Effects of near-zero Dirac eigenmodes on axial U(1) symmetry at finite temperature

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for JLQCD collaboration
1. Introduction
Chiral symmetry breaking in QCD ($N_f=2$, $m_{ud}=0$)

$T = 0$

\[
\frac{SU(2)_L \times SU(2)_R \times U(1)_V \times U(1)_A}{SSB} \quad \text{Remains}
\]

$T > T_c$

\[
SU(2)_V \rightarrow SU(2)_L \times SU(2)_R \quad \text{Restored}
\]

\[
U(1)_A \rightarrow ??
\]

Susceptibilities, Dirac Spectrum

Cossu’s talk

This Talk
Dirac Spectrum and Symmetry

\[ SU(2)_L \times SU(2)_R \]

Banks-Casher Relation

\[ |\rho(0)| = \frac{\sum}{\pi} \]

\[ U(1)_A \]

Atiyah-Singer Index Theorem

\[ n_+ - n_- = \nu \]

\[ n_{\pm} : \# \text{ of chiral zero-modes} \]

Dirac low modes are important for both symmetries
Dirac Spectrum and Symmetry

Aoki-Fukaya-Taniguchi (2012) argued that, if we assume

- SU(2) x SU(2) is restored ( \( T > T_c \) )
- Ginsparg-Wilson relation is satisfied
- Analyticity in mass

\[ \rho = c_3 \lambda^3 + \cdots \]

**Spectrum starts from cubic power**

\[ U(1)_A \text{ anomaly is invisible in the (pseudo) scalar correlators} \]

\[ \text{(Vol} \rightarrow \infty) \]

\[ m_{ud} \rightarrow 0 \]

\*G. Cossu et al (JLQCD 2013) reported a gap in the Dirac spectrum
Cohen(1996) argued that:

If the chiral zero-mode's effect is ignored, and if there is a gap in the Dirac spectrum

\[ \rightarrow U(1)_A \text{ breaking susceptibility} \]

\[ = \chi_\pi - \chi_\delta \]

\[ = \int_0^\infty d\lambda \frac{4m^2 \rho(\lambda)}{(m^2 + \lambda^2)^2} = 0 \]
(Controversial) Previous lattice studies

<table>
<thead>
<tr>
<th>Group</th>
<th>Action</th>
<th>Vol.</th>
<th>Gap</th>
<th>U(1)$_A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>JLQCD(2013)</td>
<td>Overlap Fixed Topology</td>
<td>L=16</td>
<td>Yes</td>
<td>Restored</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\rho \sim \lambda^3 + \ldots$</td>
<td></td>
</tr>
<tr>
<td>Ohno et al (2011)</td>
<td>HISQ</td>
<td>L=32</td>
<td>No</td>
<td>Violated</td>
</tr>
<tr>
<td>LLNL/RBC (2013)</td>
<td>Domain-wall</td>
<td>L=16, 32</td>
<td>No</td>
<td>Violated</td>
</tr>
</tbody>
</table>

What makes the difference: Finite V effects? Fixed topology? Chiral symmetry?
This Work

Finite volume $\rightarrow$ Larger volume

Fixed Topology $\rightarrow$ Tunneling Allowed

Chiral symmetry $\rightarrow$ OV/DW reweighting
### Whats’ New in This work?

<table>
<thead>
<tr>
<th>Feature</th>
<th>G.Cossu et al (2013)</th>
<th>This Work</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fermion</td>
<td>Overlap</td>
<td>Mobius Domain-wall</td>
</tr>
<tr>
<td>GW-rel. Cost</td>
<td>Exact</td>
<td>$m_{\text{res}} \approx 1\text{MeV}$ or lower</td>
</tr>
<tr>
<td>Lat. Size</td>
<td>16</td>
<td>16, 32</td>
</tr>
<tr>
<td>Topology tunneling</td>
<td>Frozen</td>
<td>Allowed</td>
</tr>
<tr>
<td>Comment</td>
<td></td>
<td>We also try reweighting to OV</td>
</tr>
</tbody>
</table>
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3. Domain-wall Dirac spectrum
4. Violation of Ginsparg-Wilson relation
5. (Reweighted) overlap Dirac spectrum
6. Summary
2. Mobius DW
# Mobius Domain Wall


Overlap: 
$$D_N(m) = \frac{1 + m}{2} + \frac{1 - m}{2} \gamma_5 \text{sgn}(H_K).$$

(Satisfy Ginsparg-Wilson relation)

<table>
<thead>
<tr>
<th>Domain Wall</th>
<th>Mobius DW</th>
</tr>
</thead>
<tbody>
<tr>
<td>4-dim eff. operator</td>
<td>$D^4 = \frac{1 + m}{2} + \frac{1 - m}{2} \gamma_5 \frac{T^{-Ls} - 1}{T^{-Ls} + 1}.$</td>
</tr>
<tr>
<td>$T^{-1} = \frac{1 + H_T}{1 - H_T}$</td>
<td>$D^4 = \frac{1 + m}{2} + \frac{1 - m}{2} \gamma_5 \prod_{s}^{Ls} \frac{T_s^{-1} - 1}{T_s^{-1} + 1}.$</td>
</tr>
<tr>
<td>$H_T = \frac{D_W}{2 + D_W}$</td>
<td>$T_s^{-1} = \frac{1 + \omega_s H_M}{1 - \omega_s H_M}.$</td>
</tr>
<tr>
<td>$H_M = \frac{b D_W}{2 + c D_W}$</td>
<td></td>
</tr>
</tbody>
</table>

New parameter $b$, $c$

Parameter $L_s$

(Ls→∞ : OV)

$L_s, b, c$

(Ls→∞ : OV)

b and c make $m_{\text{res}}$ small

(b=2, c=1, $10^{-1}$-$10^{-3}$ smaller $m_{\text{res}}$ for Ls=12)
Lattice set up

Gauge action: tree level **Symoznik**
Fermion : Mobius DW(b=2, c=1, Scaled Shamir + Tanh) w/ **Stout** smearing(3)
code : lrolro++ (G. Cossu et al.)
Resource : BG/Q (KEK)

<table>
<thead>
<tr>
<th>$L^3 \times L_t$</th>
<th>$\beta$</th>
<th>$m_{ud}$ (MeV)</th>
<th>$L_s$</th>
<th>$m_{res}$ (MeV)</th>
<th>Temp. (MeV)</th>
<th>Note</th>
</tr>
</thead>
<tbody>
<tr>
<td>$16^3 \times 8$</td>
<td>4.07</td>
<td>30</td>
<td>12</td>
<td>2.5</td>
<td>180</td>
<td>488 Conf. every 50 Trj.</td>
</tr>
<tr>
<td>$16^3 \times 8$</td>
<td>4.07</td>
<td>3.0</td>
<td>24</td>
<td>1.4</td>
<td>180</td>
<td>319 Conf. every 20 Trj.</td>
</tr>
<tr>
<td>$16^3 \times 8$</td>
<td>4.10</td>
<td>32</td>
<td>12</td>
<td>1.2</td>
<td>200</td>
<td>480 Conf. every 50 Trj.</td>
</tr>
<tr>
<td>$16^3 \times 8$</td>
<td>4.10</td>
<td>3.2</td>
<td>24</td>
<td>0.8</td>
<td>200</td>
<td>538 Conf. every 50 Trj.</td>
</tr>
<tr>
<td>$32^3 \times 8$</td>
<td>4.10</td>
<td>32</td>
<td>12</td>
<td>1.7</td>
<td>200</td>
<td>175 Conf. every 20 Trj.</td>
</tr>
<tr>
<td>$32^3 \times 8$</td>
<td>4.10</td>
<td>16</td>
<td>24</td>
<td>1.7</td>
<td>200</td>
<td>294 Conf. every 20 Trj.</td>
</tr>
<tr>
<td>$32^3 \times 8$</td>
<td>4.10</td>
<td>3.2</td>
<td>24</td>
<td>-</td>
<td>200</td>
<td>88 Conf. every 10 Trj.</td>
</tr>
</tbody>
</table>
Topological charge changes along HMC

\[ L = 16, \ \beta = 4.10, \ m = 0.01, \ L_s = 12 \]
Tc Estimation

Polyakov & Chiral condensate

Chiral Condensate

Above Tc (T=200MeV)

Around Tc (T=180MeV)

Vol. dependence of Polyakov loop
Decreasing of Chiral condensate
3. Domain-wall Dirac spectrum
Observable

Histogram of Dirac operator

\[ H_m \psi_i = \lambda_i^m \psi_i \]

\[ H_m = \gamma_5 [(1 - m_{ud}) D^4 + m_{ud}] \]

\[ D^4 = [\mathcal{P}^{-1} (D^5_{DW}(m = 1))^{-1} D^5_{DW}(m_{ud}) \mathcal{P}]_{11} \]
3. Histogram for DW(T ~ Tc)

$T = 180\text{MeV} \sim T_c (L = 16)$

$\rho(\lambda)$

$\rho(\lambda) \, a^3$

$\lambda$

$\lambda - m_{ud} a$

$\lambda - m_{ud} a$

$m_{ud} = 30\text{MeV}$

$m_{res} = 2.5\text{MeV}$

$m_{ud} = 3.0\text{MeV}$

$m_{res} = 1.4\text{MeV}$

Gap? Finite $V$ effect?
3. Histogram for DW (above $T_c$)

$T=200\text{MeV}>T_c$ ($L=16$)

\[ \rho(\lambda) \]

\[ \rho(\lambda) \]

$|\lambda^{mal} - m_{uda}|$

$|\lambda^{mal} - m_{uda}|$

$m_{ud}=32\text{MeV}$

$m_{ud}=3.2\text{MeV}$

$m_{res}=1.2\text{MeV}$

$m_{res}=0.8\text{MeV}$

Gap? Finite V effect?
3. Histogram for DW(above $T_c$)

$T=200\text{MeV} > T_c \ (L=32)$

Very small but non-zero $\Rightarrow$ **Gap is not apparent**

$U(1)$ looks broken
Short summary

$L=32$, $T=200$ MeV $m_{ud}=3.2$MeV No clear Gap

$U(1)_A$ looks broken

Consistent with LLNL/RBC(2013). Then, What is the difference from OV(JLQCD)?

Finite $V$?

topology tunneling?

Violation of Ginsparg-Wilson relation?
4. Violation of Ginsparg-Wilson relation
Violating of Ginsparg-Wilson relation for each mode

\[ g_i \equiv \frac{\psi_i^\dagger \gamma_5 [D \gamma_5 + \gamma_5 D - 2D \gamma_5 D] \psi_i}{\lambda_i^m} \left[ \frac{(1 - m_{ud})^2}{2(1 + m_{ud})} \right] \]

\( g_i \) should be zero if GW is satisfied

Cf.

\[ m_{\text{res}} = \frac{\sum_i \lambda_i^m (1 + m_{ud})}{(1 - m_{ud})^2 (\lambda_i^m)^2} \sum_i \frac{1}{(\lambda_i^m)^2} g_i \]
Low-modes have significant violation of Ginsparg Wilson relation
5. (Reweighted) Overlap Dirac spectrum
Reweighting to OV

\[
\langle O \rangle_{ov} = \left\langle O \frac{\det D^2_{ov}(m_{ud})}{\det D^2_{DW}(m_{ud})} \frac{\det D^2_{DW}(1/2a)}{\det D^2_{ov}(1/2a)} \rightangle_{DW}
\]

We can measure OV quantity by using DW configuration

\[
\begin{align*}
\langle \rho(\lambda_{DW}) \rangle_{DW} \\
\langle \rho(\lambda_{ov}) \rangle_{DW} \\
\langle \rho(\lambda_{ov}) \rangle_{ov}
\end{align*}
\]

let's compare them!

partially quenched OV

rewighted overlap
$T=200\text{MeV},\ m_{ud}=32\text{MeV}$

**L32**

- **DW**
  - $\rho(\lambda a)^3$ vs. $|\lambda^{\text{mal}} - m_{ud} a|$ for L32 with DW-sHtTanh-32x8x12-b4.10-M1.00-mud0.01

- **Partially Quenched OV**
  - $\rho(\lambda a)^3$ vs. $|\lambda^{\text{mal}} - m_{ud} a|$ for L32 HovTanhthre0.24m0.01

- **Reweighted OV**
  - Reweighting not available

**L16**

- **DW**
  - $\rho(\lambda a)^3$ vs. $|\lambda^{\text{mal}} - m_{ud} a|$ for L16 with DW-sHtTanh-16x8x12-b4.10-M1.00-mud0.01

- **Partially Quenched OV**
  - $\rho(\lambda a)^3$ vs. $|\lambda^{\text{mal}} - m_{ud} a|$ for HovTanhthre0.35-Beta4.10-m0.01

- **Reweighted OV**
  - $\rho(\lambda a)^3$ vs. $|\lambda^{\text{mal}} - m_{ud} a|$ for HovTanhthre0.35-Beta4.10-m0.01

**Domain-wall and overlap: visible difference.**
T=200MeV, $m_{ud}=3.2$MeV

Consistent with LLNL/RBC 2013

Consistent with JLQCD 2013

Isolated chiral zero-modes
**T=200MeV, m_{ud}=3.2MeV**

<table>
<thead>
<tr>
<th>Observation</th>
<th></th>
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</thead>
<tbody>
<tr>
<td>• Strong violation of Ginsparg-Wilson relation in the low lying mode</td>
<td>• the histograms (DW vs OV) look different</td>
</tr>
<tr>
<td>• Overlap Dirac operator has isolated chiral zero-modes + gap. (DW vs pqOV)</td>
<td>• Exactly chiral zero-modes should disappear in the large volume limit</td>
</tr>
<tr>
<td>• The gap looks stable as Volume increases. (Partially quenched OV L=16 vs L32)</td>
<td>• This gap may suggest $U(1)_A$ symmetry restoration</td>
</tr>
</tbody>
</table>

• We need to confirm this in L=32 overlap (or DW with better chirality) simulations.
6. Summary
Summary

We have studied eigenvalue distribution of DW and (rewighted)overlap Dirac operators above Tc

1. Mobius Domain-wall spectrum
   => $U(1)_A$ is broken. consistent with LLNL/RBC(2013)

2. We found significant violation of chiral symmetry of low-lying modes even when $m_{\text{res}}$ is small.

3. OV/DW reweighting shows gap for lighter mass
   => $U(1)_A$ restoration? consistent with JLQCD(2013)

4. More study of finite volume effect is necessary.
   (OV/DW reweighting works only for smaller lattice)
Backup
$T=200\text{MeV}, \ m_{ud}=16\text{MeV}$

(beta=4.10 m=0.005)

L32

$DW$  

$\rho(\lambda) a^3$

$0.0005$  

$0.001$  

$0.0015$  

$I\lambda^m a - m_{ud} a$

L16

$DW$  

$\rho(\lambda) a^3$

$0.0005$  

$0.001$  

$0.0015$  

$I\lambda^m a - m_{ud} a$

Partially Quenched OV

$DW$  

$\rho(\lambda) a^3$

$0.0005$  

$0.001$  

$0.0015$  

$I\lambda^m a - m_{ud} a$

Reweighted OV

Reweighting not available

$DW$  

$\rho(\lambda) a^3$

$0.0005$  

$0.001$  

$0.0015$  

$I\lambda^m a - m_{ud} a$
Reweighting to OV with UV suppressing determinant

\[ R_{UVS} = \left( \frac{\det \gamma_5 D_{ov}(m_{ud})}{\det \gamma_5 D_{DW}(m_{ud})} \right)^2 \left( \frac{\det \gamma_5 D_{DW}(M)}{\det \gamma_5 D_{ov}(M)} \right)^2 \]

DW/OV reweighting is UV surpassing determinant. unphysical mode suppressed by heavy unphysical modes M~O(1/a).
- **L16 b4.07**
  - $m_{\text{res}} = 2.5 \text{MeV}$
  - $g_i$ vs $\text{E-val DW-sHtTanh-16x8x12-b4.07-M1.00-m0.01}$

- **L16 b4.10**
  - $m_{\text{res}} = 1.2 \text{MeV}$
  - $g_i$ vs $\text{E-val DW-sHtTanh-16x8x12-b4.10-M1.00-m0.01}$

- **L32 b4.10**
  - $m_{\text{res}} = 1.7 \text{MeV}$
  - $g_i$ vs $\text{E-val DW-sHtTanh-32x8x12-b4.10-M1.00-m0.01}$

- **ma = 0.01**
  - $m \sim 30 \text{MeV}$

- **ma = 0.001**
  - $m \sim 3 \text{MeV}$
  - $m_{\text{res}} = 1.4 \text{MeV}$
  - $g_i$ vs $\text{E-val DW-sHtTanh-16x8x24-b4.10-M1.00-m0.001}$

  - $m_{\text{res}} = 0.8 \text{MeV}$
  - $g_i$ vs $\text{E-val DW-sHtTanh-16x8x24-b4.1.00-m0.001}$

  - $m_{\text{res}} = 1.7 \text{MeV}$
  - $g_i$ vs $\text{E-val DW-sHtTanh-32x8x24-b4.10-M1.00-m0.005}$

  - $g_i$ vs $\text{E-val DW-sHtTanh-32x8x24-b4.1.00-m0.001}$
\[ f(x) = a + c^* x^*^3 \]

Variance of residuals (reduced chi-square) = WSSR/ndf  : 1.33016

Final set of parameters  
<table>
<thead>
<tr>
<th>Asymptotic Standard Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a )  = 0.000132414 +/- 6.752e-05  (50.99%)</td>
</tr>
<tr>
<td>( c )  = 6.76224 +/- 1.104  (16.32%)</td>
</tr>
</tbody>
</table>
Large violation of GW-rel

$m_{\text{res}}$(Next to lowest) history

History of $m_{\text{res}}$ from $g_{ij}$: plot CP-smeared-SymDW-sHtTanh-16x8x24-b4.10-M1.00-mud0.001

Date: 2014/06/04 12:12:19

$Q_{\text{top}}(f_{ij}; \text{Cut off}=2)$

$m_{\text{res}}(g_{ij}; \text{Cut off}=2)$

$m_{ud}$
Figure 9: Residual mass with the scaled-Shamir kernel and tanh approximation. The results with $L_s = 6, 8, 12, 16$ are plotted as a function of the scale parameter $b$. $c=1$

\[ H_M = \gamma_5 \frac{bD_W}{2 + cD_W}, \]