mixing

Davoudi and Savage, Phys. Rev. 86 (2012) 054505

Operator mixing on the lattice

Aim:

systematically connect nonperturbative, smeared lattice calculations to continuum physics

Operator product expansion

Wilson's idea: operator product expansion (OPE) Wilson, Phys. Rev. 179 (1969) 1499

nonlocal operator ~ (perturbative) coefficients x product of local operators

Operator product expansion

Wilson's idea: operator product expansion (OPE)

Wilson, Phys. Rev. 179 (1969) 1499

nonlocal operator ~ (perturbative) coefficients x product of local operators

For example, in free scalar field theory

operator products

$$\phi(x)\phi(0) = \frac{1}{4\pi x^2}\mathbb{I} + 1\,\phi^2(0) + x^\mu \partial_\mu \phi^2(0) + \frac{1}{2}x^\mu x^\nu \partial_\mu \partial_\nu \phi^2(0) + \ldots$$

Wilson coefficients

[here the OPE is just a Laurent expansion]

Operator product expansion

Wilson's idea: operator product expansion (OPE) Wilson, Phys. Rev. 179 (1969) 1499

nonlocal operator ~ (perturbative) coefficients x product of local operators

For example, in free scalar field theory

$$\phi(x)\phi(0) = \frac{1}{4\pi x^2}\mathbb{I} + 1\,\phi^2(0) + x^\mu \partial_\mu \phi^2(0) + \frac{1}{2} x^\mu x^\nu \partial_\mu \partial_\nu \phi^2(0) + \ldots$$

Interactions modify the Wilson coefficients

$$\phi(x)\phi(0) = \frac{1}{4\pi x^2}\left(1 + a_{\mathbb{I}} \log(x^2\mu^2)\ldots\right)\mathbb{I} + \left(1 + a_{\phi^2} \log(x^2\mu^2)\ldots\right)\phi^2(0) + \ldots$$

... but not their leading-*x* behaviour
(determined by operator mass dimension)

Operator product expansion

(Formally) convenient to separate leading-x behaviour

$$\phi(x)\phi(0) = \frac{1}{x^2} c_{\mathbb{I}} \mathbb{I} + c_{\phi^2}\, \phi^2(0) + x^\mu c_{\partial_\mu \phi^2} \partial_\mu \phi^2(0) + x^\mu x^\nu c_{\partial_\mu \partial_\nu \phi^2} \partial_\mu \partial_\nu \phi^2(0) + \ldots$$

$$c_{\mathbb{I}} = \frac{1}{4\pi} + a_{\mathbb{I}} \log(x^2 \mu^2) + \ldots$$

In general

$$O(x) \overset{x \to 0}{\sim} \sum_k d_k(x^2)\, c_k(x,\mu)\, \mathcal{O}_R^{(k)}(0,\mu)$$

Operator relation - understood as acting in matrix element with N external fields

$$\langle \Omega | O(x) \tilde{\phi}(p_1) \ldots \tilde{\phi}(p_N) | \Omega \rangle \overset{x \to 0}{\sim} \sum_k d_k(x^2)\, c_k(x,\mu) \langle \Omega | \mathcal{O}_R^{(k)}(0,\mu) \tilde{\phi}(p_1) \ldots \tilde{\phi}(p_N) | \Omega \rangle$$

Smeared operator product expansion

Replace product of local operators with locally smeared operators (sOPE)

operator ~ (perturbative) coefficients x product of locally smeared operators

$$O(x) \overset{x\to 0}{\sim} \sum_k d_k(x^2)\bar{c}_k(x,\mu,\tau)\overline{\mathcal{O}}_R^{(k)}(0,\mu,\tau)$$

Smeared operator product expansion

Replace product of local operators with locally smeared operators

operator ~ (perturbative) coefficients x product of locally smeared operators

$$O(x) \overset{x\to 0}{\sim} \sum_k d_k(x^2)\overline{c}_k(x,\mu,\tau)\overline{\mathcal{O}}_R^{(k)}(0,\mu,\tau)$$

smearing scale τ

bar denotes smeared coefficients and operators

Smearing implemented via gradient flow

- nonperturbative matrix elements finite in continuum limit at fixed physical τ
- partially restores rotational symmetry
- removes operator mixing due to hypercubic lattice symmetry

Smeared operator product expansion

For example, consider the two-point function with OPE

$$\phi(x)\phi(0) = \frac{1}{x^2} c_{\mathbb{I}} \mathbb{I} + c_{\phi^2}\, \phi^2(0) + \mathcal{O}(x)$$

which becomes

$$\phi(x)\phi(0) = \frac{1}{x^2} \overline{c}_{\mathbb{I}} \mathbb{I} + \overline{c}_{\phi^2}\, \overline{\phi}^2(\tau, 0) + \mathcal{O}(x)$$

Gradient flow

Deterministic evolution of fields in "flow time" τ toward classical minimum

$$\frac{\partial}{\partial\tau}\overline{\phi}(\tau,x) = \partial^2\overline{\phi}(\tau,x) \qquad \overline{\phi}(\tau=0,x) = \phi(x)$$

Lüscher, Commun. Math. Phys. 293 (2010) 899

Exact solution possible with Dirichlet boundary conditions

$$\overline{\phi}(\tau,x) = e^{\tau\partial^2}\phi(x) \qquad \widetilde{\overline{\phi}}(\tau,p) = e^{-\tau p^2}\widetilde{\phi}(p) \qquad s_{\mathrm{rms}} = \sqrt{8\tau}$$

N.B. $[\,\tau\,] = 2$

ideal testing ground for sOPE

Renormalised theory on the boundary requires no further renormalisation

Lüscher and Weisz, JHEP 1102 (2011) 51

Smeared Wilson coefficients

Calculate Wilson coefficients in standard manner:

- for example, consider again the sOPE for the two-point function

$$\phi(x)\phi(0) = \frac{1}{x^2}\overline{c}_{\mathbb{I}}\mathbb{I} + \overline{c}_{\phi^2}\,\overline{\phi}^2(\tau,0) + \mathcal{O}(x)$$

Define Green functions via operators "embedded" in matrix element

- rearrange sOPE

$$\begin{aligned}\overline{c}_{\mathbb{I}}(x,\tau)\langle\Omega|\mathbb{I}\tilde{\phi}_R(p_1)\tilde{\phi}_R(p_2)|\Omega\rangle &= \langle\Omega|\phi(x)\phi(0)\tilde{\phi}_R(p_1)\tilde{\phi}_R(p_2)|\Omega\rangle \\ &\quad - \overline{c}_{\phi^2}(x,\tau)\langle\Omega|\overline{\phi}(0,\tau)\phi(0)\tilde{\phi}_R(p_1)\tilde{\phi}_R(p_2)|\Omega\rangle\end{aligned}$$

- work at tree-level and expand to order m^2

$$\overline{c}_{\mathbb{I}}^{(0)}(x,\tau) = \Big\{\langle\Omega|\phi(x)\phi(0)\tilde{\phi}_R(p_1)\tilde{\phi}_R(p_2)|\Omega\rangle - \langle\Omega|\overline{\phi}(0,\tau)\phi(0)\tilde{\phi}_R(p_1)\tilde{\phi}_R(p_2)|\Omega\rangle\Big\}_{\text{to }\mathcal{O}(m^2)}$$

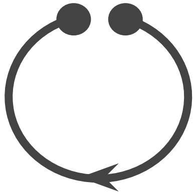
Smeared Wilson coefficients

So we have

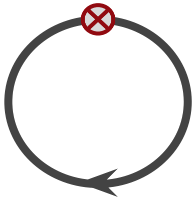
$$\bar{c}_{\mathbb{I}}^{(0)}(x,\tau) = \Big\{ \langle \Omega | \phi(x)\phi(0)\tilde{\phi}_R(p_1)\tilde{\phi}_R(p_2)|\Omega\rangle^{(0)} - \langle \Omega | \overline{\phi}(0,\tau)\phi(0)\tilde{\phi}_R(p_1)\tilde{\phi}_R(p_2)|\Omega\rangle^{(0)} \Big\}_{\text{to } \mathcal{O}(m^2)}$$

- graphically

$\bar{c}_{\mathbb{I}}^{(0)}(x,\tau) = \Big\{$



$-$



$\Big\}_{\text{to } \mathcal{O}(m^2)}$

$$\bar{c}_{\mathbb{I}}^{(0)}(x,\tau) = \left\{ \int \frac{d^4q}{(2\pi)^4}\frac{e^{iq\cdot x}}{q^2+m^2} - \int \frac{d^4q}{(2\pi)^4}\frac{e^{-q^2\tau}}{q^2+m^2} \right\}_{\text{to } \mathcal{O}(m^2)}$$

Smeared Wilson coefficients

- expanding in the mass and carrying out integrals

$$\bar{c}_{\mathbb{I}}^{(0)}(x,\tau) = \frac{1}{(2\pi)^2}\left[\frac{1}{x^2} - \frac{1}{4\tau} + \frac{m^2}{4}\left(1-\gamma_E + \log\left(\frac{4\tau}{x^2}\right)\right)\right]$$

- compare to the Wilson coefficient in the original OPE

$$c_{\mathbb{I}}^{(0)}(x,\mu) = \frac{1}{(2\pi)^2}\left[\frac{1}{x^2} - \frac{m^2}{4}\left(\gamma_E + \log\left(\pi^2\mu^2 x^2\right)\right)\right]$$

Beyond tree-level things get slightly trickier…

Smeared Wilson coefficients

Working at one-loop, the rearranged sOPE

$$\bar{c}_{\mathbb{I}}(x,\tau)\langle\Omega|\mathbb{I}\tilde{\phi}_R(p_1)\tilde{\phi}_R(p_2)|\Omega\rangle = \langle\Omega|\phi(x)\phi(0)\tilde{\phi}_R(p_1)\tilde{\phi}_R(p_2)|\Omega\rangle - \bar{c}_{\phi^2}(x,\tau)\langle\Omega|\overline{\phi}(0,\tau)\phi(0)\tilde{\phi}_R(p_1)\tilde{\phi}_R(p_2)|\Omega\rangle$$

becomes

$$\bar{c}_{\mathbb{I}}^{(1)}(x,\tau) = \langle\Omega|\phi(x)\phi(0)\tilde{\phi}_R(p_1)\tilde{\phi}_R(p_2)|\Omega\rangle^{(1)} - \Big[\langle\Omega|\overline{\phi}(0,\tau)\phi(0)\tilde{\phi}_R(p_1)\tilde{\phi}_R(p_2)|\Omega\rangle^{(1)} + \bar{c}_{\phi^2}^{(1)}(x,\tau)\langle\Omega|\overline{\phi}(0,\tau)\phi(0)\tilde{\phi}_R(p_1)\tilde{\phi}_R(p_2)|\Omega\rangle^{(0)}\Big]$$

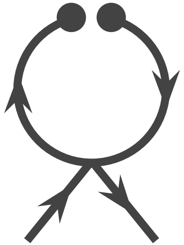
so we must first determine $\bar{c}_{\phi^2}^{(1)}(x,\tau)$

We have

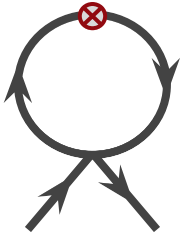
$$\bar{c}^{(1)}_{\phi^2}(x,\tau) = \left\{ \langle \Omega | \phi(x)\phi(0)\tilde{\phi}_R(p_1)\tilde{\phi}_R(p_2)|\Omega\rangle^{(1)} - \langle \Omega | \overline{\phi}(0,\tau)\phi(0)\tilde{\phi}_R(p_1)\tilde{\phi}_R(p_2)|\Omega\rangle^{(1)} \right\}_{\text{to } \mathcal{O}(m^0)}$$

- graphically

$$\bar{c}^{(1)}_{\phi^2}(x,\tau) = \Bigg\{$$



$$-$$



$$\Bigg\}_{\text{to } \mathcal{O}(m^0)}$$

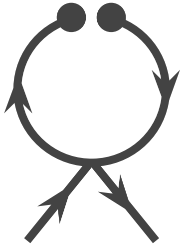
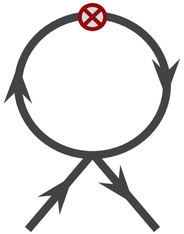
$$\bar{c}^{(1)}_{\phi^2}(x,\tau) = \left\{ \frac{1}{(p_1^2+m^2)(p_2^2+m^2)} \left[-\frac{\lambda}{2} \int_q \frac{e^{iq\cdot x} - e^{-q^2\tau}}{(q^2+m^2)((q-p_1-p_2)^2+m^2)} \right] \right\}_{\text{to } \mathcal{O}(m^0)}$$

Smeared Wilson coefficients

We have

$$\bar{c}^{(1)}_{\phi^2}(x,\tau) = \left\{ \langle \Omega | \phi(x)\phi(0)\tilde{\phi}_R(p_1)\tilde{\phi}_R(p_2)|\Omega\rangle^{(1)} - \langle \Omega | \overline{\phi}(0,\tau)\phi(0)\tilde{\phi}_R(p_1)\tilde{\phi}_R(p_2)|\Omega\rangle^{(1)} \right\}_{\text{to } \mathcal{O}(m^0)}$$

- graphically

$$\bar{c}^{(1)}_{\phi^2}(x,\tau) = \Bigg\{$$  $$-$$  $$\Bigg\}_{\text{to } \mathcal{O}(m^0)}$$

$$\bar{c}^{(1)}_{\phi^2}(x,\tau) = \left\{ \frac{1}{(p_1^2+m^2)(p_2^2+m^2)} \left[-\frac{\lambda}{2} \int_q \frac{e^{iq\cdot x} - e^{-q^2\tau}}{(q^2+m^2)((q-p_1-p_2)^2+m^2)} \right] \right\}_{\text{to } \mathcal{O}(m^0)}$$

momentum independence in small-x limit requires

$$\tau \propto x^2$$

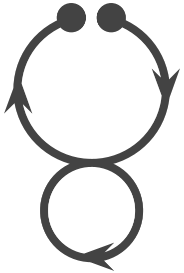
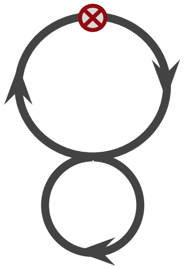
With

$$\bar{c}^{(1)}_{\phi^2}(x,\tau) = \frac{1}{2}\left(1 - \gamma_E + \log\left(\frac{4\tau}{x^2}\right)\right)$$

we need to determine remaining contribution

$$\bar{c}^{(1)}_{\mathbb{I}}(x,\tau) = \left\{ \langle \Omega | \phi(x)\phi(0)\tilde{\phi}_R(p_1)\tilde{\phi}_R(p_2)|\Omega\rangle^{(1)} - \langle \Omega | \overline{\phi}(0,\tau)\phi(0)\tilde{\phi}_R(p_1)\tilde{\phi}_R(p_2)|\Omega\rangle^{(1)} \right\}_{\text{to } \mathcal{O}(m^4)}$$

- graphically

$$\bar{c}^{(1)}_{\mathbb{I}}(x,\tau) = \Bigg\{$$  $$-$$  $$\Bigg\}_{\text{to } \mathcal{O}(m^4)}$$

Smeared Wilson coefficients

With

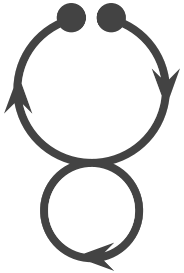
$$\bar{c}_{\phi^2}^{(1)}(x,\tau) = \frac{1}{2}\left(1 - \gamma_E + \log\left(\frac{4\tau}{x^2}\right)\right)$$

we need to determine remaining contribution

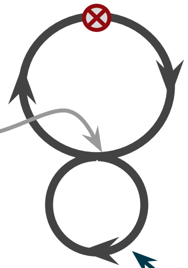
$$\bar{c}_{\mathbb{I}}^{(1)}(x,\tau) = \left\{\langle\Omega|\phi(x)\phi(0)\tilde{\phi}_R(p_1)\tilde{\phi}_R(p_2)|\Omega\rangle^{(1)} - \langle\Omega|\overline{\phi}(0,\tau)\phi(0)\tilde{\phi}_R(p_1)\tilde{\phi}_R(p_2)|\Omega\rangle^{(1)}\right\}_{\text{to } \mathcal{O}(m^4)}$$

- graphically

$\bar{c}_{\mathbb{I}}^{(1)}(x,\tau) = \Big\{$



$-$



$\Big\}_{\text{to } \mathcal{O}(m^4)}$

interaction vertex
at flow time zero

loop integral requires renormalisation
introduce new scale μ

Smeared Wilson coefficients

Tree-level calculation demonstrates

- recover leading-x behaviour

One-loop calculation demonstrates

- momentum independence of Wilson coefficients requires $\tau \propto x^2$
- quantum effects generate renormalisation scale dependence μ

Renormalisation group equations enable us to study scale dependence

Renormalisation group equations

Consider renormalisation group (RG) equations for connected Green functions

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu} = \mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial g} - \gamma_m m^2\frac{\partial}{\partial m^2}$$

Green function of N external scalar fields

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}G_N^{(\mathrm{conn})} = -\frac{N}{2}\gamma\, G_N^{(\mathrm{conn})}$$

Green function of renormalised operator coupled to N external scalar fields

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}G_N^{(\mathrm{conn})}(\phi_R^2) = \left(\gamma_m - \frac{N}{2}\gamma\right)\, G_N^{(\mathrm{conn})}(\phi_R^2)$$

Renormalisation group equations

Applying the operator

$$\mathcal{O}_{\mathrm{RG}} = \mu\frac{\mathrm{d}}{\mathrm{d}\mu} + \left(\frac{N}{2}+1\right)\gamma$$

to the OPE

$$G_{N+2}^{(\mathrm{conn})} = c_{\phi^2}(\mu x)\, G_N^{(\mathrm{conn})}(\phi_R^2) + \mathcal{O}(x)$$

nothing other than

$$\phi(x)\phi(0) = c_{\phi^2}\, \phi^2(0) + \mathcal{O}(x)$$

leads to the RG equation for the Wilson coefficient

$$\left[\mu\frac{\mathrm{d}}{\mathrm{d}\mu} + (\gamma+\gamma_m)\right] c_{\phi^2}(\mu x) = 0$$

anomalous dimension = difference between anomalous dimensions of non-local and local operators

Renormalisation group equations

For the sOPE we have

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} \to \mu \frac{\mathrm{d}}{\mathrm{d}\mu} + \kappa \frac{\mathrm{d}}{\mathrm{d}\kappa}$$

$$\tau = \kappa x^2$$

We now act with

$$\mathcal{O}_{\mathrm{RG}} = \mu \frac{\mathrm{d}}{\mathrm{d}\mu} + \left(\frac{N}{2} + 1\right)\gamma \to \mu \frac{\mathrm{d}}{\mathrm{d}\mu} + \kappa \frac{\mathrm{d}}{\mathrm{d}\kappa} + \left(\frac{N}{2} + 1\right)$$

on the sOPE

$$G_{N+2}^{(\mathrm{conn})} = \overline{c}_{\phi^2}(\mu x)\, G_N^{(\mathrm{conn})}(\overline{\phi}_R^2) + \mathcal{O}(x)$$

we obtain

$$\left[\mu \frac{\mathrm{d}}{\mathrm{d}\mu} + \kappa \frac{\mathrm{d}}{\mathrm{d}\kappa} + \gamma + \overline{\gamma}_{\phi^2}\right] \overline{c}_{\phi^2}(\mu x, \kappa) = 0$$

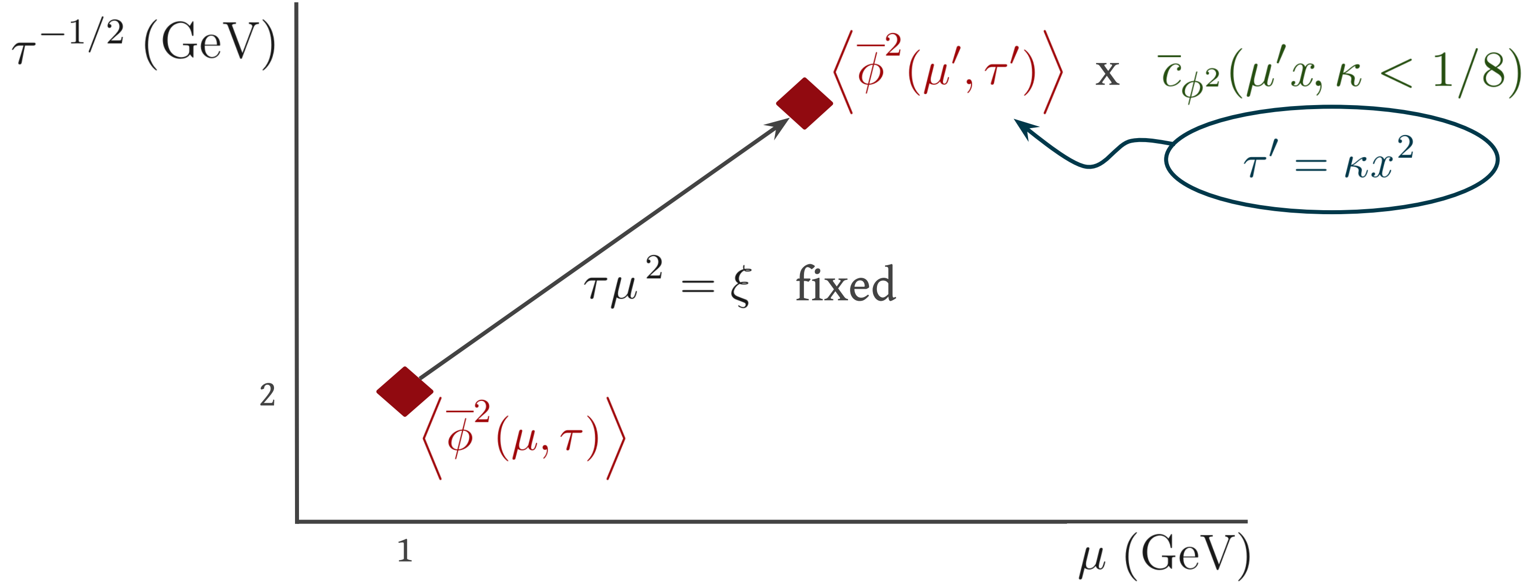
anomalous dimension = difference between anomalous dimensions of non-local and (smeared) local operators

Renormalisation group equations

Wilson coefficients and matrix elements a function of two scales

scale invariance ties scales together

Match to nonperturbative lattice calculations



sOPE summary

Introduced locally smeared operator product expansion

Scalar field theory demonstrates

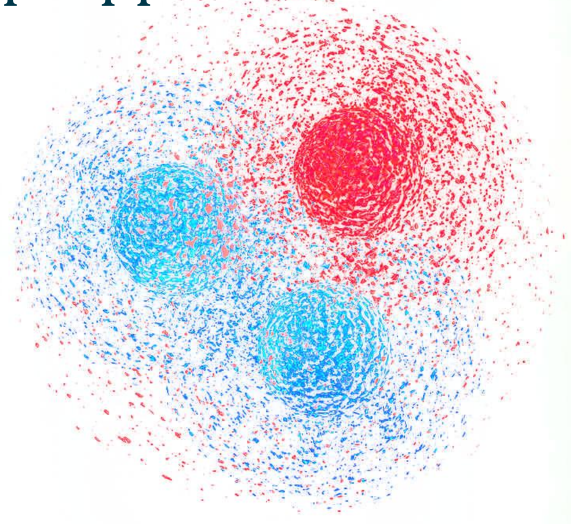
- momentum independence of coefficients connects smearing radius to space-time separation
- quantum effects generate renormalisation scale dependence

Renormalisation group considerations

- tie together smearing and renormalisation scale

Systematic method to incorporate smeared operators in lattice calculations

Next up: application to DIS



Gradient flow a well-established tool for QCD

- non-linear flow time equations complicate analysis
- flow time evolution still classical

Lüscher and Weisz, JHEP 1102 (2011) 51

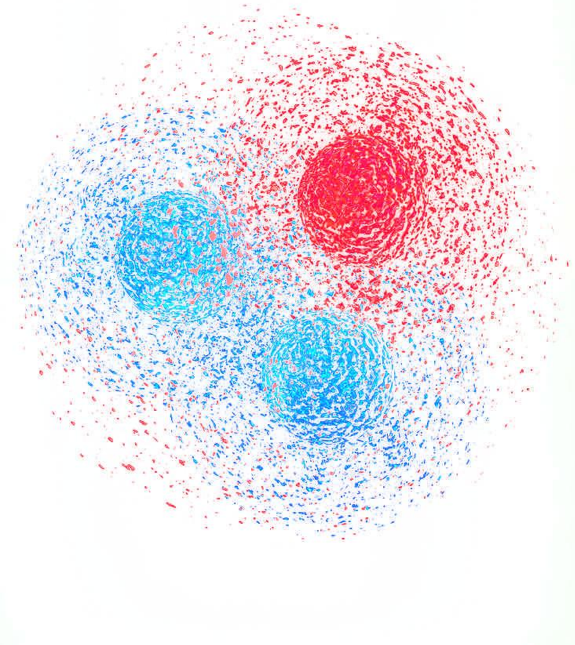
Demonstrate effectiveness

- determine Wilson coefficients at one loop
- calculate twist-2 matrix elements nonperturbatively

$$\overline{q}\gamma_\mu D_\nu D_\rho q \sim \frac{1}{a^2}\overline{q}\gamma_\mu q \longrightarrow \overline{q}\gamma_\mu D_\nu D_\rho q \sim \frac{1}{\tau}\overline{q}\gamma_\mu q$$

- apply nonperturbative step-scaling procedure

Thank you



cjmonahan@wm.edu