Grassmann Tensor Renormalization Group Study of Lattice QED with Theta Term in Two Dimensions

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Background

Tensor Renormalization Group (TRG)

M. Levin and C. P. Nave, Phys. Rev. Lett. **99** 120601 (2007).

- can avoid the numerical sign problem!
- requires too much computational resources in 4 dimensions.

  It is beyond the reach of current computer facilities.

Grassmann Tensor Renormalization Group (GTRG)

Z.-C. Gu, F. Verstraete and X.-G. Wen, arXiv:1004.2563 [cond-mat.str-el].

- extension to fermionic systems.

We apply the GTRG to 2-dimensional Lattice QED including fermions.

This is a pilot study toward lattice QCD.

Tensor network model

We translate lattice gauge theory into a tensor network model.

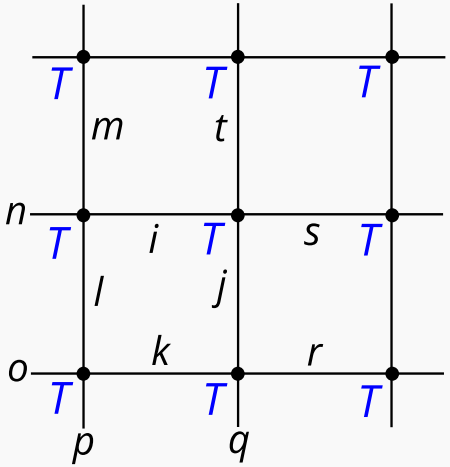
$$Z = \int d\psi\, d\bar{\psi} dU\, e^{-S_f[\psi,\bar{\psi}]-S_g[U]} \qquad \text{partition function}$$

e^{-S_f} is decomposed by using properties of Grassmann numbers.
e^{-S_g} is expanded by the character expansion.

Y. Liu, Y. Meurice, M. P. Qin, J. Unmuth-Yockey, T. Xiang, Z. Y. Xie, J. F. Yu and H. Zou, Phys. Rev. D **88** 056005 (2013).

$$e^{\beta\cos\varphi_p} = \sum_{k=-\infty}^{\infty} e^{ik\varphi_p} I_k(\beta)$$

I_k : modified Bessel function



After integrating out all link variables,

$$Z = \int \sum_{i,j,k,l,\cdots} T_{i,m,n,l} T_{s,t,i,j} T_{r,j,k,q} T_{k,l,o,p} \cdots \qquad \text{tensor network}$$

$T_{i,j,k,l}$ includes Grassmann numbers ψ, $\bar{\psi}$, $d\psi$, $d\bar{\psi}$. $T_{i,j,k,l}$ is a G-even number.

With topological term

Including the θ term

$$S_g \to S_g - i\theta Q$$

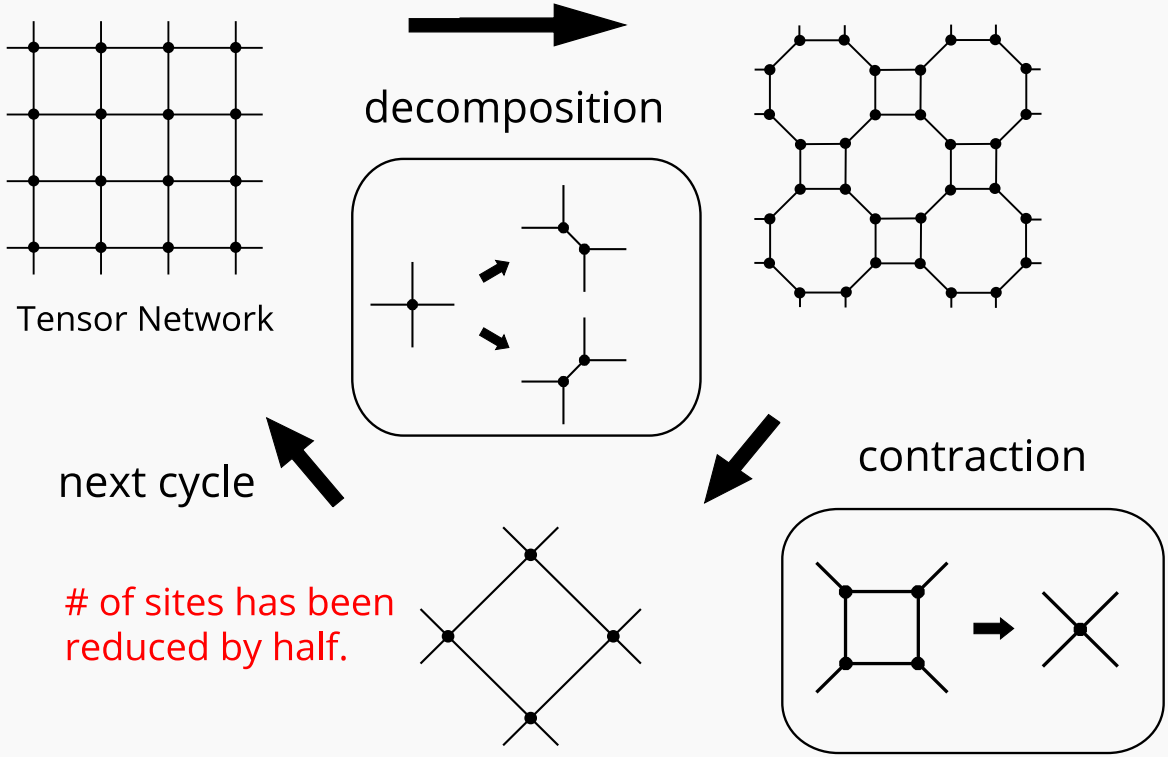
The topological charge Q is an integer even on lattice.

character expansion including the θ term

A. S. Hassan, M. Imachi and H. Yoneyama, Prog. Theor. Phys. **93** 161 (1995).

$$\exp\left\{\beta\cos\varphi_p + i\frac{\theta}{2\pi}q_p\right\} = \sum_{m=-\infty}^{\infty} e^{im\varphi_p} \sum_{l=-\infty}^{\infty} I_l(\beta)\frac{2\sin\frac{\theta+2\pi(m-l)}{2}}{\theta+2\pi(m-l)}$$

Tensor renormalization group



The key idea is low-rank approximation by the SVD.

$$T_{i,j,k,l} \simeq \sum_{m=1}^{D} U_{(i,j),m} \Lambda_m V_{m,(k,l)}$$ truncated by a number D!

We keep only large singular values.

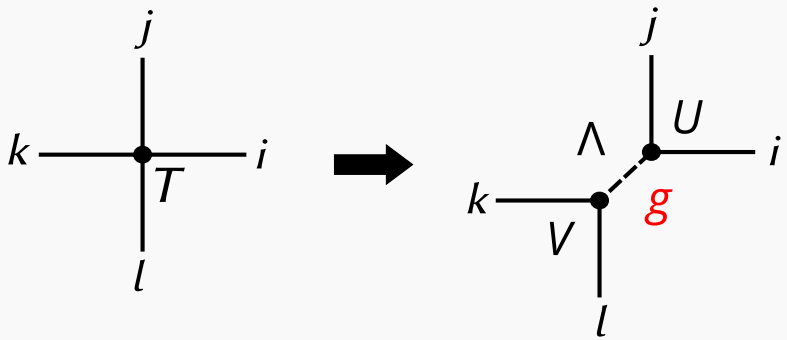
Grassmann version of SVD

The SVD is modified in order to keep G-evenness.

Z.-C. Gu, F. Verstraete and X.-G. Wen, arXiv:1004.2563 [cond-mat.str-el].

Z.-C. Gu, Phys. Rev. B **88** 115139 (2013).

The Grassmann version of the Kronecker delta is inserted.



$$g_{m,n} \equiv \delta_{m,n}\, \bar{\xi}^{n_f}\, \xi^{m_f}, \qquad \int d\xi^{m_f}\, d\bar{\xi}^{n_f}\, g_{m,n} = 1, \qquad m_f = n_f = \begin{cases} 1 & U \text{ and } V \text{ are G-odd.} \\ 0 & U \text{ and } V \text{ are G-even.} \end{cases}$$

$$T_{i,j,k,l} = \sum_{m,n} \int \underbrace{(U_{(i,j),m}\, d\xi^{m_f})}_{\text{G-even}}\, \Lambda_m\, g_{m,n}\, \underbrace{(d\bar{\xi}^{n_f} V_{n,(k,l)})}_{\text{G-even}}$$

Numerical test 1

$\beta - \kappa$ phase diagram with one flavor of the Wilson fermion

$\det D$ can be negative near κ_c.

- free fermion ($\beta = \infty$)

  $\kappa_c = 0.25$

  Ising universality class ($\alpha = 0,\ \nu = 1$)

- strong coupling limit ($\beta = 0$)

  Ising universality class ($\alpha = 0,\ \nu = 1$)

  H. Gausterer and C. B. Lang, Nucl. Phys. B **455**, 785 (1995).

  U. Wenger, Phys. Rev. D **80**, 071503 (2009).

- finite coupling

  $\alpha = 1,\ \nu = \frac{2}{3}$

  V. Azcoiti, G. Di Carlo, A. Galante, A. F. Grillo and V. Laliena, Phys. Rev. D **53**, 5069 (1996).

  No phase transition

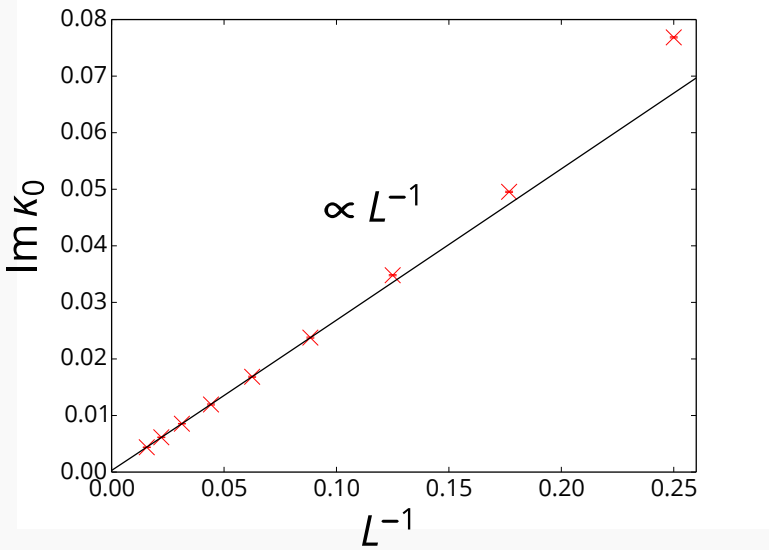
  R. Kenna, C. Pinto and J. Sexton, arXiv:hep-lat/9812004.

We verify the disagreement by using the GTRG.

Numerical test 1

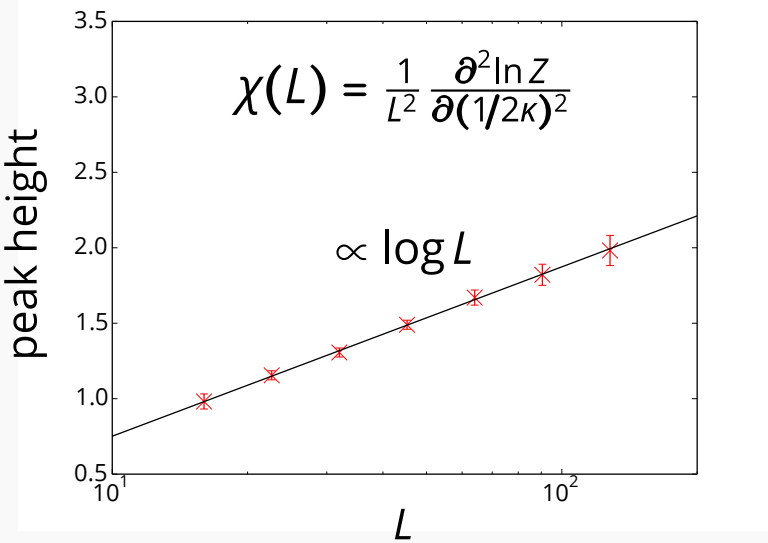
$\beta = 10.0,\ L \leq 64,\ D = 96$

Partiton function zero in complex κ plane



$\nu = 0.995(19)$

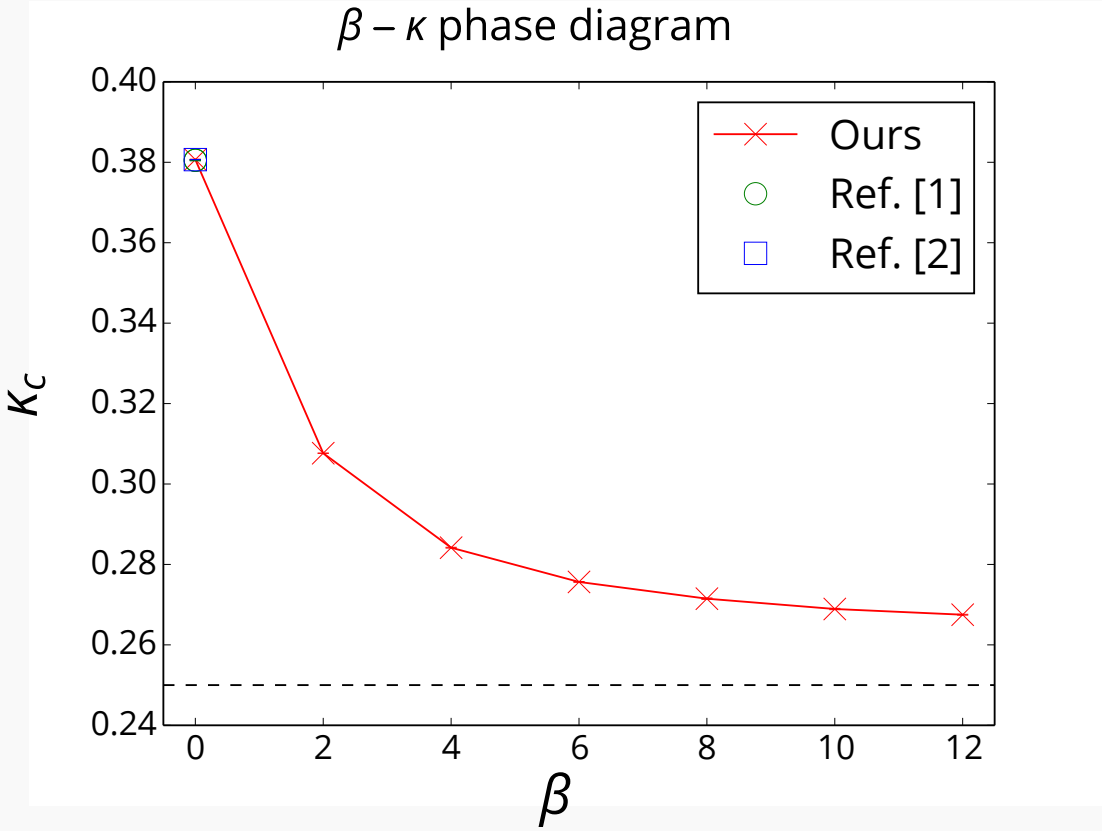
Peak height of Chiral susceptibility



$\alpha \simeq 0$

Our result supports the Ising universality class even at finite coupling.

Numerical test 1



[1] H. Gausterer and C. B. Lang, Nucl. Phys. B **455**, 785 (1995).

[2] U. Wenger, Phys. Rev. D **80**, 071503 (2009).

Our result is very consistent with others in the strong coupling limit and approaching 0.25 as β increases.

Numerical test 2

Phase structure at $\theta = \pi$

The θ term gives a pure imaginary contribution to the action.

There are some previous studies based on the Hamiltonian LGT.

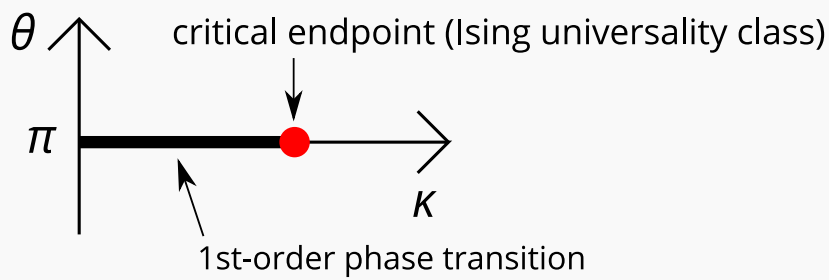
C. J. Hamer, J. Kogut, D. P. Crewther and M. M. Mazzolini, Nucl. Phys. B **208**, 413 (1982).

A. J. Schiller and J. Ranft, Nucl. Phys. B **225**, 204 (1983).

T. Byrnes, P. Sriganesh, R. J. Bursill and C. J. Hamer, Phys. Rev. D **66**, 013002 (2002).

Our approach is based on the Euclidean path-integral formulation.

expected phase structure

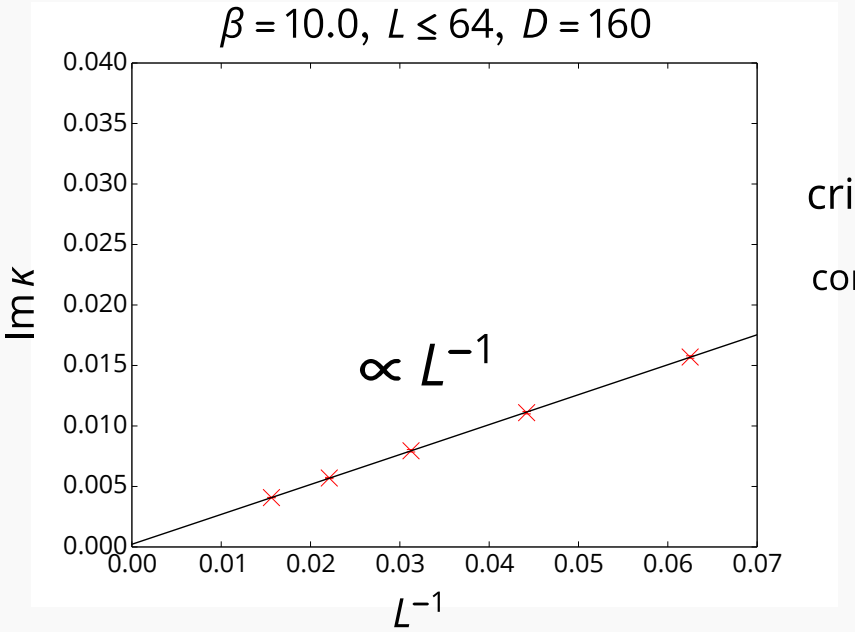


Numerical test 2

Partition function zero analysis in the complex κ plane at $\theta = \pi$

The value of κ_c is very different from the one in the case $\theta = 0$.

For $\beta = 10.0$, $\kappa_c = 0.24144(13)$ smaller than 0.25!



critical exponent $\nu = 0.999(38)$

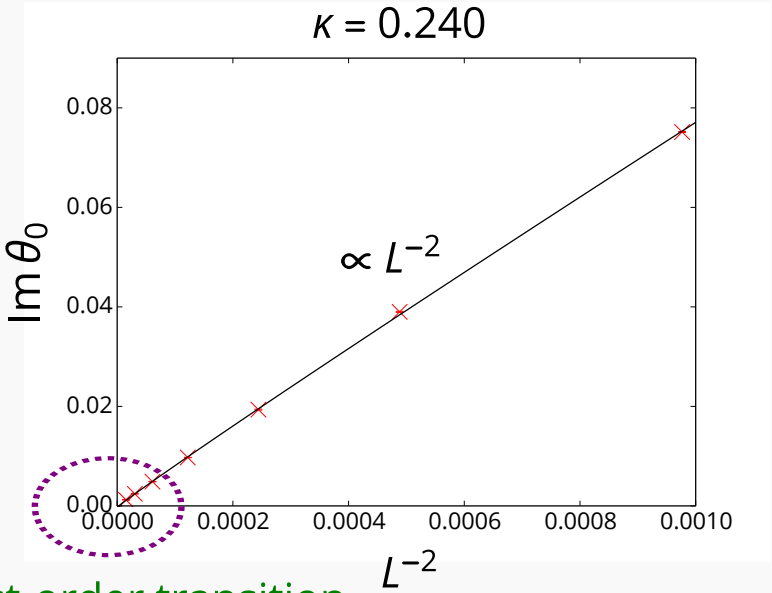
consistent with the Ising universality class

Numerical test 2

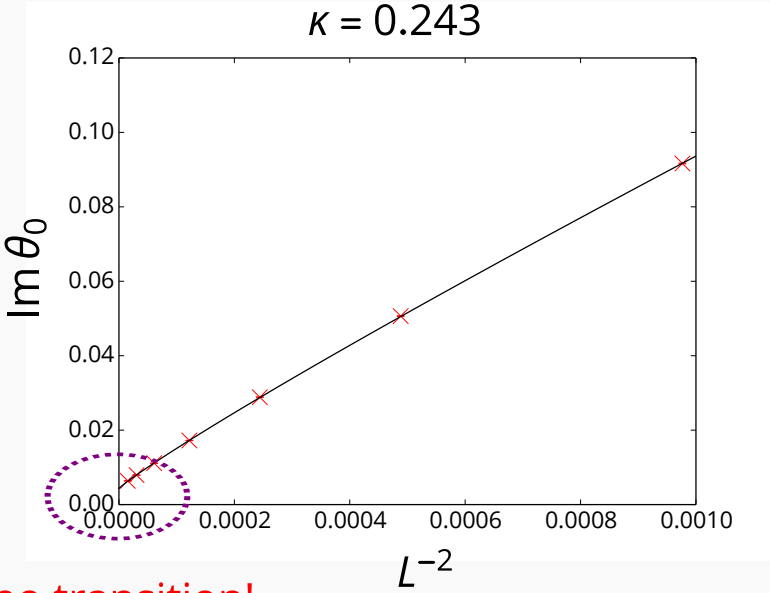
Partition function zero analysis in the complex θ plane

All zeros lie in the line $\mathrm{Re}\,\theta = \pi$.

$$\beta = 10.0,\ L \leq 256,\ D = 160$$



1st-order transition



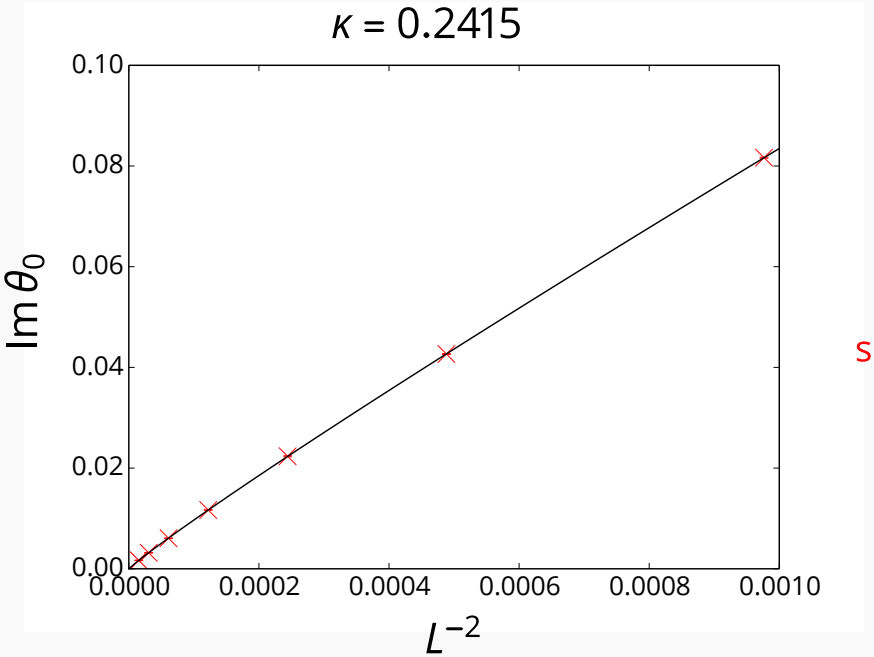
no transition!

The 1st-order phase transition exists where $\kappa < \kappa_c (= 0.24144(13))$.

κ_c indicates the critical endpoint.

Numerical test 2

same analysis near κ_c



$\propto L^{-y}$

y=1.8711(54)

slightly different from 2

θ is regarded as an external field parameter.

$$y = \frac{2\delta}{1+\delta} = \frac{2\nu-\beta}{\nu} = 1.875 \quad \text{(Ising universality class)}$$

Summary and Outlook

We have applied the GTRG to 2d lattice QED and investigated the phase structure.

- One flavor of Wilson fermion

  Our result agrees with others in the strong coupling limit.

  We conclude that the phase transition belongs to the Ising universality class even at finite coupling.

- With θ term

  We have succeeded in reproducing the expected phase structure.

  The GTRG enables us to distinguish the order of the phase transition.

  The remaining task is taking the continuum limit.

  Our critical endpoint should be same as the one obtained by the Hamiltonian LGT in the continuum limit.