Asymptotic Safety on the lattice: The nonlinear O(N) Sigma model

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Asymptotic Safety

RG approach to QFT

- theory at scale $k$ described by effective average action $\Gamma_k$
- upper (ultraviolet) cutoff $\Lambda$: $\Gamma_k = S_{mic}$
- use RG flow to integrate out fluctuations until lower cutoff $k = \lambda$
- fundamental theory valid on all scales, limit $\Lambda \to \infty$ and $\lambda \to 0$ exists
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Application to Classical Gravity

- unified theory requires quantization of spacetime metric
- at IR cutoff: Einstein-Hilbert action
- perturbative approach leads to severe divergences
- theory not renormalizable in perturbative way
Asymptotic Safety

- **Asymptotic Safety Scenario** [Weinberg '80]: gravity non-perturbatively renormalizable
- Needs **nongaussian (ultraviolet) fixed point with finite number of relevant directions** to protect UV from unphysical divergences
- **Asymptotic freedom** (QCD): theory approaches gaussian fixed point for $k \to \infty$

[cf. Reuter, Saueressig '02] [cf. Christiansen, Litim, Pawlowski, Rodigast '12]
Setting the stage

- nontrivial UV fixed point suspected in \((D > 2)\) nonlinear sigma models by perturbation theory and functional RG calculations [Codello, Percacci '08] [Flore, Wipf, Zanusso '12]

\[ S = \frac{1}{2g^2} \int d^Dx \ \partial_\mu \vec{\phi} \partial^\mu \vec{\phi}, \text{ where } \vec{\phi}^2 = 1 \]

- Flow diagram determinable from the lattice via Monte Carlo Renormalization Group (MCRG) techniques
Setting the stage

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**O(N) nonlinear Sigma model**

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- Flow diagram determinable from the lattice via Monte Carlo Renormalization Group (MCRG) techniques

**Our Setup**

- local HMC with O(N)-valued fields to generate Markov Chain
- blockspin transformation to integrate out fluctuations
- canonical demon method to determine effective couplings
  \([\text{Hasenbusch, Pinn, Wieczerkowski '95}]\)
RG picture on the lattice

- (discrete) lattice momenta cut off by inverse lattice spacing $a^{-1}$ and inverse linear box size $(aN)^{-1}$
- lattice simulation equivalent to integrating out all fluctuations inbetween
- correlation functions determined by direct measurement

\[ \text{UV cutoff} \quad \frac{1}{a} \quad \text{lattice simulation} \quad \frac{1}{(aN)} \quad \text{IR cutoff} \]

- Blockspin transformation: $a \rightarrow 2a$, $N \rightarrow N/2$

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**Observables**

- **(discrete) beta function** $\tilde{\beta}_i = \tilde{\beta}(g_i) = g^{\text{blocked}}_i - g_i$

- **(discrete) stability matrix** $S_{ij} = \frac{\partial \tilde{\beta}_i}{\partial g_j}$
Systematic Errors

- **finite volume** effects not visible for largest lattices considered ($32^3$)
- **discretisation errors** small near critical line
- effective action in demon method leads to **truncation errors**
- half group property of RG transformation $R_s$ (blockspin + demon) violated

\[ R_s \circ R_s \neq R_s^2 \]
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Option 1: Add more operators!

- follow derivative expansion; up to four operators, i.e. all possible operators up to fourth order in the momenta

$$\Phi \tilde{x} \propto P(\exp\{C \cdot \Phi \tilde{x} \sum_{x \in \Lambda} \tilde{x} \phi \})$$

\[\text{Hasenfratz, Hasenfratz, Heller, Karsch '84}\]

- parametrize $C = \sum c_i g_i$ such that for $g \to \infty$, $C \to \infty$ holds
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**Option 1: Add more operators!**

- follow derivative expansion ; up to four operators, i.e. all possible operators up to fourth order in the momenta

**Option 2: Use improved blockspin transformation!**

\[
\Phi_{\bar{x}} \propto P\left(\exp\left\{ C \cdot \Phi_{\bar{x}} \sum_{x \in \Lambda_{\bar{x}}} \phi_x \right\}\right) \quad \text{[Hasenfratz, Hasenfratz, Heller, Karsch '84]}
\]

- parametrize \(C = \sum_i c_i g_i\) such that for \(g \to \infty\), \(C \to \infty\) holds
$D = 2$: Asymptotic Freedom
- lattice beta function does not depend on the lattice size (in this truncation)
- considerable variation w.r.t. the optimization parameter $c_0$
- crucial implications: artificial fixed points emerge / true fixed points vanish due to truncation
Optimization of the RG transformation

- MC simulation allows to directly measure lattice correlation functions which are free from truncation errors
- blockspin transformation does not change the IR physics
- idea: simulate the ensemble with truncated couplings
- systematic truncation effects show as differences in the correlation functions of the original and truncated ensemble
Optimization of the RG transformation II

- **RG picture**: trajectories of the RG flow are **attracted by the renormalized trajectory**, which connects the fixed points of the RG flow.

- **Idea**: location of the renormalized trajectory depends on RG scheme.

- Optimization equivalent to an RG scheme where the renormalized trajectory is closest to a given truncation.

[cf. Shenker, Tobochnik '80]
1-Parameter effective action

- compare **two-point** correlation lengths, **ignore all higher** correlation functions
- **significant deviations** from $\xi_{64}/\xi_{32} = 2$ for large and small optimization parameter
- choose $c_0^{opt} = 2.8$ which is **closest to the expected value**
- beta function for $c_0^{opt} = 2.8$ approaches the known large-N value $\ln(2)/6\pi$
- additional zero crossing is an artifact of the truncation
- high-temperature fixed point at zero coupling and low-temperature fixed point at infinite coupling
trajectories flow from the low-temperature fixed point (LT FP) in the UV to the high-temperature fixed point (HT FP) in the IR
both operators are IR-irrelevant
low-temperature FP corresponds to Gaussian FP (asymptotic freedom)
$D = 3$: Asymptotic Safety
-0.1 -0.05 0 0.05 0.1 0 0.05 0.1 0.15 0.2 0.25 0.3

\[ \beta \quad 0 \quad g \quad 2.00 \quad 3.35 \quad 5.00 \]

- zero crossing at finite coupling with a UV-attractive direction indicates non-Gaussian fixed point
1-Parameter effective action

- critical exponent $\nu$ of the correlation length extracted from slope of the beta function
- $\nu$ deviates from expected behaviour for $N > 6$
2-Parameter effective action

- High temperature FP: spins randomly aligned, absolute disorder
- Low temperature FP: spins uniformly aligned, absolute order
- Non-Gaussian FP: 1 IR-relevant, 1 IR-irrelevant direction
- Critical line: separates symmetry broken and unbroken regime
- Renormalized trajectory: attractor for the RG trajectories
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- **renormalized trajectory**: attractor for the RG trajectories
starting from the usual Hamiltonian for the **Heisenberg ferromagnet** there are three possible IR fixed points of the RG flow

**universality** : non-gaussian FP corresponds to **Wilson-Fisher FP** of the linear $O(N)$ sigma model
• Heisenberg ferromagnet is an effective theory that is defined at some finite UV cutoff
• initial goal: theory that is IR- and UV-complete
- complete theory "lives" on the renormalized trajectory
- non-gaussian fixed point acts as a **UV fixed point** with one relevant direction
- asymptotic safety scenario realized (in this truncation ...)
2-Parameter effective action

\[ \nu \approx -1 \text{ (high temperature FP)} \]

\[ \nu \approx 1 \text{ (low temperature FP)} \]

\[ \nu = 0.62 \text{ (non-Gaussian FP)} \]

\[ \theta' \propto \nu^{-1} \]
2-Parameter effective action

high temperature FP: $\nu \approx -1$
2-Parameter effective action

- High temperature FP: $\nu \approx -1$
- Low temperature FP: $\nu \approx 1$
2-Parameter effective action

high temperature FP: \( \nu \approx -1 \)
low temperature FP: \( \nu \approx 1 \)
non-Gaussian FP: \( \nu = 0.62(3) \)
1-Parameter effective action

- significant improvement for the critical exponent $\nu$ of the correlation length
- large-N limit inconclusive (numerical effort grows with N)
3-Parameter effective action

- only irrelevant coupling is added
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Summary

- MCRG+Demon Method suitable to obtain global flow diagram
- in principle arbitrarily long trajectories on moderate lattices
- systematic errors of truncation reduced by optimization scheme

D=2

- flow diagram shows low-temperature (Gaußian) FP in the UV and high-temperature FP in the IR (asymptotic freedom reproduced)

D=3

- flow diagram reveals two trivial IR fixed points (absolute order and absolute disorder) and one nontrivial UV fixed point (Wilson-Fisher-FP)
- fixed point structure stable against change of truncation
- only one relevant direction → asymptotic safety scenario realized

Outlook

- applicable to fermionic models (e.g. Thirring model)